THEORY AND PRACTICE
OF
ALTERNATING CURRENTS

GENERAL PRINCIPLES, CIRCUITS
INSTRUMENTS, MEASUREMENTS

BY

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PREFACE

THE original intention of the author, when undertaking this work, was to write a single volume general textbook dealing with the theory and practice of alternating current electrical engineering. As the work progressed, however, it was realized that the scope of the subject was too great for adequate treatment in a single volume. Accordingly, the present volume is devoted to general principles, circuits, polyphase systems, non-sinusoidal wave-forms, the magnetization of iron, instruments, measurements, and an elementary treatment of the initial conditions in the simpler electric circuits.

The author is of opinion that a thorough grounding in the principles of alternating currents is essential before proceeding to the study of alternating current machines and apparatus. This portion of the subject has, therefore, been treated on a broader basis than is the case in some textbooks.

Among the special features of the present volume are an extended application of the circle (locus) diagram to series, parallel series-parallel circuits; the reduction of the general circuit to its equivalent series-parallel form and the development of a locus diagram therefor; the calculation of currents in unbalanced polyphase circuits and the determination of the neutral point potential in the case of unbalanced star-connected three-phase circuits; the theory of the principal types of measuring instruments employed in alternating current engineering; and a large number of worked examples in the text.

The deduction of the circle, or current locus, diagram for a general circuit involves the application of geometrical inversion, which principle has not hitherto received much consideration in textbooks for the English-speaking students, although it has received considerable development in Arnold and La Cour's *Wechselstromtechnik*, a portion of which has been translated into English by Professor Stanley Parker Smith.[1] The author has given considerable attention to the application of this (inversion) principle to circle diagrams for the simple, or fundamental, circuit, as well as its application to the general circuit. As an extension of the methods here developed the student is referred to the Advanced Course of

[1] *Theory and Calculation of Electric Currents* (1913). (Longmans Green & Co.)

Lectures in Engineering given, in 1921, by Professor Miles Walker under the auspices of the University of London. Some interesting applications of the principle of inversion to the predetermination of the performance of special types of induction motors were included in the lectures.[1]

In connection with the worked examples in the text, analytical, graphical and complex algebraic methods of solution have been employed. In some cases alternative methods of solution are given in order to show the student the steps involved in the application of the alternative methods.

The thanks of the author are due to his colleagues and numerous friends who have made suggestions, criticisms, and given help during the preparation of the work. Special thanks are due to Messrs. H. C. Mann, B.Sc. (Eng.), A.M.I.E.E., F. W. Harvey, B.A., B.Sc., and B. Hague, M.Sc. (Eng.), D.I.C., A.M.I.E.E.

The author is also indebted to a number of firms who have generously supplied him with drawings, photographs and data relating to measuring instruments. Among those to whom the special thanks of the author are due are the following: Messrs. Nalder Bros. & Thompson, Messrs. Tinsley & Co., The Cambridge Instrument Co., The Weston Electrical Instrument Co., The Western Electric Co., The British Thomson-Houston Co., The Metropolitan Vickers Electrical Co., Messrs. Crompton & Co., Messrs. Kelvin, Bottomley & Baird, and Messrs. Ferranti.

The thanks of the author are also due to the Senate of the University of London for permission to use questions from the Final Engineering (B.Sc.) examination papers, and to the City and Guilds of London Institute for permission to use questions from the Electrical Engineering examination papers.

A. T. DOVER.

[1] The lectures were printed in the *Electrician*, v. xc. (1921), pp. 216, 247, 302, 391, 418, 451, 478. They were subsequently extended and published in book form—*Methods of Controlling the Speed and Power Factor of Induction Motors*

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ABBREVIATIONS

NOTE. The abbreviations and symbols employed in this volume are generally in accordance with the recommendations of the International Electro-technical Commission.

Amperes	amp., A.
Centimetre	cm.
Electro-motive force	E.M.F.
Centigrade	C.
Foot, feet	ft.
Inch	in.
Henry	H.
Farad	F.
Kilowatt	kW.
Kilogramme	kg.
Gramme	gm.
Gramme-centimetre	gm-cm.
Magneto-motive force	M.M.F.
Milli-ampere	mA.
Milli-henry	mH.
Micro-farad	μF.
Micro-micro-farad	$\mu\mu$F.
Microhm	μO.
Millimetre	mm.
Pound	lb.
Potential difference	P.D.
Ohm	O.
Revolutions per second	revs. per sec.
Root-mean-square	R.M.S.
Second	sec.
Square centimetre	cm.^{2}
Square millimetre	mm.^{2}
Square inch	in.^{2}
Volts	V.
Watts	W.
Yard	yd.

CHIEF SYMBOLS

A = Coefficient of sine term in Fourier series. Constant.
a = Area of cross section.
B = Flux density.
= Coefficient of cosine term in Fourier series. Constant.
b = Susceptance.
C = Capacity.
= Complex constant.
D = Value of determinant.
= Distance between centres
d Diameter.
E_e = E.M.F. (root-mean-square value).
E_m = E.M.F. (maximum value of quantity varying with respect to time).
e = E.M.F. (instantaneous value)
F = Force.
= Ampere turns.
f = Frequency.
g = Conductance.
I = Current (root-mean-square value).
I_m = Current (maximum value of quantity varying with respect to time).
i = Current (instantaneous value).
J, j = Symbolic operators for vectors.
K, k = Constants.
k_f = Form factor.
L = Inductance.
L, l = Length.
M = Mutual inductance.
N = Number of turns.
n = Angular speed in revolutions per second.
= Number of phases in polyphase systems.
P = Power.
p = Power (instantaneous value).

$\mathcal{P}$ = Permeance.
Q = Electrostatic charge.
R = Resistance.
r = Radius.
S = Reluctance.
T = Period. Tension.
$\mathfrak{T}$ = Torque.
t = Time (instantaneous value).
V = Voltage, potential difference.
v = Voltage, potential difference (instantaneous value).
= Linear velocity.
W = Energy.
X = Reactance.
x = Abscissa of point with respect to co-ordinate axes.
Y = Admittance.
y = Ordinate of point with respect to co-ordinate axes.
Z = Impedance.

GREEK SYMBOLS

α = Coefficient of linear expansion.
= Current density.
$\alpha, \beta, \gamma, \theta, \psi$ = Angles.
δ = Small quantity. Thickness
Δ = Small quantity Delta connection of three-phase circuit.
Σ = Sign denoting summation of series.
ϵ = Base of natural logarithms = 2·7318. . .
η = Hysteretic coefficient.
= Efficiency.
Φ = Flux.
φ = Phase difference.
Θ = Temperature.
κ = Dielectric constant.
μ = Permeability.
ξ = Amplitude factor.
= Eddy-current loss coefficient
= Young's modulus of elasticity.

π = Ratio of circumference to diameter of circle (= 3·14 . . .).

ρ = Specific resistance.

σ = Density.

τ = Time constant (= L/R).

ω = Angular velocity.

ALGEBRAIC SYMBOLS

$\neq$ = Not equal to.

$>$ = Greater than.

$<$ = Less than.

$\angle$ = Angle.

! = Factorial (e.g. 5 ! = 5 × 4 × 3 × 2 × 1).

∞ = Infinity.

NOTES. **Maximum values** of quantities varying with respect to time are denoted by the subscript m attached to the symbol of the quantity.

Vector diagrams. All vector diagrams have been drawn for counter clockwise rotation.

E.M.F. vectors are represented by an ordinary arrow-head.
Flux vectors are represented by a double arrow-head.
Ampere-turn vectors are represented by a solid arrow-head.
Current vectors are represented by a closed arrow head.

Vector quantities are denoted by dotted italic upper-case symbols, thus $\dot{E}$, $\dot{I}$, $\dot{Y}$, $\dot{Z}$; their *rectangular components* are denoted by heavy-faced italic lower-case symbols and numerals, vertical components being compounded with the symbolic operator j, thus $\dot{E} = \boldsymbol{110} + j\,\boldsymbol{191}$.

Phase rotation. In the analytical treatment, and vector diagrams, of polyphase systems the clockwise direction of phase rotation, or phase sequence, is adopted, except where statements are made to the contrary.

Phase and line quantities in polyphase systems. Symbols denoting "phase" E.M.Fs. and currents usually have Roman numeral subscripts, thus e_{I}, e_{II}; those denoting "line" E.M.F's, and currents have Arabic numeral subscripts, thus v_{1-2}, v_{2-3}, i_1, i_2.

Suggested course of reading. Students to whom alternating-current work is new should concentrate on the first six chapters, omitting the sections in small type at the first reading. This should be followed by a study of Chapters VIII, IX, after which consideration should be given to Chapters XII, XIII, XV (omitting, if desired, the more advanced sections in small type). The second reading should include Chapters VII, X, XI, XIV.

More advanced students may omit Chapters VII and XVI at the first reading.

THEORY AND PRACTICE OF ALTERNATING CURRENTS

CHAPTER I

GENERAL CONSIDERATIONS AND DEFINITIONS

Definitions. An alternating current, or E.M.F., is one which changes periodically in magnitude and direction.

The *graphical representation of an alternating current* is shown in Fig. 1, in which abscissæ denote time and ordinates current. During the time interval AB the current increases from zero to its positive maximum value; it decreases to zero during the interval BC, attains its maximum negative value at the instant D, and returns to zero at E. The negative half-wave is usually an exact reproduction of the positive half-wave with the sign reversed, and the succeeding waves are identical with the initial wave.

The complete set of changes through which the current, or E.M.F., passes is called a *cycle*, and the time interval during which these changes occur is called a *period*. For example, the full-line curve in Fig. 1 represents a cycle, and the time interval AE represents a period.

The number of cycles per second is called the *frequency*. Hence if T denote the time, in seconds, of a period, the frequency (f) is equal to $1/T$.

The graphical representation of Fig. 1 refers to steady conditions in an alternating-current circuit. In special cases Fig. 1 is also representative of the initial conditions occurring when a circuit is connected to a source of alternating E.M.F., but more generally the initial conditions give rise to transient phenomena and the waves are unsymmetrical.*

Application of alternating currents in practice. When electric power is required in large quantities it is always produced in the alternating-current form, as the alternating-current generator can be built in much larger sizes than the continuous-current generator and gives satisfactory operation at the high speeds at which large steam turbines have a high efficiency. Moreover, when electrical

* A few special cases of transient phenomena are considered in Chap. XVI. For detailed consideration of the subject see Steinmetz's *Transient Phenomena*

energy is generated in the alternating-current form the transmission, distribution, and utilization pressures are not limited to that of the generator, as by means of stationary transformers the energy may be transformed, at high efficiency, to pressures either higher or lower than that of the generator. Thus the pressures adopted for transmission and distribution may be chosen entirely from economic considerations without reference to the generator pressure, which is governed by a number of technical considerations connected with design.

Production of alternating E.M.Fs. Alternating E.M.Fs. may be produced either dynamically, by the relative motion of an electric circuit and a magnetic circuit, or statically, by the variation of

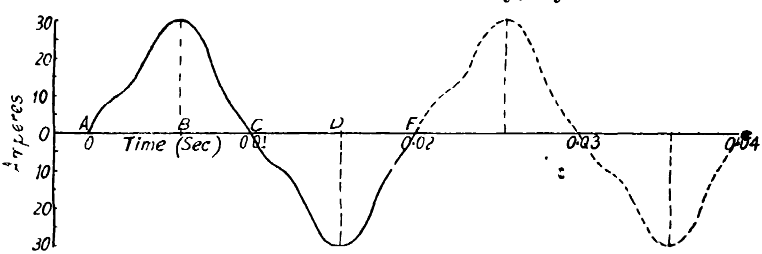


FIG. 1.— Representation, in Rectangular Co ordinates, of an Alternating Current

flux in a stationary magnetic circuit which is interlinked with a stationary electric circuit. In both cases the magnitude of the E.M.F. at any instant is proportional to the rate of change of the linkage of flux with the electric circuit, i.e. $e \propto N . d\Phi/dt$, where e is the instantaneous value of the E.M.F., N the number of turns in the electric circuit, and $d\Phi/dt$ the rate of change of the flux. An E.M.F. of 1 volt is produced when the rate of change of linkages per second is equal to 10^8. Thus

$$e = 10^{-8} N \, d\Phi/dt$$

In alternating-current engineering we are concerned with E.M.Fs. produced both dynamically and statically. For instance, in the majority of circuits carrying alternating currents and in electromagnetic apparatus and transformers, E.M.Fs. are produced statically; but in all alternating-current machines operating on load both dynamically- and statically-produced E.M.Fs. occur simultaneously. In these cases the dynamically-produced E.M.F. is usually called the *E.M.F. of rotation* or the *generated E.M.F.*, and the statically-produced E.M.F. is called the *E.M.F. of pulsation* or the *induced E.M.F.*

Production of E.M.F. in a simple alternating-current generator. Let a single conductor be rotated at constant angular velocity in a uniform magnetic field, the axis of revolution being perpendicular to the magnetic lines as indicated in Fig. 2. And let

B = flux density (in lines per cm^2.) of the magnetic field,
l = length (in cm.) of the conductor in the field,
ω = angular velocity (in radians per second) of the conductor,
r = radius (in cm.) of the path in which the conductor moves,
$v = r\omega$ = linear velocity (in cm. per second) of the conductor.

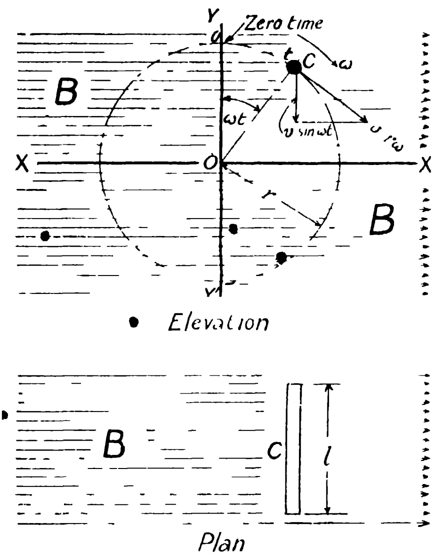


FIG. 2.—Pertaining to the Production of E.M.F. in a Simple Alternator

Further, let time, in seconds, be measured from a plane of reference (YOY', Fig. 2) which contains the axis of revolution and is perpendicular to the magnetic lines.

Then at any instant t, when the conductor occupies the position C, Fig. 2, the component of the linear velocity perpendicular to the flux is

$$r\omega \sin \angle\ YOC = r\omega \sin \omega t.$$

If the conductor were to continue its motion in this direction the flux cut in one second would be

$$B\ l\ r\ \omega \sin \omega t.$$

Hence at this particular instant, t, the E.M.F. generated in the conductor is

$$e = B\,l\,r\,\omega \sin \omega t \times 10^{-8} \text{ volts}$$
$$= \tfrac{1}{2}\,\Phi\,\omega \sin \omega t \times 10^{-8} \text{ volts} \quad . \quad . \quad . \quad (1)$$

where Φ $(= 2Blr)$ is the flux cut by the conductor during one half of a revolution.

Thus the E.M.F. varies as a sine function of the time-angle ωt and if E.M.F. and time be plotted in rectangular co-ordinates we obtain the curve shown in Fig. 3, which is called a sine curve. An E.M.F. which varies in this manner is called a *sinusoidal E.M.F.* (sometimes the term *simple harmonic E.M.F.* is used) and the generator in which it is produced is called a *sine-wave generator.*

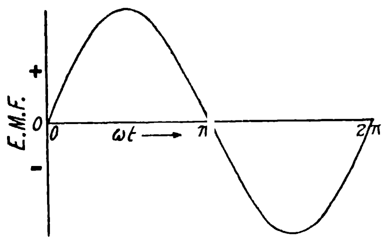


FIG. 3. Representation of Sinusoidal E.M.F.

The maximum, or crest, value of the E.M.F. occurs when the direction of motion of the conductor is perpendicular to the flux—i.e., when $\omega t = \frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, etc.— and its value (E_m) is

$$E_m = \tfrac{1}{2}\,\Phi\,\omega \times 10^{-8}.$$

Hence, equation (1) may be written

$$e = E_m \sin \omega t \quad . \quad . \quad . \quad . \quad (2)$$

If instead of a single conductor we have a coil consisting of a single turn, and the plane of the coil passes through the axis of rotation, the E.M.F generated in the coil at any instant will equal twice that generated in one conductor, or coil-side, so that in this case*

$$e = \Phi\,\omega \sin \omega t \times 10^{-8} \quad . \quad . \quad . \quad (1a)$$

* This equation may be obtained directly as follows—

Let Φ = flux passing through the coil when the latter is perpendicular to the magnetic lines (i.e. when the plane containing the coil coincides with the plane of reference). Then the flux (φ) passing through the coil at the instant t is

$$\varphi = \Phi \cos \omega t,$$

and the instantaneous E.M.F. generated in the coil is

$$e = -10^{-8} \times d\varphi/dt = -10^{-8} \times \frac{d}{dt}(\Phi \cos \omega t)$$
$$= \Phi\,\omega \sin \omega t \times 10^{-8}$$

The sine wave of E.M.F. is therefore produced naturally in the simplest type of alternator. The type of construction represented in Fig. 2 would, however, not be commercially practicable on account of the poor utilization of the magnetic and electric materials. Commercial alternators are constructed with iron magnetic circuits and slotted armatures for the purpose of reducing the magnetic reluctance and the ampere-turns required for excitation. The slotted construction results in deviations from the sine wave, even when the flux is distributed sinusoidally in the air gap, and accordingly features have to be introduced into the construction to correct or neutralize the defects due to the slotted construction.

Forms of E.M.F. equation. In each of the above cases the E.M.F. passes through a complete set of changes during each revolution of the conductor, or coil. Hence, the frequency (f) is equal to the number of revolutions per second (n), and the period (T) is equal to $1/n$.

Now the angular velocity $(\omega) = 2\pi n = 2\pi f$, so that $f = \omega/2\pi$. Equation (2) may therefore be expressed in the following forms—

$$e = E_m \sin \omega t \quad . \quad . \quad . \quad . \quad (2)$$

$$= E_m \sin 2\pi f t \quad . \quad . \quad . \quad . \quad (2a)$$

$$= E_m \sin (2\pi/T)t \quad . \quad . \quad . \quad . \quad (2b)$$

In these equations we observe that—

(1) *the maximum value* (also called the crest value or amplitude) *of the E.M.F. is given by the coefficient of the sine of the time-angle*;

(2) *the frequency is given by* : (coefficient of time $\div 2\pi$).

For example, if the equation to an alternating E.M.F. is

$$e = 100 \sin 314\, t$$

the maximum value of the E.M.F. is 100, and the frequency is $(314/2\pi) = 50$.

Similarly if the equation takes the form

$$e = I_m \sqrt{(R^2 + 9\omega^2 L^2)} \sin 3\omega t,$$

the maximum E.M.F. is given by $E_m = I_m \sqrt{(R^2 + 9\omega^2 L^2)}$ and the frequency by $3\omega/2\pi$.

Phase. Consider now the effect of adding additional turns to the above coil. These turns may be arranged either radially in the same plane, as in Fig. 4, or side by side on the surface of a cylindrical core, as in Fig. 6. In the former case the time-angle (ωt) is the same for each turn, but the instantaneous E.M.Fs. generated in the several turns differ in magnitude, since these turns do not cut the flux at the same speed. If the turns are connected in series

the total E.M.F. of the coil at any instant is equal to the sum of the instantaneous E.M.Fs. in the several turns and is given by

$$
\begin{aligned}
e &= e_1 + e_2 + e_3 + \dots \\
&= (\Phi_1\omega \sin \omega t + \Phi_2\omega \sin \omega t + \Phi_3\omega \sin \omega t + \dots) \times 10^{-8} \\
&= (E_{1m} + E_{2m} + E_{3m} + \dots) \sin \omega t \qquad . \qquad . \qquad (3)
\end{aligned}
$$

where Φ_1, Φ_2, Φ_3, ... denote the maximum fluxes enclosed by the several turns, and E_{1m}, E_{2m}, E_{3m}, ... denote the corresponding maximum E.M.Fs. A graphical representation of the E.M.Fs. for the case of a three-turn coil is shown in Fig. 5, from which it will

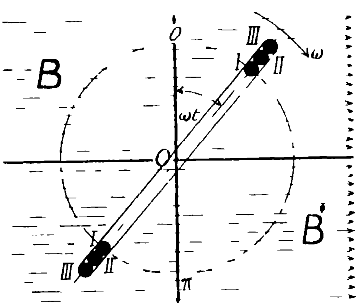


FIG. 4

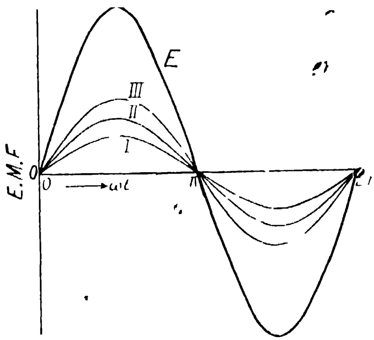


FIG. 5

Pertaining to the Production of E.M.F. in a Simple Alternator with Co-planar Coils

be observed that the instantaneous value of the resultant E.M.F., as well as that of each of the individual E.M.Fs., is proportional to $\sin \omega t$. Thus all the E.M.Fs. become zero at the same instant and reach their maxima together, i.e. *the E.M.Fs. are in phase with one another.* Therefore, alternating quantities of the same frequency have the same phase, or are in phase, when their zero values—or their maximum or other corresponding values—occur at the same instant.

With the conditions shown in Fig. 6 each turn cuts the same maximum flux (Φ) during a revolution, but the time-angle is different for each turn. If the turns are connected in series the total E.M.F. generated in the coil is given by

$$
\begin{aligned}
e &= e_1 + e_2 + e_3 + \dots \\
&= \{\Phi\omega \sin \omega t + \Phi\omega \sin(\omega t - \alpha) + \Phi\omega \sin[\omega t - (\alpha + \beta)] + \dots\} \times 10^{-8} \\
&= \Phi\omega \times 10^{-8}\{\sin \omega t + \sin(\omega t - \alpha) + \sin[\omega t - (\alpha + \beta)] + \dots\} \\
&= E_m \{\sin \omega t + \sin(\omega t - \alpha) + \sin(\omega t - (\alpha + \beta)) + \dots\} \quad . \quad (4)
\end{aligned}
$$

where ωt is the time-angle referred to the first turn, and α, β, ... are the angles by which the planes containing the several turns are displaced from one another (see Fig. 6).

A graphical representation of these conditions is given in Fig. 7, which refers to the case of a coil having three turns. In this case the maxima of the several E.M.Fs. occur at different instants, i.e. *the E.M.Fs. are out of phase with one another*.

In simple harmonic motion the term "phase" is used to denote the point or stage in the period, considered in relation to a standard position or to the instant of starting, to which the oscillation has

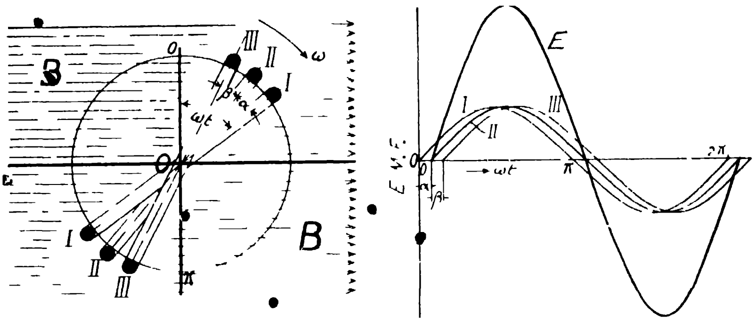


FIG. 6 FIG. 7

Pertaining to the Production of E.M.F. in a Simple Alternator with Distributed Coils

advanced. For example, in Fig. 2 the phase of the conductor at time t is given by the angle ωt. Hence this angle is sometimes called the "phase angle." In electrical engineering, however, we are concerned more with the *relative* phases of alternating quantities, with respect to a quantity of reference, rather than their *absolute* phases, and hence the term "phase difference" is more appropriate. Moreover the term "phase" has other meanings than that given above.

Phase difference. The phase difference between two alternating quantities is measured by the time-angle between their zero values. When the quantities have the same frequency* this angle is constant for corresponding values (e.g. zero, maximum, etc.) of the quantities. For the case represented in Fig. 7 the phase difference between the E.M.Fs. in turns I and II is α, and is equal to the angle by which these turns are displaced from each other. Similarly, the phase

* The meaning of "phase difference" when applied to quantities of different frequencies is considered in Chap. X.

difference between the E.M.Fs. in turns II and III is β, while that between the E.M.Fs. in turns I and III is $(\alpha + \beta)$. These expressions, however, need qualification, as they give no indication of which of the E.M.Fs. first reaches its maximum value. To supply this deficiency the terms "lead" and "lag" are employed.

Lead denotes that the maximum (or zero) value of an alternating quantity occurs earlier than the corresponding value of a second quantity.

Lag denotes that the maximum (or zero) value occurs later than the corresponding value of a second quantity.

Phase difference accordingly is usually expressed as either the "angle of lead" or the "angle of lag." For example, in Fig. 7 the E.M.F. wave I is leading with respect to II, the "angle of lead" being α; conversely, III is lagging with respect to II, the "angle of lag" being β

If in Fig. 6 the time-angle is referred to turn II instead of to turn I, as above, and this angle is now denoted by $\omega t'$, then on substituting $(\omega t' + \alpha)$ for ωt in equation (4) we obtain

$$e = E_m \{\sin(\omega t' + \alpha) + \sin \omega t' + \sin(\omega t' - \beta) + \ldots\} \quad (5)$$

The E.M.F. in turn I is now given by

$$e'_1 = E_m \sin(\omega t' + \alpha),$$

that in turn II is given by

$$e'_2 = E_m \sin \omega t';$$

while that in turn III is given by

$$e'_3 = E_m \sin(\omega t' - \beta).$$

Now the E.M.F. in turn I is leading, and that in turn III is lagging, with respect to the E.M.F. in turn II. Hence *a plus (+) sign employed in connection with phase difference denotes "lead" and a minus (−) sign denotes "lag."*

Thus in general we can determine from the equation of an alternating quantity: (1) its maximum value, (2) its frequency, (3) its phase difference with respect to another quantity of the same frequency.

Examples. Determine the maximum value and frequency of the following alternating E.M.Fs and currents. Determine also the phase differences between the respective E.M.Fs. and currents—

(a) $\begin{cases} e = 3250 \sin 157t \\ i = 35 \sin (157t - 15°) \end{cases}$ (b) $\begin{cases} e = E'_m \sin 5\omega t \\ i = 5\omega C E'_m \sin (5\omega t + \frac{1}{2}\pi) \end{cases}$

(c) $\begin{cases} e = \omega L I'_m \sin (\omega t + \frac{1}{2}\pi) \\ i = I'_m \sin \omega t \end{cases}$ (d) $\begin{cases} e = E''_m \sin (\omega t + a) \\ i = E''_m/\sqrt{(R^2+X^2)} \sin (\omega t - \tan^{-1} X/R) \end{cases}$

Denoting, in each case, the maximum value of the E.M.F. by E_m, the maximum value of the current by I_m, the frequency by f, and the phase difference by φ, we have

(a) $E_m = 3250$;	$I_m = 35$;	$f = 157/2\pi = 25$;	$\varphi = -15^\circ$
(b) $E_m = E'_m$;	$I_m = 5\omega CE'_m$;	$f = 5\omega/2\pi$;	$\varphi = +\frac{1}{2}\pi = +90^\circ$
(c) $E_m = \omega LI'_m$;	$I_m = I'_m$;	$f = \omega/2\pi$;	$\varphi = -\frac{1}{2}\pi = -90^\circ$
(d) $E_m = E''_m$;	$I_m = E''_m/\sqrt{(R^2 + X^2)}$;	$f = \omega/2\pi$;	$\varphi = -(\alpha + \tan^{-1}X/R)$

Observe that both frequency and phase difference are determined from the time-angle, the former being given by ($1/2\pi \times$ coefficient of time), and the latter by the constant term, if any, of the time-angle. Also observe that in expressing phase difference, the E.M.F. is taken as the quantity of reference. Thus in (c) the E.M.F. leads the current by 90°, therefore the current lags behind the E.M.F. by this amount: in (d) the E.M.F. is zero when $\omega t = -\alpha$, and the current is zero when $\omega t = +\tan^{-1} X/R$, so that the phase difference between E.M.F. and current is $(\alpha + \tan^{-1} X/R)$, lagging.

Properties of sine curves. The study of alternating-current phenomena requires a knowledge of the properties of sine curves. A fundamental property of such curves is that any ordinate is proportional to the sine of the corresponding abscissa. Other properties are (1) that the mean ordinate of a half-wave is equal to $2/\pi$ times the maximum ordinate; (2) that the square root of the mean of the squared ordinates of a half-wave or a complete wave is equal to $1/\sqrt{2}$ times the maximum ordinate. We shall now show how these quantitative relations are obtained.

Mean ordinate, or arithmetic mean value, of a sine curve. Let the curve be represented by the equation

$$e = E_m \sin (2\pi t/T)$$

where E_m denotes the maximum ordinate and T the period. Then the mean ordinate is given by

$$E_{av} = \frac{\text{Area included between the curve and abscissa axis for one half-period}}{\text{Half-period}}$$

$$= \int_0^{\frac{1}{2}T} \frac{e \cdot dt}{\frac{1}{2}T} = \frac{2}{T}\int_0^{\frac{1}{2}T} E_m \sin\left(\frac{2\pi}{T}\right) t \cdot dt$$

$$= \frac{2}{T} \cdot \frac{T}{2\pi} \cdot E_m \left[-\cos\left(\frac{2\pi}{T}\right) t\right]_0^{\frac{1}{2}T}$$

$$= \frac{E_m}{\pi} [-\cos \pi + \cos 0]$$

$$= (2/\pi)E_m = 0{\cdot}636\ E_m$$

An *approximation to the value of the mean ordinate* can be obtained arithmetically by determining the mean value of the sines of angles

between 0° and 180°. Thus taking angles at intervals of 15° we have

Angle	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Sine	0·259	0·5	0·707	0·866	0·966	1·0	0·966	0·866	0·707	0·5	0·259	0

Total = 7·56 ; Mean = 7·56/12 = 0·633

The arithmetic mean value of an alternating current is of little importance in practice, as the heating effect due to a current is proportional, not to the square of the mean value of the current, but to the mean of the squared instantaneous values of the current as explained later. However, a knowledge of the mean value of an alternating E.M.F. is frequently required in calculations connected with alternating-current machines.

Root-mean-square (R.M.S.)* value of a sine curve. The root-mean-square value is obtained by determining the square root of the mean value of the squared ordinates for a cycle or half cycle. Let the curve be represented by the equation

$$i = I_m \sin(2\pi/T)t$$

The square of any ordinate (i) is then

$$i^2 = I^2_m \sin^2 (2\pi/T)t,$$

and if I^2 denote the mean value of the squares of the ordinates during a period, we have

$$\begin{aligned} I^2 &= \frac{1}{T}\int_0^T I_m{}^2 \sin^2\left(\frac{2\pi}{T}\right)t\,.\,dt \\ &= \frac{I_m{}^2}{T}\int_0^T \tfrac{1}{2}(1-\cos 2(2\pi/T)t)\,dt \\ &= \frac{I_m{}^2}{2T}\left[t-\frac{T}{4\pi}\sin\left(\frac{4\pi}{T}\right)t\right]_0^T \\ &= \tfrac{1}{2}I_m{}^2 \end{aligned}$$

Whence $I = \sqrt{\tfrac{1}{2}I_m{}^2} = I_m/\sqrt{2} = 0{\cdot}707\,I_m$

An approximation to the R.M.S. value can be obtained arithmetically by determining the mean value of the squares of the sines of angles between 0° and 180° and taking the square root of this quantity. Thus taking angles at intervals of 15° we have

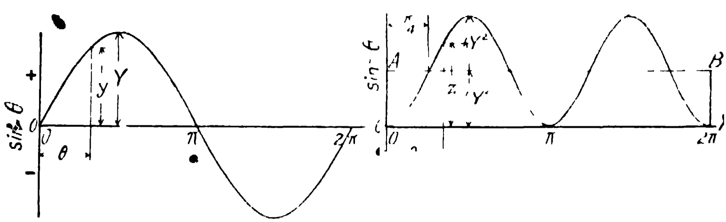
Angle	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Sine	0·259	0·5	0·707	0·866	0·966	1·0	0·966	0·866	0·707	0·5	0·259	0
(Sine)^2	0·067	0·25	0·5	0·75	0·933	1·0	0·933	0·75	0·5	0·25	0·067	0

Sum of (sine)^2 = 6·0, Mean of (sine)^2 = 6·0/12 = 0·5, $\sqrt{\{\text{Mean of (sine)}^2\}}$ = 0·707

* In this book the square root of the mean value of the squared ordinates of a periodic curve is called the "root-mean-square" (R.M.S.) value. The terms "effective" and "virtual" are also used in practice, the former being widely used in America.

Graphical representation of R.M.S. value of a sine curve. In Fig. 8 (a) is shown a sine curve and at (b) in the same Fig. is shown a curve the ordinates of which are proportional to the squares of the corresponding ordinates of the sine curve (a). The horizontal line AB is drawn such that its height is equal to the mean ordinate of curve (b), i.e. the area of rectangle $OABX$ = area between curve and the abscissa axis. Hence OA is the mean value of the squared ordinates of the sine curve (a) and $\sqrt{OA}$ is the R.M.S. value of (a).

It can be shown that curve (b) is a sine curve having AB as axis and a frequency twice that of (a). Thus if any ordinate of (a) is given by $y = Y \sin \theta$, then the corresponding ordinate of (b) is $z = y^2 = Y^2 \sin^2 \theta$. Now $\sin^2 \theta$ may be expressed as $(\frac{1}{2} - \frac{1}{2} \cos 2\theta)$.



(a). (b)

FIG. 8. Graphical Determination of R.M.S. Value of a Sine Curve

which, since $\cos \theta = -\sin (\theta - \frac{1}{2}\pi)$, reduces to $[\frac{1}{2} + \frac{1}{2} \sin (2\theta - \frac{1}{2}\pi)]$. Hence $z = \frac{1}{2} Y^2 + \frac{1}{2} Y^2 \sin (2\theta - \frac{1}{2}\pi)$. Now $\frac{1}{2} Y^2 \sin (2\theta - \frac{1}{2}\pi)$ is a sine curve having a maximum ordinate $\frac{1}{2} Y^2$, a frequency twice that of curve (a), its zero at $2\theta = 90°$, or $\theta = 45°$, and its axis at a distance $\frac{1}{2} Y^2$ above the abscissa axis. Obviously the mean ordinate of this curve, taken over a cycle, is zero. Therefore the mean value of curve (b) is $\frac{1}{2} Y^2$, whence the R.M.S. value of (a) is $(1/\sqrt{2}) Y = (1/\sqrt{2}) \times$ maximum ordinate of (a).

Examples. Give the R.M.S. value and frequency of the following :-

(a) $\sqrt{(R^2 + X^2)} \sin (\omega t + \frac{1}{3}\pi)$;

(b) $(A + B) \sin (3\omega t - a)$;

(c) $A \sin \omega t + B \cos \omega t$;

(d) $141{\cdot}4 \sin 377t$.

The maximum value of the quantities (a), (b), (d) may be written down directly, but that of quantity (c) must be deduced as follows –

Multiply and divide each term by $\sqrt{(A^2 + B^2)}$ thus

$$\sqrt{(A^2 + B^2)} \frac{A}{\sqrt{(A^2 + B^2)}} \sin \omega t + \sqrt{(A^2 + B^2)} \frac{B}{\sqrt{(A^2 + B^2)}} \cos \omega t$$

Now if $B/A = \tan \varphi$, $A/\sqrt{(A^2 + B^2)} = \cos \varphi$, and $B/\sqrt{(A^2 + B^2)} = \sin \varphi$. Hence expression ($c$) may be written

$$\sqrt{(A^2 + B^2)} \{\sin \omega t \cos \varphi + \cos \omega t \sin \varphi\} = \sqrt{(A^2 + B^2)} \sin(\omega t + \varphi)$$

Therefore the R.M.S. values and frequencies are

Quantity.	R.M.S. Value	Frequency.
$\sqrt{(R^2 + X^2)} \sin (\omega t + \frac{1}{3}\pi)$	$\sqrt{\frac{1}{2}(R^2 + X^2)} = 0.707\sqrt{(R^2 + X^2)}$	$\omega/2\pi$
$(A + B) \sin (3\omega t - \alpha)$	$(A + B)/\sqrt{2} = 0.707 (A + B)$	$3\omega/2\pi$
$A \sin \omega t + B \cos \omega t$	$\sqrt{\frac{1}{2}(A^2 + B^2)} = 0.707\sqrt{(A^2 + B^2)}$	$\omega/2\pi$
$141.4 \sin 377t$	$141.4/\sqrt{2} = 100$	$377/2\pi = 60$

Practical importance of R.M.S. value of alternating quantities. The heat produced in a conductor carrying a current is proportional to the square of the current. Hence, if i denotes the instantaneous value of the current in amperes, R, the resistance of the conductor in ohms, the energy (dW) expended in heat during the time dt is

$$dW = i^2 R \, . \, dt \text{ joules}$$

and the mean rate at which energy is expended (i.e. the mean heating effect) during the time T is

$$P = \int_0^T \frac{dW}{T} = \frac{1}{T}\int_0^T i^2 R \, . \, dt$$

If the current follows a sine law

$$i = I_m \sin (2\pi/T)t$$

the mean heating effect during a period (T) is

$$P = \frac{1}{T}\int_0^T R I_m^2 \sin^2(2\pi/T)t \, . \, dt$$

$$= \frac{R I_m^2}{T}\int_0^T \sin^2\left(\frac{2\pi}{T}\right)t \, . \, dt$$

$$= \tfrac{1}{2} R I_m^2 = R(I_m/\sqrt{2})^2 = RI^2$$

where $I (= I_m/\sqrt{2})$ is the R.M.S. value of the current.

Thus the mean heating effect due to a sinusoidal alternating current of maximum value I_m is the same as that due to a continuous current equal to $0{\cdot}707\ I_m$.

The R.M.S. value of an alternating current is therefore of considerable importance in practice. Moreover, as will be shown later in Chapter XII, the indications of ammeters and voltmeters give R.M.S. values of current and E.M.F. respectively. In commercial electrical engineering when values of current and E.M.F. are specified, these values always refer to the R.M.S. values.

Form-factor of a sine curve. The form-factor of a periodic curve

is the ratio (R.M.S. value/arithmetic mean value). Hence for a sine curve the form-factor (k_f) is

$$k_f = \frac{1}{\sqrt{2}} \div \frac{2}{\pi} = \frac{\pi}{2\sqrt{2}} = 1{\cdot}11$$

A knowledge of the form-factor of a curve therefore enables the R.M.S. value to be obtained from the arithmetic mean value, or *vice versâ*.

Example. A conductor rotates at a uniform speed of n r.p.s. in a two pole magnetic field. The flux per pole is Φ lines and the flux distribution is sinusoidal. Obtain an expression for the R.M.S. value of the E.M.F. generated in the conductor.

Now the mean E.M.F. generated in the conductor is equal to (flux cut per second $\times$ 10^{-8}), and since the conductor cuts the flux (Φ) twice in one revolution, the mean E.M.F. (E_{av}) is

$$E_{av} = 2\,\Phi\,n \times 10^{-8} \text{ volts.}$$

But since the velocity is constant the E.M.F. is, at any instant, proportional to the flux density of the field in which the conductor is moving. Therefore the E.M.F. follows a sine law and its R.M.S value (E) is

$$E = 1{\cdot}11\,E_{av} = 2{\cdot}22\,\Phi\,n \times 10^{-8} \text{ volts.}$$

Amplitude- or peak-factor of a sine curve. The amplitude-factor of a periodic curve is the ratio (maximum value/R.M.S.value) Hence for a sine curve the amplitude-factor (ξ) is

$$\xi = 1 \div (1/\sqrt{2}) = \sqrt{2} = 1{\cdot}414$$

The amplitude-factor is only used in connection with E.M.F. waves and is of importance in insulation testing and high-voltage apparatus, as the maximum stress to which the insulation is subjected is proportional to the maximum or crest value of the E.M.F. A knowledge of the crest value of the E.M.F. is also necessary when measuring iron losses, as the iron loss depends upon the maximum value of the flux.

If E is the R.M.S. value of the applied E.M.F. as given by the reading of a voltmeter and ξ is the amplitude-factor, the maximum E.M.F. is

$$E_m = \xi\,E$$

In order to obviate the necessity of a knowledge of this factor when the wave-form is non-sinusoidal a special form of voltmeter (called a crest voltmeter) has been designed, by means of which the crest value of the E.M.F. is given directly by the instrument reading.*

* See *Transactions of American Institute of Electrical Engineers*, Vol 35. pp. 99, 109, *et seq.*; also p. 809.

CHAPTER II

FORMS OF REPRESENTATION

THE study of alternating-current phenomena is considerably simplified by assuming the quantities to vary sinusoidally with respect to time; in fact, the elementary principles of alternating-current circuits can only be developed in an intelligible manner by making this assumption, which, in many cases, is justifiable. Moreover, the sine curve is, in practice, considered as the standard wave-form, and deviations therefrom are classed as distorted waves.

In view of the importance of the sine wave in alternating-current engineering we shall devote the present chapter to a consideration of the various methods of representating sinusoidal quantities. These methods may be classified as follows—

I. Graphical methods.	(1) Rectangular co-ordinates.
	(2) Polar co-ordinates.
	(3) Vector diagrams.
II Analytical methods.	(4) Trigonometric functions.
	(5) Complex algebra.

REPRESENTATION BY RECTANGULAR CO-ORDINATES

In this method time is measured horizontally, as abscissae, and instantaneous values of the alternating quantity are measured vertically, as ordinates. Examples are given in Chapter I, Figs. 1, 3, 5, 7, 8.

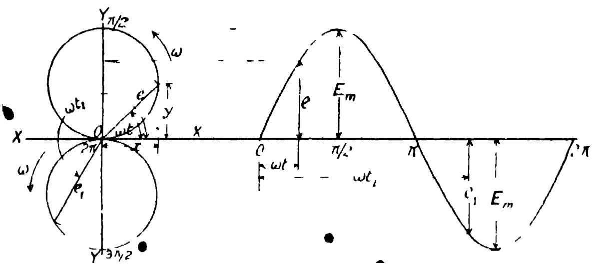
The method can be also employed for representing non-sinusoidal quantities.

REPRESENTATION BY POLAR CO-ORDINATES

In this method the alternating quantity is represented by a rotating line—called a radius-vector– of variable length. Time is measured by the *angle* between this line and a reference axis. The *length* of the line at any position represents the instantaneous value *of the quantity at the time corresponding to this position.* The fixed point about which the line rotates is called the *pole.* The positive direction of rotation is counter-clockwise, and one revolution corresponds to one cycle of the alternating quantity.

When this method of representation is applied to sinusoidal

quantities the locus of the free end of the radius-vector forms two circles (one corresponding to each half-cycle) which touch each other, the line joining the centres and point of contact passing through the pole. The diameter of each circle is equal to the maximum value of the alternating quantity. If time is measured from the horizontal, or (X), axis and the zero value of the quantity occurs at zero time, the centres of the circles lie in the vertical, or (Y), axis as shown in Fig. 9 (*a*), which is a polar representation of the sine curve of Fig. 3, reproduced in Fig. 9 (*b*).



(*a*) (*b*)

FIG. 9. Polar Representation of a Sine Curve

Proof. Let the equation to the alternating quantity be

$$e = E_m \sin \omega t.$$

Then the length of the rotating radius-vector at any instant (t) is equal to e, and the angle between it and the axis of reference is equal to ωt. Let x, y, be the co-ordinates of the free end of the vector referred to rectangular axes XOX, YOY, passing through the pole O (see Fig. 9). Then

$$x = e \cos \omega t, \qquad y = e \sin \omega t,$$

$$x^2 + y^2 = e^2 \cos^2 \omega t + e^2 \sin^2 \omega t = e^2 = e(E_m \sin \omega t) = E_m y.$$

Hence $x^2 + y^2 - E_m y = 0$.

This is the equation to a circle, referred to rectangular coordinates, of radius $\frac{1}{2}E_m$, having its centre in the vertical (Y) axis at a distance $\frac{1}{2}E_m$ from the origin (i.e. the circle is tangential to the horizontal (X) axis). Therefore the locus of the free end of the rotating vector is a circle.

If the zero value of the quantity does not occur at zero time the line of centres is displaced from the vertical by an angle corresponding to the time at which the zero value of the quantity occurs. *Lead* is represented by *a displacement in the clockwise direction* (see Fig. 10 (*a*)) ; *lag* is represented by a displacement in the counter-clockwise direction (see Fig. 10 (*b*)).

Applications of polar co-ordinates to the representation of non-sinusoidal quantities. Non-sinusoidal quantities may also be represented by polar co-ordinates, but in this case the locus of the rotating radius-vector will form two irregular figures, which will be

symmetrical if the two half-waves of the alternating quantity are symmetrical. Fig. 11 shows the non-sinusoidal curve of Fig. 1 plotted in polar co-ordinates.

In the case of symmetrical waves the equivalent sine wave—i.e. the sine wave which has the same R.M.S. value and frequency

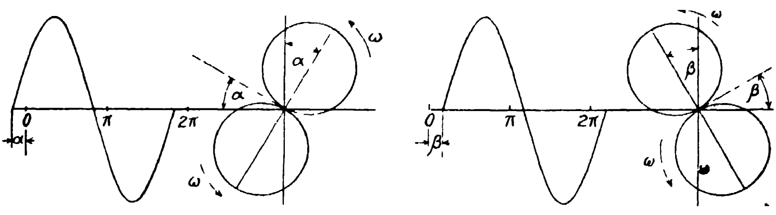


FIG. 10.—Representation of Lead (a) and Lag (b) by Polar Co-ordinates

as the non-sinusoidal wave is obtained by drawing a circle having an area equal to that of one of the irregular figures. The diameter

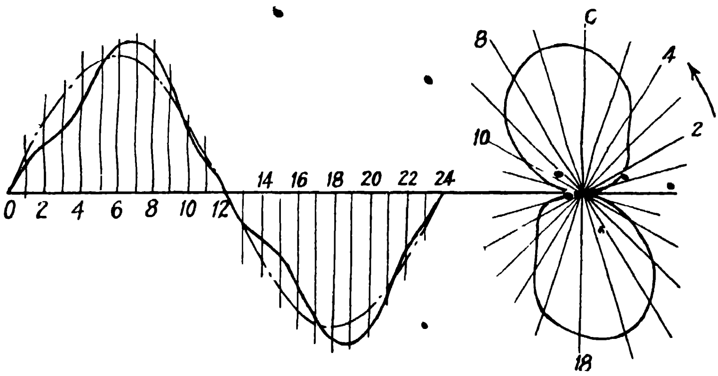


FIG. 11.—Representation of Non-sinusoidal Curve by Polar Co ordinates. The Chain-dotted Curve is the Equivalent Sine Curve

of this circle gives the maximum value of the equivalent sine wave and hence the R.M.S. value of the distorted wave is given by

$$(1/\sqrt{2}) \times \text{diameter of circle, or}$$

$$\sqrt{2} \times \text{radius of circle.}$$

Proof. Let the value of the periodic quantity at time t be denoted by e and its phase by ωt (see Fig. 9). Then the increase in polar area for an increment of time dt (i.e. an increment $d\omega t$ in phase) is

$$\tfrac{1}{2}e.e\, d\omega t = \tfrac{1}{2}e^2\, d\omega t.$$

Hence the area of the polar diagram corresponding to half a period of the distorted wave is

$$\int_0^{\pi} \tfrac{1}{2} e^2\, d\omega t = \int_0^{\frac{1}{2}T} \tfrac{1}{2} e^2 d\left(\frac{2\pi}{T}\right) t = \frac{\pi}{T}\int_0^{\frac{1}{2}T} e^2\, dt$$

Let r denote the radius of circle which has an area equal to that of the polar diagram. Then

$$\pi r^2 = \frac{\pi}{T}\int_0^{\frac{1}{2}T} e^2\, dt$$

and $$r\sqrt{2} = \sqrt{\left(\frac{1}{\frac{1}{2}T}\int_0^{\frac{1}{2}T} e^2\, dt\right)} = \text{R.M.S. value of distorted wave.}$$

Representation by Vector Diagrams

A vector diagram is a graphical representation of one or more vector quantities, a vector quantity being a physical quantity which has *direction* as well as magnitude. Examples of vector quantities are force, velocity, magnetic flux, E.M.F., current, ampere-turn.

Such quantities are only completely specified when particulars of their magnitudes, direction, and sense are given.

Vector quantities are represented graphically by straight lines called *vectors.* The *length* of the line represents the *magnitude* of the quantity; the *inclination* of the line with respect to some axis of reference denotes the *direction* of the quantity, and an *arrow-head,* usually placed at one end of the line, indicates the *sense* in which the quantity acts.

In electrical engineering we may require to represent a number of vector quantities in the same vector diagram: it is necessary, therefore, to adhere to a system of nomenclature in order that the several vectors may be readily distinguished from one another. The vector nomenclature in this treatise is as follows*-

E.M.F. vectors are represented by an ordinary arrow-head	——→
Flux vectors are represented by a double arrow-head	—↠
Ampere-turn vectors are represented by a solid arrow-head	– – –➤
Current vectors are represented by a closed arrow-head	——▷

With alternating quantities varying harmonically, two methods of vector representation may be adopted, viz. (1) the polar vector diagram, (2) the crank vector diagram.

Polar vector diagram. In this diagram a fixed vector, representing the maximum value of the quantity, occupies a definite

* This nomenclature is the same as that employed in vector diagrams in the author's *Electric Traction* and *Electric Motors and Control Systems.* (Pitman.)

position relative to an axis of reference, one end of the vector being located at the origin. *Instantaneous values* of the quantity are represented by the projections of the vector on a rotating line (called the "time line") which rotates, in a plane containing the vector, with uniform angular velocity. The counter-clockwise direction of rotation is positive, and *time* is measured from the horizontal axis in the first quadrant. Hence if the zero value of the quantity occurs at zero time the fixed vector lies in the vertical axis, but otherwise this vector will be displaced from the vertical axis by an angle equal to the value of the time-angle at which zero time occurs; *lead* being represented by an angle measured in the clockwise direction, and *lag* by an angle in the counter-clockwise

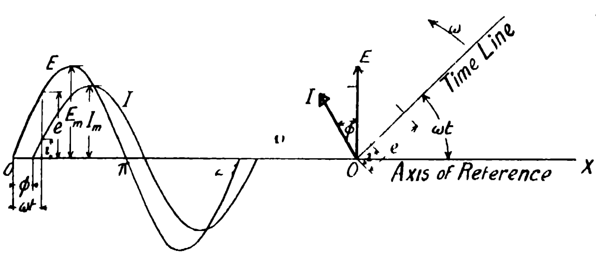


FIG. 12 Polar Vector Diagram (*b*) for the Sinusoidal E M F. and Current Represented by (*a*)

direction. A positive phase-angle must be measured in the clockwise direction because if one quantity is leading relative to another quantity, the revolving time line must meet the vector of the former before that of the latter. Fig. 12 shows the polar vector diagram for a current, I, lagging φ with respect to an E.M.F. E

Crank vector diagram. In this diagram a vector, equal to the maximum value of the quantity, rotates with uniform angular velocity, ω, about a fixed point which is the origin of rectangular axes. *Time* is measured from the horizontal axis in the first quadrant, the counter-clockwise direction of rotation being positive *Instantaneous values* of the quantity are represented by the projections of the vector on the vertical axis. *Phase* is given by the angle, measured in the counter-clockwise direction, which the rotating vector makes with the time axis. Hence the phase difference between two quantities, is represented by the angle between their vectors, an angle measured in the counter-clockwise (+) direction denoting *lead*, and one measured in the opposite

direction denoting *lag*. Fig. 13 shows a crank vector diagram representing the same conditions as the polar diagram of Fig. 12.

Polar and crank vector diagrams compared. In both polar and crank vector diagrams *time* is measured by the angle, from the horizontal axis, in the counter-clockwise (+) direction, but *phase difference* is measured in the clockwise direction in the polar diagram and in the counter-clockwise direction in the crank diagram. Thus a vector representing a lagging quantity is shown in one diagram on one side of the vector of reference and in the other diagram on the opposite side of the latter. Hence when the complex algebraic, or symbolic, method of representation is employed ambiguity in signs is likely to occur if the polar and crank

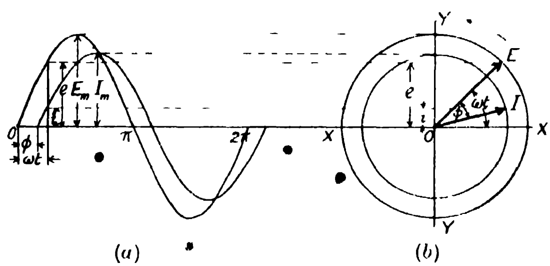


FIG. 13. Crank Vector Diagram (*b*) for the Sinusoidal E.M.F. and Current Represented by (*a*)

vector diagrams are used indiscriminately. To remove this ambiguity the crank vector diagram, in which alternating quantities are represented by rotating vectors, has been adopted by the International Electro-technical Commission as the standard form of representation for electrical vector quantities. All vector diagrams and complex algebraic expressions in this book conform to this standard, it being understood that such diagrams and expressions are used solely in connection with quantities varying sinusoidally.

Addition and subtraction of vector quantities. In problems connected with alternating-current circuits we may be concerned with a number of E.M.Fs. or currents of the same frequency but of different phases, and may wish to obtain the resultant E.M.F. or current. The quantities, if sinusoidal, may be represented by a number of rotating vectors having a common axis of rotation and displaced from one another by invariable angles which are equal to the phase differences between the respective quantities. The instantaneous value of the resultant E.M.F., or current, is given by the algebraic sum of the projections of the vectors on the vertical

axis. The maximum value of the resultant is obtained by compounding the several vectors according to the parallelogram and polygon laws.

The *parallelogram law of vectors* states—

If two co-planar vectors, not lying in the same straight line, meet at a point and a parallelogram be constructed having these vectors as adjacent sides, then their resultant is given in magnitude and direction by the diagonal which passes through the point of intersection of the vectors.

The *polygon law of vectors* states—

If a number of co-planar vectors, not lying in the same straight line, meet at a point and an open polygon be constructed having one side formed by one of the vectors and the remaining sides equal and parallel to the remaining vectors, taken in order, then their resultant is given in magnitude and *reversed* direction by the closing side of the polygon.

To prove these laws it is necessary to show that the sum of the vertical projections of the vectors at a given instant is equal to the vertical projection of their resultant at that instant. Thus in the case of the parallelogram of vectors let OA, OB, Fig. 14 (*a*), represent two rotating vectors having a phase difference φ, and let OC the diagonal of the parallelogram $OACB$ represent their resultant. The vertical projections of OA, OB OC are given by Oa, Ob, Oc, respectively.

From Fig. 14 we have

$$Oc = Oa + ac.$$

But ac is the vertical projection of AC, and as AC is equal and parallel to OB, therefore $ac = Ob$. Hence

$$Oc = Oa + ac = Oa + Ob$$

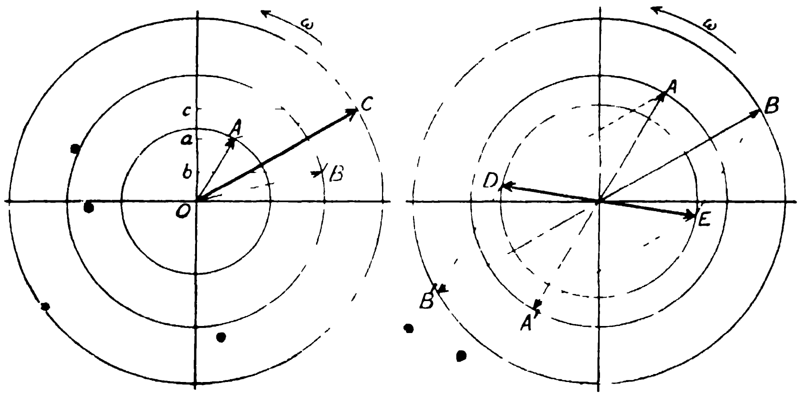
and the proposition is proved.

The polygon law of vectors may be proved in a similar manner.

If the *vector difference* of the quantities is required, the vector representing one of the quantities is reversed, and this reversed vector is compounded with the other vector according to the parallelogram law. For example, if in Fig. 14 (*b*), OA and OB are two vectors and their difference $(A - B)$ is required, the vector OB is reversed, i.e. another vector OB' is drawn equal to, but in the reverse direction to OB, and OA and OB' are compounded according to the parallelogram law, thus giving the resultant OD. Then OD represents the difference of the vector quantities A, B. Similarly

if the difference $B - A$ is required the vector OA is reversed and the resultant OE is obtained.

Vector representation of root-mean-square values of alternating quantities. In alternating-current circuits we are chiefly concerned with R.M.S. values of current and E.M.F., and their phase differences. Provided that the law of variation of the quantities is known, a knowledge of their instantaneous values throughout a period is unnecessary and is of little interest in practice. Now with sinusoidal quantities the R.M.S. value bears a fixed relation to the maximum value. Hence R.M.S. values may be determined directly from the vector diagram by a suitable change of scale. Therefore, when the resultant of a number of currents, or E.M.Fs., is to be determined we may represent the quantities by *fixed* vectors, the lengths of which represent the R.M.S. values of the quantities and the angular displacements of the vectors with respect to one vector, or quantity of reference represent their phase differences with respect to this quantity. *Lead* is represented by an angle in the counter-clockwise direction, and *lag* is represented by an angle in the clockwise direction.



(*a*) (*b*)

FIG. 14. -Parallelogram of Vectors Applied to the Addition (*a*) and Subtraction (*b*) of Two Vectors OA, OB

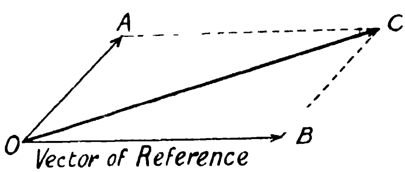


FIG. 15. -Vector Diagram for Constant Quantities

Thus the crank diagram of Fig. 14 (*a*) may be replaced by the stationary vector diagram of Fig. 15, which is virtually a diagram

for constant quantities, as would be employed in connection with applied mechanics.

Example. Represent the following quantities by vectors—

$$5 \sin (2\pi ft - 1)\,;\ 3 \cos (2\pi ft + 1)\,;\ 2 \sin (2\pi ft + 2{\cdot}5)\,;\ 4 \sin (2\pi ft - 1).$$

Add these vectors together and express the result in the form—

$A \sin (2\pi ft + \varphi)$. [C. and G., 1918.]

Observe that all the quantities have the same frequency, f. Hence the vectors representing them may be added together. Before doing so, however, it will be convenient to express all the quantities as sine functions. Thus the second quantity $3 \cos (2\pi ft) + 1)$,

may be written $\quad 3 \sin (2\pi ft + 1 + \frac{1}{2}\pi) - 3 \sin (2\pi ft + 1 + 1{\cdot}57)$
$\qquad = 3 \sin (2\pi ft + 2{\cdot}57)$

The maximum value of each quantity and its phase difference with respect to the quantity of reference, viz. $X \sin 2\pi ft$, is given in the following table—

Quantity.	Maximum value.	Phase difference with respect to $X \sin 2\pi ft$. Radians.	Degrees.
(*a*) $5 \sin (2\pi ft - 1)$	5	-1	$-57{\cdot}3$
(*b*) $3 \cos (2\pi ft + 1)$	3	$+2{\cdot}57$	$+147{\cdot}2$
(*c*) $2 \sin (2\pi ft + 2{\cdot}5)$	2	$+2{\cdot}5$	$+143{\cdot}2$
(*d*) $4 \sin (2\pi ft - 1)$	4	-1	$-57{\cdot}3$

The vector diagram is shown in Fig. 16, in which OX represents the vector of reference, and OA, OB, OC, OD represent the four quantities a, b, c, d respectively. The resultant, or sum, of these vectors is obtained by compounding them according to the polygon law and is represented by OG, which is 4·81 units long and lags 82°, or (82/57·3) = 1·43 radians, with respect to the vector of reference. Hence the sum of the quantities is

$$4{\cdot}81 \sin (2\pi ft - 1{\cdot}43).$$

Representation by Trigonometrical Functions

We have already shown that in the case of quantities varying harmonically the instantaneous values are proportional to the sine, or in some cases to the cosine, of the time-angle. Hence in the analytical treatment of these quantities they are expressed as a trigonometrical function (sine or cosine) of the time-angle, e.g. $e = E_m \sin \omega t$; $i = I_m \sin (\omega t - \varphi)$; $i = I_m \cos \omega t$.

This method is particularly suitable for dealing with instantaneous values, as it enables the relationship between current and E.M.F.

for the simpler circuits to be deduced analytically and the quantitative relations between these quantities to be calculated accurately as shown in the following chapters. By means of Fourier's series the method can be extended to the calculations of circuits when the wave-form of the supply E.M.F. is non-sinusoidal (see Chapter X).

As both maximum and R.M.S. values can, in the case of sinusoidal quantities, readily be obtained from the instantaneous value, the trigonometrical method is largely employed in the calculations of electric circuits and is developed further in the following chapters.

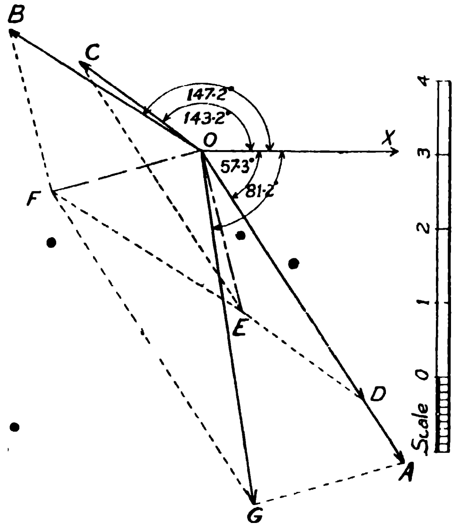


FIG. 16.—Graphical Solution and Vector Diagram for Example (p. 22)

REPRESENTATION BY COMPLEX ALGEBRA

(*Symbolic Notation*)

This form of representation is an algebraic method of representing vector quantities and enables the operations which are carried out graphically in a vector diagram to be performed analytically. It is, moreover, of considerable value in the solution of alternating-current problems, as the results obtained are of the same order of accuracy as those obtained by trigonometrical methods although the calculations are usually simpler and less tedious.

The basis of the method is the representation of vector quantities by their components in the direction of arbitrarily chosen axes of

reference. For example, the vector OE_1, Fig. 17, may be completely specified by stating that its horizontal component is $\boldsymbol{a}_1$ and its vertical component is $\boldsymbol{b}_1$. Instead of stating these conditions verbally we may express them symbolically, thus

$$\dot{E}_1 = \boldsymbol{a}_1 + j\boldsymbol{b}_1$$

where j indicates that the component $\boldsymbol{b}_1$ is perpendicular to the component $\boldsymbol{a}_1$.

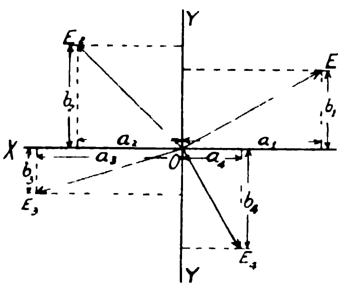


FIG. 17. Components of Vectors Along Axes of Reference

This method of representation is similar to that adopted in analytical geometry for representing co-planar points in terms of their rectangular co-ordinates but it admits of more extensive application owing to the two components, or co-ordinates, being connected by an algebraic sign, the use of which requires the introduction of the distinguishing symbol j to prevent these components being treated as ordinary algebraic quantities.

Axes of reference. The horizontal axis, XOX', Fig. 17, is called the *real* or *in-phase axis*. Components in this axis are called *in-phase components*: they are positive when measured to the right of O and negative when measured in the reverse direction.

The vertical axis, YOY', Fig. 17, is called the *imaginary*, or *quadrature, axis*. Components in this axis are called *quadrature components*: they are positive when measured upwards and negative when measured downwards. These components are distinguished from those in the horizontal axis by being compounded with the symbol j.

For example, in Fig. 17, the vector OE_1 is represented symbolically

by the expression	$\dot{E}_1$	$= \boldsymbol{a}_1 + j\boldsymbol{b}_1$,
the vector OE_2 by	$\dot{E}_2$	$= -\boldsymbol{a}_2 + j\boldsymbol{b}_2$,
the vector OE_3 by	$\dot{E}_3$	$= -\boldsymbol{a}_3 - j\boldsymbol{b}_3$,
and the vector OE_4 by	$\dot{E}_4$	$= \boldsymbol{a}_4 - j\boldsymbol{b}_4$.

Similarly, vectors having directions along the axes of reference are represented by

$$\dot{I}_1 = \boldsymbol{g}_1 \quad : \quad \dot{I}_2 = j\boldsymbol{g}_2 \quad : \quad \dot{I}_3 = -\boldsymbol{g}_3 \quad : \quad \dot{I}_4 = -j\boldsymbol{g}_4$$

We may therefore consider j as an operator, the application of

which to a vector, or a vector component, causes a rotation through 90° in the positive (counter-clockwise) direction. The double application of the operator rotates a vector through 180° and reverses its sense, e.g. the double application of j to the vector $\dot{I} = \boldsymbol{g}$ gives the vector $\dot{I}' = jj\boldsymbol{g} = -\boldsymbol{g}$. Hence operation by jj or j^2 is equivalent to multiplication by -1, so we may regard the effect of j^2 as being equivalent numerically to -1 and write

$$j^2 (\ ) = -1 (\ )$$

Whence $$j (\ ) = \sqrt{-1} (\ )$$

i.e. j has numerically the effect of the imaginary root of -1.*

In ordinary algebra quantities having the factor $\sqrt{-1}$ are called imaginary. This term may also be applied to vector components in the vertical axis, since when considering a uni-directional quantity such as a length, measured along a given axis, any measurement taken along a perpendicular axis has no physical meaning and must therefore be regarded as imaginary. But when dealing with electrical quantities the term "quadrature component" is preferable.

Complex quantities. A quantity which is represented by two components along perpendicular axes is called a complex quantity. In this book complex quantities, i.e. vector quantities and complex numbers, are denoted by dotted italic capitals, thus $\dot{E}$; vector components are denoted by bold italic lower-case characters, thus $\boldsymbol{a}$, $\boldsymbol{b}$; simple magnitudes or scalar quantities are denoted by upper and lower-case italic characters, thus E, I, a, b.

For example, the vector OE_1, Fig. 17, is represented by the complex quantity

$$\dot{E}_1 = \boldsymbol{a}_1 + j\boldsymbol{b}_1 ;$$

its *absolute value*, or magnitude, is given by

$$E_1 = \sqrt{(a_1{}^2 + b_1{}^2)} ;$$

and its *phase*, or inclination, (φ), to the horizontal axis is given by

$$\varphi = \tan^{-1} b_1/a_1 .$$

φ is also called the *argument* of the complex quantity.

Again, if a vector in the horizontal axis is represented by

$$\dot{I}_1 = \boldsymbol{g}_1 + j0 = \boldsymbol{g}_1$$

its absolute value (I_1) is

$$I_1 = \sqrt{(g_1{}^2 + 0^2)} = g_1$$

and its argument is zero.

* Mathematicians denote $\sqrt{-1}$ by ι, but in electrical engineering it is necessary to use j for this quantity, as otherwise confusion might arise between the symbol ι and that (i) used to denote current.

Similarly for other vectors in the axes of reference,

$$I_2 = 0 + jg_2 = jg_2;\ I_2 = \sqrt{(0^2 + g_2^2)} = g_2;\ \varphi = \tfrac{1}{2}\pi$$
$$I_3 = -g_3 + j0 = -g_3;\ I_3 = \sqrt{(-g_3^2 + 0^2)} = g_3;\ \varphi = \pi$$
$$I_4 = 0 - jg_4 = -jg_4;\ I_4 = \sqrt{(0^2 + (-g_4)^2)} = g_4;\ \varphi = \tfrac{3}{2}\pi (\text{or} - \tfrac{1}{2}\pi)$$

Polar and exponential forms of expressing complex quantities. The method of representing a complex number, or quantity, by its rectangular components is called the *rectangular form of representation*. These components may also be expressed in the polar form. Thus, in Fig. 17,

$$a_1 = E_1 \cos\varphi, \qquad b_1 = E_1 \sin\varphi;$$

whence

$$\begin{aligned} \dot{E}_1 &= E_1 \cos\varphi + jE_1 \sin\varphi \\ &= E_1 (\cos\varphi + j\sin\varphi) \end{aligned}$$

This is one of the polar forms of expressing complex quantities and is usually called the *trigonometrical form*.

The exponential form, $\dot{E}_1 = E_1\varepsilon^{j\varphi}$, where φ is in circular measure, may be derived by expressing sin φ and cos φ in their trigonometrical series and substituting these in the above expression for $\dot{E}_1$. Thus, if φ is in circular measure,

$$\begin{aligned} \sin\varphi &= \varphi - (\varphi^3/3!) + (\varphi^5/5!) - (\varphi^7/7!) + \dots \\ \cos\varphi &= 1 - (\varphi^2/2!) + (\varphi^4/4!) - (\varphi^6/6!) + \dots \end{aligned}$$

Hence

$$\begin{aligned} \dot{E}_1 &= E_1\left[\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \dots\right) + j\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \dots\right)\right] \\ &= E_1\left(1 + j\varphi - \frac{\varphi^2}{2!} - j\frac{\varphi^3}{3!} + \frac{\varphi^4}{4!} + j\frac{\varphi^5}{5!} - \dots\right) \\ &= E_1\left(1 + j\varphi + \frac{j^2\varphi^2}{2!} + \frac{j^3\varphi^3}{3!} + \frac{j^4\varphi^4}{4!} + \frac{j^5\varphi^5}{5!} + \dots\right), \end{aligned}$$

since numerically $j^2 = -1$, $j^4 = +1$, $j^6 = -1$, etc.

Now $$1 + j\varphi + \frac{j^2\varphi^2}{2!} + \frac{j^3\varphi^3}{3!} + \frac{j^4\varphi^4}{4!} + \frac{j^5\varphi^5}{5!} + \dots$$

is the expansion for $\varepsilon^{j\varphi}$. Hence the former expression for $\dot{E}_1$ may be written

$$\dot{E}_1 = E_1\varepsilon^{j\varphi}$$

where ε (= 2·71828 . .) is the base of natural logarithms.

Instead of employing the exponential factor $\varepsilon^{j\varphi}$ to express the polar form of a vector a proposal has been made* to employ the factor $J^{\varphi/\frac{1}{2}\pi}$, where J is an operator the application of which to a vector rotates the latter through a right angle with respect to the horizontal axis. The index of J denotes the fraction, or multiple, of a right angle through which the vector is rotated, a plus (+) sign denoting rotation in the counter-clockwise, or positive, direction and a minus (–) sign denoting rotation in the clockwise direction. For example, J^m, $m > 0$, signifies a rotation of $\frac{1}{2}\pi m$ radians, or $90m°$, in the counter-clockwise direction; J^{-m}, a rotation of $\frac{1}{2}\pi m$ radians, or $90m°$, in the clockwise direction. Similarly $J^{\frac{1}{2}}$, $J^{\frac{2}{3}}$, $J^{\frac{1}{3}}$, signify rotations, in the counter-clockwise direction, of $\frac{1}{2} \times \frac{1}{2}\pi = \frac{1}{4}\pi$, or 45°, $\frac{2}{3} \times \frac{1}{2}\pi = \frac{1}{3}\pi$, or 60°; $\frac{1}{3} \times \frac{1}{2}\pi = \frac{1}{6}\pi$, or 30°, respectively; and J^{-3}, $J^{-\frac{1}{6}}$, J^{-1}, signify rotations, in the clockwise direction, of $3 \times \frac{1}{2}\pi = \frac{3}{2}\pi$, or 270°; $\frac{1}{6} \times \frac{1}{2}\pi = \frac{1}{12}\pi$, or 15°; $\frac{1}{2}\pi$, or 90°, respectively.

This form of representation enables complex quantities to be expressed in simpler mathematical expressions than the exponential form. It can be shown that the two forms are equivalent† and that J follows the ordinary laws of algebra in the same manner as j.

In addition to the above polar forms for expressing a vector quantity a conventional form, $\dot{E} = E\underline{/\ \varphi}$, is frequently adopted in practice. This form is purely conventional and does not possess the mathematical significance of the other forms: it simply denotes that the vector quantity $\dot{E}$ has an absolute value equal to E, and is inclined at the angle φ to the axis of reference, the plus (+) sign denoting an angle in the counter-clockwise direction and the minus (–) sign denoting an angle in the clockwise direction. This notation possesses the advantage that numerical results may be expressed directly in the polar form without the introduction of algebraic symbols. For example, a vector $\dot{E}$, of absolute value equal to 100, and inclined at an angle of 60° to the horizontal axis is represented by

$$\dot{E} = 100\underline{/60°}$$

* "Application of a polar form of complex quantities to the calculation of alternating-current phenomena," by Prof. N. S. Diamant. *Transactions of the American Institute of Electrical Engineers*, V. 35, p. 957.

† From De Moivre's theorem we have the identities

$$\cos\varphi + j\sin\varphi \equiv \cos\tfrac{1}{2}\pi m + j\sin\tfrac{1}{2}\pi m \equiv (\cos\tfrac{1}{2}\pi + j\sin\tfrac{1}{2}\pi)^m = J^m,$$

i.e. the operator $J^m (m > 0)$ is identical with $(\cos\varphi + j\sin\varphi)$ where $\varphi = \frac{1}{2}\pi m$.

Summarizing, we have the following ways of expressing symbolically complex numbers and vector quantities—

(1) the rectangular form $\dot{E} = \boldsymbol{a} \pm j\boldsymbol{b}$
(2) the trigonometrical form $\dot{E} = E(\cos\varphi \pm j\sin\varphi)$
(3) the exponential form $\dot{E} = E\varepsilon^{\pm j\varphi}$
(4) the polar form $\dot{E} = EJ^{\pm(\varphi/\frac{1}{2}\pi)}$
(5) the conventional form $\dot{E} = E/\varphi$

Conjugate complex quantities. When two conjugate quantities have the same absolute values and arguments which are equal in magnitude but of opposite sign, they are called *conjugate quantities* Examples: $\boldsymbol{a} + j\boldsymbol{b}$, $\boldsymbol{a} - j\boldsymbol{b}$; $-\boldsymbol{a} + j\boldsymbol{b}$, $-\boldsymbol{a} - j\boldsymbol{b}$; $E^{j\varphi}$, $E^{-j\varphi}$, $EJ^{(\varphi/\frac{1}{2}\pi)}$, $EJ^{-(\varphi/\frac{1}{2}\pi)}$; E/φ, $E/-\varphi$.

Two conjugate complex quantities therefore correspond to two points, in the plane of the co-ordinates, which are images of each other with respect to the horizontal axis.

Addition and subtraction of complex quantities. These operations are effected by the same rules which govern their application to ordinary algebraic quantities, but the in-phase and quadrature components must be treated separately. For example, the *sum of two complex quantities* $-\dot{E}_1 = \boldsymbol{a}_1 + j\boldsymbol{b}_1$ and $\dot{E}_2 = \boldsymbol{a}_2 + j\boldsymbol{b}_2$ is given by the complex quantity

$$\dot{E} = \dot{E}_1 + \dot{E}_2 = (\boldsymbol{a}_1 + j\boldsymbol{b}_1) + (\boldsymbol{a}_2 + j\boldsymbol{b}_2) = \boldsymbol{a}_1 + \boldsymbol{a}_2 + j(\boldsymbol{b}_1 + \boldsymbol{b}_2)$$

The numerical value of this quantity is given by

$$E = \sqrt{[(a_1 + a_2)^2 + (b_1 + b_2)^2]}$$

and its inclination (φ) to the axis of reference is given by

$$\varphi = \tan^{-1}(b_1 + b_2)/(a_1 + a_2)$$

A graphical representation of the process is shown in Fig. 18(a). Similarly with more than two quantities we have

$$\begin{aligned} \dot{E} = \dot{E}_1 + \dot{E}_2 + \dot{E}_3 + \ldots &= (\boldsymbol{a}_1 + j\boldsymbol{b}_1) + (\boldsymbol{a}_2 + j\boldsymbol{b}_2) \\ &\quad + (\boldsymbol{a}_3 + j\boldsymbol{b}_3) + \ldots \\ &= \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 + \ldots \\ &\quad + j(\boldsymbol{b}_1 + \boldsymbol{b}_2 + \boldsymbol{b}_3 + \ldots) \end{aligned}$$

$$E = \sqrt{[(a_1 + a_2 + a_3 + \ldots)^2 + (b_1 + b_2 + b_3 + \ldots)^2]}$$

$$\varphi = \tan^{-1}(b_1 + b_2 + b_3 + \ldots)/(a_1 + a_2 + a_3 + \ldots)$$

The result is obviously independent of the order of the quantities.

If the polar form of representation is employed we have

$$E = E_1 J^m + E_2 J^n;$$

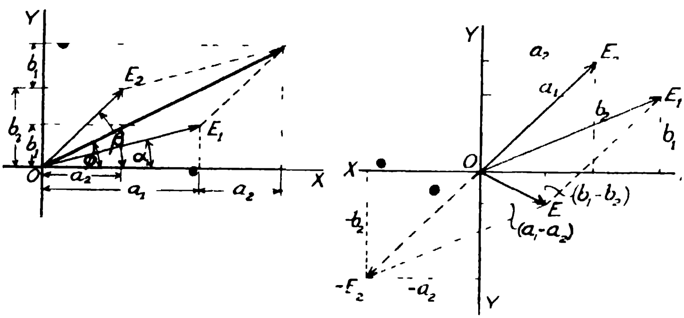
where $m = \alpha/\frac{1}{2}\pi$, and $n = \beta/\frac{1}{2}\pi$; α, β being the inclinations, in radians, of the vectors E_1, E_2, respectively, to the axis of reference.

The numerical value of this quantity, from the geometry of Fig. 18(a), is

$$E = \sqrt{\{E_1^2 + E_2^2 + 2E_1E_2 \cos(\tfrac{1}{2}\pi n - \tfrac{1}{2}\pi m)\}} \quad [n > m]$$

The inclination (φ) to the axis of reference is given by

$$\varphi = \tan^{-1}\{(E_1 \sin \tfrac{1}{2}\pi m + E_2 \sin \tfrac{1}{2}\pi n)/(E_1 \cos \tfrac{1}{2}\pi m + E_2 \cos \tfrac{1}{2}\pi n)\}$$



(a) (b)

FIG. 18.—Addition (a) and Subtraction (b) of Vectors by Treatment of their Components Along the Axes of Reference

When the sum of more than two complex quantities is required it is generally more convenient to employ the rectangular form.

The *difference of two complex quantities* $E_1 = a_1 + jb_1$, and $E_2 = a_2 + jb_2$ is given by the complex quantity

$$E = E_1 - E_2 = (a_1 + jb_1) - (a_2 + jb_2)$$
$$= a_1 - a_2 + j(b_1 - b_2)$$

The numerical value of this quantity is

$$E = \sqrt{[(a_1 - a_2)^2 + (b_1 - b_2)^2]}$$

and its inclination to the axis of reference is given by

$$\varphi = \tan^{-1}(b_1 - b_2)/(a_1 - a_2).$$

A graphical representation of the process is shown in Fig. 18(b).

These operations are applicable to vector quantities as well as complex numbers.

Examples. (1) Find the sum and difference of the vectors $\dot{E}_1 = \mathbf{20 + j40}$ and $\dot{E}_2 = \mathbf{15 + j10}$.

The sum is given by the complex expression

$$\dot{E} = \dot{E}_1 + \dot{E}_2 = \mathbf{20 + 15 + j(40 + 10)}$$
$$= \mathbf{35 + j50}$$

Hence $E = \sqrt{(35^2 + 50^2)} = 61$

$\varphi = \tan^{-1} 50/35 = 55°$

The vector difference is given by the complex expression

$$\dot{E}' = \dot{E}_1 - \dot{E}_2 = \mathbf{20 - 15 + j(40 - 10)}$$
$$= \mathbf{5 + j30}$$

Hence $E' = \sqrt{(5^2 + 30^2)} = 30{\cdot}4$

$\varphi' = \tan^{-1} 30/5 = 80{\cdot}6°$

(2) Find the sum of the vectors $\dot{I}_1 = \mathbf{10 - j5}$; $\dot{I}_2 = \mathbf{-2 + j15}$; $\dot{I}_3 = \mathbf{20 + j10}$; $\dot{I}_4 = \mathbf{-4 - j30}$.

The vector sum is given by the complex expression

$$\dot{I} = \dot{I}_1 + \dot{I}_2 + \dot{I}_3 + \dot{I}_4 = \mathbf{(10 - 2 + 20 - 4) + j(-5 + 15 + 10 - 30}$$
$$= \mathbf{24 - j10}$$

Hence $I = \sqrt{(24^2 + 10^2)} = 26$

$\varphi = \tan^{-1} 10/24 = 21{\cdot}8°$ or $338{\cdot}2°$

(3) Find the sum of the vectors $\dot{Z}_1 = 10J^{0{\cdot}66}$, $\dot{Z}_2 = 7J^{0{\cdot}26}$

The vector sum is given by

$$\dot{Z} = \sqrt{[10^2 + 7^2 + 2 \times 10 \times 7 \cos\{90(0{\cdot}66 - 0{\cdot}26)\}°]}$$
$$= \sqrt{(100 + 49 + 140 \cos 36°)}$$
$$= \sqrt{262{\cdot}2} = 16{\cdot}2$$
$$\varphi = \tan^{-1}[10 \sin(0{\cdot}66 \times 90)° + 7 \sin(0{\cdot}26 \times 90)°]/[10 \cos(0{\cdot}66 \times 90) + 7 \cos(0{\cdot}26 \times 90)°]$$
$$= \tan^{-1} 11{\cdot}387/11{\cdot}52 = 44°\ 40'$$

(4) Add together the quantities

$5 \sin(2\pi ft - 1)$; $3 \cos(2\pi ft + 1)$; $2 \sin(2\pi ft + 2{\cdot}5)$; $4 \sin(2\pi ft - 1)$

[Note.—The graphical solution to this example is given on p. 23.]

The maximum value of each quantity and its phase difference with respect to an arbitrary quantity of reference—$X \sin 2\pi ft$—are given on p. 22, and from these data we can calculate the rectangular components of each quantity. Thus—

Quantity.		Rectangular components.		Quantity expressed symbolically in rectangular form.
Trigonometrical form.	Conventional form.	Horizontal.	Vertical.	
$5 \sin(2\pi ft - 1)$	$5\underline{/-57{\cdot}3°}$	$5 \cos(-57{\cdot}3)° = 2{\cdot}7$	$5 \sin(-57{\cdot}3)° = -4{\cdot}207$	$\mathbf{2{\cdot}7 - j4{\cdot}207}$
$3 \cos(2\pi ft + 1)$	$3\underline{/147{\cdot}2°}$	$3 \cos 147{\cdot}2° = -2{\cdot}522$	$3 \sin 147{\cdot}2° = 1{\cdot}625$	$\mathbf{-2{\cdot}522 + j1{\cdot}625}$
$2 \sin(2\pi ft + 2{\cdot}5)$	$2\underline{/143{\cdot}2°}$	$2 \cos 143{\cdot}2° = -1{\cdot}6$	$2 \sin 143{\cdot}2° = 1{\cdot}198$	$\mathbf{-1{\cdot}6 + j1{\cdot}198}$
$4 \sin(2\pi ft - 1)$	$4\underline{/-57{\cdot}3°}$	$4 \cos(-57{\cdot}3)° = 2{\cdot}161$	$4 \sin(-57{\cdot}3)° = -3{\cdot}366$	$\mathbf{2{\cdot}161 - j3{\cdot}366}$

Hence the sum is given by the vector quantity

$$\dot{Z} = (2{\cdot}7 - 2{\cdot}522 - 1{\cdot}6 + 2{\cdot}161) + j(-4{\cdot}207 + 1{\cdot}625 + 1{\cdot}198 - 3{\cdot}366)$$
$$= 0{\cdot}739 - j4{\cdot}75$$

$$\therefore \quad Z = \sqrt{(0{\cdot}739^2 + 4{\cdot}75^2)} = 4{\cdot}81$$
$$\varphi = \tan^{-1}(-4{\cdot}75/0{\cdot}739) = -81{\cdot}2^\circ$$

The sum of the above quantities is therefore given by $4{\cdot}81\underline{/-81{\cdot}2^\circ}$.

The results obtained on p. 22 by the graphical construction agree fairly well with these results.

Multiplication and division of complex numbers. These operations are effected by ordinary algebraic rules. For example, the product of the numbers $\dot{X} = a_1 + jb_1$ and $\dot{Y} = a_2 + jb_2$ is given by the complex number $\dot{Z}$, thus

$$\dot{Z} = \dot{X}\dot{Y} = (a_1 + jb_1)(a_2 + jb_2) = a_1a_2 + ja_1b_2 + ja_2b_1 + j^2b_1b_2$$
$$= a_1a_2 - b_1b_2 + j(a_1b_2 + a_2b_1)$$

The absolute value (Z) of the product is

$$Z = \sqrt{[(a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2]}$$

and its inclination to the axis of reference is

$$\varphi = \tan^{-1}(a_1b_2 + a_2b_1)/(a_1a_2 - b_1b_2)$$

If the trigonometrical form of representation is employed, and the inclination of $\dot{X}$ and $\dot{Y}$ to the axis of reference are α, β respectively, then

$$a_1 = X\cos\alpha\,; \quad b_1 = X\sin\alpha\,; \quad a_2 = Y\cos\beta\,; \quad b_2 = Y\sin\beta.$$

Hence,

$$\dot{Z} = \dot{X}\dot{Y} = a_1a_2 - b_1b_2 + j(a_1b_2 + a_2b_1)$$
$$= XY\cos\alpha\cos\beta - XY\sin\alpha\sin\beta + j(XY\cos\alpha\sin\beta + XY\cos\beta\sin\alpha)$$
$$= XY\{\cos(\alpha + \beta) + j\sin(\alpha + \beta)\}$$

This is expressed in the polar form by

$$\dot{Z} = XYJ^{(\alpha+\beta)/\frac{1}{2}\pi}$$

In the exponential form the product is given by

$$\dot{Z} = \dot{X}\dot{Y} = XY\varepsilon^{j(\alpha+\beta)}$$

Thus the product of two complex numbers is a complex number having a magnitude equal to the product of the absolute values of the numbers and an argument equal to the sum of the arguments of the numbers.

It is apparent that the physical meaning of the result is shown better by the polar and exponential forms than by the rectangular form.

The result also holds if one of the numbers is a vector quantity, since in this case the product is another vector quantity. For example, if a sinusoidal alternating quantity, which is represented by a rotating vector, is multiplied by a complex number the result is another alternating quantity of the same frequency but of different magnitude and phase with respect to the original quantity.

If, however, two alternating quantities of the same frequency are multiplied together the result (see p. 77) is an alternating quantity of double frequency.

For example, the product $E \sin \omega t$ and $I \cos \omega t$ is

$$E \sin \omega t \,.\, I \cos \omega t = \tfrac{1}{2} EI \sin 2\omega t$$

The physical meaning of the product is considered in Chapter V.

Division of complex numbers and quantities. The quotient of two complex numbers is another complex number, and that of two vector quantities is a complex number; but if a vector quantity is divided by a complex number the result, is another vector quantity of different magnitude and phase to the original vector quantity. For example, the operation on the vector quantity $\dot{E} = \boldsymbol{a} + j\boldsymbol{b}$ by the complex number $\dot{Z} = \boldsymbol{r} + j\boldsymbol{x}$ is given by the vector quantity

$$I = \frac{\dot{E}}{\dot{Z}} = \frac{\boldsymbol{a} + j\boldsymbol{b}}{\boldsymbol{r} + j\boldsymbol{x}} = \frac{(\boldsymbol{a} + j\boldsymbol{b})(\boldsymbol{r} - j\boldsymbol{x})}{(\boldsymbol{r} + j\boldsymbol{x})(\boldsymbol{r} - j\boldsymbol{x})} = \frac{(\boldsymbol{a} + j\boldsymbol{b})(\boldsymbol{r} - j\boldsymbol{x})}{r^2 + x^2}$$

$$= \frac{\boldsymbol{ar} + \boldsymbol{bx}}{r^2 + x^2} + j\left(\frac{\boldsymbol{br} - \boldsymbol{ax}}{r^2 + x^2}\right)$$

[Observe that the denominator $\boldsymbol{r} + j\boldsymbol{x}$ is rationalized by introducing the conjugate expression $\boldsymbol{r} - j\boldsymbol{x}$.]

The absolute value of this quantity is

$$I = \sqrt{\left\{\left(\frac{ar + bx}{r^2 + x^2}\right)^2 + \left(\frac{br - ax}{r^2 + x^2}\right)^2\right\}}$$

and its inclination to the axis of reference is given by

$$\varphi = \tan^{-1}(br - ax)/(ar + bx)$$

The physical meaning of the result is shown better when the polar and exponential forms are employed. Thus the operation on the vector quantity $\dot{E} = E\varepsilon^{j\varphi_1}$ by the complex number $\dot{Z} = Z\varepsilon^{j\varphi_2}$ is given by the vector quantity

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \frac{E\varepsilon^{j\varphi_1}}{Z\varepsilon^{j\varphi_2}} = \frac{E}{Z}\varepsilon^{j(\varphi_1 - \varphi_2)}$$

Similarly, for the polar form, we have

$$I = \frac{E\,J^{(\varphi_1/\frac{1}{2}\pi)}}{Z\,J^{(\varphi_2/\frac{1}{2}\pi)}} = \frac{E}{Z} J^{(\varphi_1 - \varphi_2)/\frac{1}{2}\pi}$$

In the special case when the argument of the vector quantity is zero, the result is given by

$$I = \frac{E}{Z} = \frac{E}{Z\varepsilon^{j\varphi_2}} = \frac{E}{Z}\varepsilon^{-j\varphi_2}$$

or in the polar form by

$$I = \frac{E}{Z} J^{-(\varphi_2/\frac{1}{2}\pi)}$$

and in the rectangular form by

$$I = \frac{E}{Z} = \frac{E}{r+jx} = \frac{E(r-jx)}{r^2+x^2} = \frac{Er}{r^2+x^2} - j\left(\frac{Ex}{r^2+x^2}\right)$$

$$I = \sqrt{\left[\left(\frac{Er}{r^2+x^2}\right)^2 + \left(\frac{Ex}{r^2+x^2}\right)^2\right]} = E\sqrt{\left(\frac{1}{r^2+x^2}\right)}$$

$$\varphi_2 = \tan^{-1} - x/r$$

Another special case of interest is where one complex quantity is the reciprocal of the other. Thus let the complex quantity $Y = Y\varepsilon^{j\beta}$ be the reciprocal of the complex quantity $Z = Z\varepsilon^{j\alpha}$. Then by definition

$$YZ = 1 = Y\varepsilon^{j\alpha} \times Z\varepsilon^{j\beta} = YZ\varepsilon^{j(\alpha+\beta)}$$

Hence $\alpha + \beta = 0$

or $\beta = -\alpha$

i.e. the reciprocal of the vector Z has a length equal to Z and is inclined at an angle equal to $-\alpha$ to the axis of reference.

If the rectangular form is employed the physical meaning of the result is not so apparent. Thus if the vector Z is represented by $Z = r + jx$, then the reciprocal vector Y is represented by

$$Y = \frac{1}{Z} = \frac{1}{r+jx} = \frac{r-jx}{(r+jx)(r-jx)}$$

$$= \frac{r}{r^2+x^2} - j\frac{x}{r^2+x^2}$$

The length of the reciprocal vector is therefore equal to

$$Y = \sqrt{\left\{\left(\frac{r}{r^2 + x^2}\right)^2 + \left(\frac{x}{r^2 + x^2}\right)^2\right\}}$$

$$= \sqrt{\left(\frac{1}{r^2 + x^2}\right)}$$

$$= 1/Z$$

and its inclination to the axis of reference is given by

$$\varphi = \tan^{-1} - x/r$$

Examples. (1) Find the quotients of each of the following pairs of numbers

(*a*) $3 + j3$, $5 - j2$, (*b*) 5, $1 + j1$, (*c*) $-2 + j2$, $-3 - j4$.

Denoting the quotients by Y_a, Y_b, Y_c, and the arguments of these quantities by φ_a, φ_b, φ_c, respectively, we have

$$Y_a = \frac{3 + j3}{5 - j2} = \frac{(3 + j3)(5 + j2)}{5^2 + 2^2} = \frac{3 \times 5 - 3 \times 2 + j(3 \times 2 + 3 \times 5)}{29}$$

$$= 0{\cdot}31 + j0{\cdot}724$$

$$Y_a = \sqrt{(0{\cdot}31^2 + 0{\cdot}724^2)} = 0{\cdot}788$$

$$\varphi_a = \tan^{-1} 0{\cdot}724/0{\cdot}31 = 66{\cdot}8^\circ$$

$$\therefore Y_a = 0{\cdot}788 \underline{/66{\cdot}8^\circ}$$

$$Y_b = \frac{5}{1 + j1} = \frac{5(1 - j1)}{1^2 + 1^2} = 2{\cdot}5 - j2{\cdot}5$$

$$Y_b = \sqrt{(2{\cdot}5^2 + 2{\cdot}5^2)} = 3{\cdot}54$$

$$\varphi_b = \tan^{-1} - 2{\cdot}5/2{\cdot}5 = -45^\circ$$

$$\therefore Y_b = 3{\cdot}54 \underline{/-45^\circ}$$

$$Y_c = \frac{-2 + j2}{-3 - j4} = \frac{(-2 + j2)(-3 + j4)}{3^2 + 4^2} = \frac{2 \times 3 - 2 \times 4 + j(-2 \times 4 + 2 \times (-3))}{25}$$

$$= -0{\cdot}08 - j0{\cdot}56$$

$$Y_c = \sqrt{(0{\cdot}08^2 + 0{\cdot}56^2)} = 0{\cdot}566$$

$$\varphi_c = \tan^{-1}(-0{\cdot}56/-0{\cdot}08) = -98{\cdot}1^\circ$$

$$\therefore Y_c = 0{\cdot}566 \underline{/-98{\cdot}1^\circ}$$

(2) Find the reciprocal of the number $5 + j8$

The reciprocal number is

$$Y = \frac{5}{5^2 + 8^2} - j\frac{8}{5^2 + 8^2} = 0{\cdot}0562 - j0{\cdot}09$$

$$Y = \sqrt{(0{\cdot}0562^2 + 0{\cdot}09^2)} = 0{\cdot}0106$$

$$\varphi = \tan^{-1} - 0{\cdot}09/0{\cdot}0562 = -58^\circ$$

$$\therefore Y = 0{\cdot}0106 \underline{/-58^\circ}$$

CHAPTER III

RESISTANCE AND INDUCTANCE

IN continuous-current circuits the relationship between E.M.F. and current is a simple one and is given by the equation $E = IR$. Now the resistance (R) of the conductors of any particular circuit is constant provided that the temperature of the conductors is constant. Hence, except for the variation of resistance caused by change of temperature, the resistance of a circuit carrying a continuous current is independent of the magnitude of the current.* Thus for a given circuit the ratio E/I is constant.

In alternating-current circuits generally this simple relation between E.M.F. and current is not applicable, as the variations of current and E.M.F. set up magnetic and electrostatic effects, respectively, which must be considered together with the resistance of the circuit when determining the quantitative relations between current and E.M.F. For example, with low-voltage circuits magnetic effects may be very large, especially when the currents are very large, but electrostatic effects are usually negligible. On the other hand, with high-voltage circuits electrostatic effects may be of appreciable magnitude, and magnetic effects are also present Hence, in obtaining the relation between E.M.F. and current, these effects must be given due consideration.

In this chapter we shall consider the manner in which the magnetic effects due to the current in a circuit affect the relationship between the applied E.M.F. and the current, reserving for a later chapter the discussion of electrostatic effects.

Inductance. An electric current in a conductor produces a magnetic field which encircles the conductor. When no other magnetic fields are present the paths of the magnetic lines are circles which are concentric with the conductor. The magnitude of the magnetic field is proportional to the magnitude of the current, and the direction of the field depends on the direction of the current. If the conductor forms part of a circuit carrying an alternating current the flux will be alternating, and the linkage of this flux with the circuit will induce therein an alternating E.M.F. The magnitude of this E.M.F. is, at any instant, proportional to the time rate of change of the flux at that instant. As the induced

* It is assumed that the distribution of the current over the cross section of the conductor remains constant at all currents.

E.M.F. is due *solely* to the magnetic effect of the current, it is called the *E.M.F. of self-induction*, or the *inductive E.M.F.*

Inductance (also called self-induction) is the property of a circuit in virtue of which a varying current causes a variation of the flux interlinked with the circuit and an E.M.F. to be induced therein.

A circuit possessing inductance is called an *inductive circuit* and one devoid of inductance is called a *non-inductive circuit*. Since it is difficult to obtain a circuit absolutely devoid of inductance, the term "non-inductive circuit" usually refers to one in which the inductance is negligible in comparison with the resistance. Examples of non-inductive circuits and apparatus: incandescent lamps, liquid and grid rheostats, concentric cables, standard low resistances constructed of concentric tubular conductors. Examples of inductive circuits: solenoids, and all electromagnetic apparatus and machinery, overhead transmission lines.

Coefficient of inductance. In an alternating-current circuit of constant magnetic reluctance the flux is directly proportional to, and is in phase with, the current. Thus

$$\Phi = \frac{0{\cdot}4\pi iN}{S} \quad \left(\frac{0{\cdot}4\pi N}{S}\right.$$

where Φ is the flux corresponding to the current i (amperes), N the number of turns through which the current passes, and S the magnetic reluctance in centimetre units. Hence, in such a circuit the E.M.F. of self-induction (e_L volts) is at any instant, proportional to the rate of change of the current, thus

$$e_L = -\frac{N}{10^8}\frac{d\Phi}{dt} = -\left(\frac{0{\cdot}4\pi N^2}{S \times 10^8}\right)\frac{di}{dt} = -L\frac{di}{dt} \quad . \quad . \quad . \quad (6)$$

where L is a constant, called the *coefficient of inductance*—or, shortly, the *inductance*—of the circuit. The minus sign is introduced because the direction of the induced E.M.F. must be such as to oppose the flow of current.

Numerically,

$$L = \frac{0{\cdot}4\pi N^2}{S \times 10^8} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$= \frac{N\Phi}{i \times 10^8} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$= \frac{e_L}{di/dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The coefficient of inductance is therefore a constant property of all circuits for which the magnetic reluctance is constant.

The *practical unit of inductance* is the *henry*. This unit is 10^9 times the C.G.S. electromagnetic unit, which is a centimetre; inductance having the dimension $[\mathsf{M}^{\frac{1}{2}}\mathsf{L}^{\frac{3}{2}}\mathsf{T}^{-2}]\mathsf{T}/[\mathsf{M}^{\frac{1}{2}}\mathsf{L}^{\frac{1}{2}}\mathsf{T}^{-1}] = \mathsf{L}$, where L, M, T denote length, mass, and time respectively.

A circuit possesses an inductance of 1 henry when

10^8 linkages of flux and turns are produced by a current of 1 ampere passing in the circuit; or when

an E.M.F. of 1 volt is induced by a current varying at the rate of 1 ampere per second.

Equation (9) gives L in practical units (i.e. henries) when e_L is expressed in volts and di/dt in amperes per second.

Examples. (1) A wooden tore (i.e. a ring of circular cross section) of 20 cm. mean diameter and 5 cm.² cross section is wound with 1000 turns of fine wire. Calculate the inductance of the winding. Also calculate the value of the induced E.M.F. when a current varying at the rate of 190 A. per second is sent through the winding.

The reluctance of the magnetic path of the wound tore is equal to

$$\frac{\text{mean magnetic length}}{\text{cross section}} = \frac{20\pi}{5} = 4\pi$$

Hence, substituting in equation (7), we have

$$L = \frac{0{\cdot}4\pi}{4\pi} \times \frac{1000^2}{10^8} = 10^{-3}\ \text{H}, \text{ or } 1 \text{ milli-henry}.$$

The induced E.M.F. is obtained from equation (9). Thus

$$\begin{aligned} e_L &= L\, di/dt \\ &= 10^{-3} \times 190 \\ &= 0{\cdot}1\ \text{V}. \end{aligned}$$

(2) Inductance of two parallel cylindrical conductors.

In this case we require the flux linked with a definite length of each conductor when the current is 1 A. The flux produced by this current is calculated from elementary principles, and to make the conditions of the problem definite we shall assume that the conductors are similar, non-magnetic, and form part of a circuit; that the current is uniformly distributed over the cross section of the conductors; and that the space surrounding them is non-magnetic. Under these conditions we may obtain the flux linked with each conductor by superimposing the separate fluxes due to the currents in each conductor.

Thus, considering the magnetic effect due to a current of i amperes in one conductor, the magnetic force (or density of magnetic field in lines per cm.²) at a point P, Fig. 19, external to the conductor, distant x cm. from its axis is

$$H_x = \frac{0{\cdot}4\pi i}{2\pi x} = \frac{0{\cdot}2 i}{x}$$

or, when $i = 1$,

$$H_x = 0{\cdot}2/x.$$

The magnetic force at a point inside the conductor, distant y cm. from its axis, is

$$H_y = \frac{0{\cdot}4\pi i(\pi y^2/\pi r^2)}{2\pi y} = 0{\cdot}2i\left(\frac{y}{r^2}\right)$$

or, when $i = 1$,

$$H_y = 0{\cdot}2(y/r^2)$$

where r = radius of the conductor.

The variation of the magnetic force in the plane perpendicular to that containing the axes of the conductors is shown in Fig. 19, curve I.

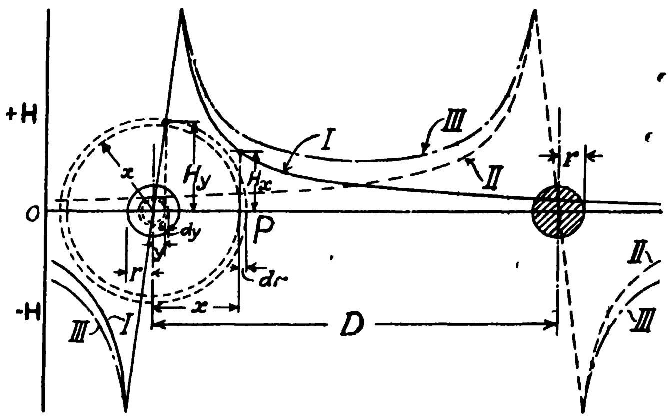


FIG. 19. -Pertaining to the Calculation of the Inductance of Parallel Cylindric Conductors

The effect of the current in the other conductor is represented by the curve II, Fig. 19. Hence when both conductors are carrying current the resultant magnetic force in the space between them is represented by the sum of curves I and II, and is shown by curve III. The area between this curve and the abscissa axis is therefore proportional to the flux linked with each cm. length of the circuit when both conductors are carrying current.

The external flux in the space between the conductors is, per cm. length of circuit and per ampere,

$$\Phi_e = 2\int_r^{D-r}(H_x \times 1)dx = 2\int_r^{D-r}\frac{0{\cdot}2}{x}\,dx = 2\times 0{\cdot}2\Big[\log_\epsilon x\Big]_r^{D-r}$$

$$= 0{\cdot}4\log_\epsilon[(D-r)/r]$$

$$= 0{\cdot}4\times 2{\cdot}3\log_{10} D/r$$

when D is large in comparison with r.

The internal flux per cm. length of each conductor, due to unit current in it, is, for an element dy distant y from the axis,

$$d\Phi_i = H_y(dy \times 1) = 0{\cdot}2(y/r^2)dy$$

This flux is linked with the portion πy^2 of the cross section of the conductor. Whence the linkage is

$$d\Phi_i \times \pi y^2/\pi r^2 = 0{\cdot}2(y^3/r^4)dy$$

Hence the total linkage for each conductor due to the internal flux is

$$\int_0^r 0{\cdot}2(y^3/r^4)dy = \frac{0{\cdot}2}{r^4}\int_0^r y^3\,dy = 0{\cdot}2/4 = 0{\cdot}05$$

Therefore the inductance per cm of circuit is

$$L = \{(0{\cdot}4 \times 2{\cdot}3 \log_{10} D/r) + 2 \times 0{\cdot}05\} \times 10^{-8}$$
$$= \{0{\cdot}92 \log_{10} D/r + 0{\cdot}1\} \times 10^{-8} \text{ henry.}$$

Mutual inductance. If a circuit (A) carrying an alternating current is in close proximity to another circuit (B), the flux due to the current in the former will interlink with both circuits and induce in them E.M.Fs., the direction of which will be the same for each circuit. Now the E.M.F. induced in circuit A acts in such a direction as to oppose the change of current in it. Hence, if circuit B is closed upon itself, the direction in which the current circulates is the same as that of the induced E.M.F., and is opposite to that of the current in A. The resultant ampere-turns due to the currents in these circuits are therefore smaller than those due to the current in A, and on the assumption of constant reluctance, the flux linked with the circuits under these conditions is smaller than that which is linked with A when B is open or removed, the current in A being the same in each case. Thus the effect of the induced current in B is equivalent to a reduction of the self induction of A. This inductive action of one circuit upon another is called *mutual inductance* or *mutual induction*.

Suppose the above circuits consist of N_1, N_2 turns, respectively, wound in close proximity upon a common non-magnetic, non-metallic core of reluctance S. Assume alternating current to be supplied to circuit A, and circuit B to be open Then the flux (Φ_1) due to a current i_1 amperes in A is

$$\Phi_1 = 0{\cdot}4\pi i_1 N_1/S$$

The number of linkages of this flux and the turns in B is

$$\Phi_1 N_2 = 0{\cdot}4\pi i_1 N_1 N_2/S$$

and the E.M.F. (e_2) induced in circuit B is

$$e_2 = -\frac{N_2}{10^8}\frac{d\Phi_1}{dt} = -\frac{0{\cdot}4\pi N_1 N_2}{10^8 \times S}\frac{di_1}{dt} = -M\frac{di_1}{dt}$$

where $M = 0{\cdot}4\pi N_1 N_2/S \times 10^8$, and is a constant.

Similarly, if alternating current is supplied to circuit B and circuit A is open, the flux (Φ_2) due to a current i_2 amperes in B is

$$\Phi_2 = 0{\cdot}4\pi i_2 N_2/S$$

The number of linkages of this flux and the turns of A is

$$\Phi_2 N_1 = 0{\cdot}4\pi i_2 N_1 N_2/S$$

and the E.M.F. (e_1) induced in circuit A is

$$e_1 = -\frac{N_1}{10^8}\frac{d\Phi_2}{dt} = -\frac{0{\cdot}4\pi N_1 N_2}{10^8 \times S}\frac{di_2}{dt} = -M\frac{di_2}{dt}$$

where $M = 0{\cdot}4\pi N_1 N_2/S \times 10^8$

The constant M is called the *coefficient of mutual induction,* or, shortly, the *mutual inductance* of the two circuits, and is a constant property of the circuits provided that the reluctance is constant.

Numerically,

$$M = \frac{0{\cdot}4\pi N_1 N_2}{10^8 \times S} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$= \frac{N_1\Phi_2}{i_2 \times 10^8} = \frac{N_2\Phi_1}{i_1 \times 10^8} \quad . \quad . \quad . \quad (11)$$

$$= \frac{e_1}{(di_2/dt)} = \frac{e_2}{(di_1/dt)} \quad . \quad . \quad . \quad (12)$$

The *practical unit of mutual inductance* is the *henry.* Thus inductance, whether self or mutual, is expressed in henries.

Two circuits possess a mutual inductance of 1 henry when

10^8 linkages of flux and turns are produced in one circuit due to a current of 1 ampere in the other circuit; or when

an E.M.F. of 1 volt is induced in one circuit by a current varying at the rate of 1 ampere per second in the other circuit.

Equation (12) gives M in henries when the induced E.M.Fs (e_1, e_2) are expressed in volts and the rate of change of current is expressed in amperes per second.

Example. A straight cylindrical wooden core 4 cm. diameter is wound over uniformly with one layer of fine wire, there being 8 turns per cm. for a length of 100 cm. Around the middle of this helix is wound a search coil of 50 turns. Calculate the mutual inductance of the coils.

As the length of the helix is great in comparison with its diameter the flux density (B) at its centre, due to current i, is

$$B = 0{\cdot}4\pi \times \text{amp. turns per cm. length}$$
$$= 0{\cdot}4\pi \times 8i$$

Hence the flux (Φ) at the middle of helix due to unit current is

$$\Phi = BA$$
$$= 0{\cdot}4\pi \times 8 \times \pi \times 2^2$$

This flux is linked with the turns of the search coil, and therefore the mutual inductance of the coils is

$$M = \frac{\Phi \times 50}{10^8} = \frac{0{\cdot}4 \times 8 \times \pi \times 2^2 \times 50}{10^8}$$
$$= 0{\cdot}063 \times 10^{-3}$$
$$= 0{\cdot}063 \text{ milli-henries.}$$

Relation between self-inductance and mutual inductance. If L_1, L_2 denote the respective self-inductances of the above circuits,

$$L_1 = 0{\cdot}4\pi N_1^2/S \times 10^8$$
$$L_2 = 0{\cdot}4\pi N_2^2/S \times 10^8$$

Hence,

$$L_1L_2 = (0{\cdot}4\pi N_1N_2/S \times 10^8)^2 = M^2$$

or

$$M = \sqrt{(L_1L_2)} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Thus the mutual inductance of two circuits is equal to the square root of the product of their self-inductances.

This equation is only strictly true when the whole of the flux due to a current in one circuit is linked with the whole of the turns of the other circuit, i.e. the turns in both circuits are coincident. In general, these two sets of turns are displaced from each other, and M is less than $\sqrt{L_1L_2}$. The ratio $M/\sqrt{L_1L_2}$ is called, in radio-telegraphy, the *coefficient of coupling*, and mutually-inductive circuits are said to be tightly or loosely coupled, according to whether the value of ratio $M/\sqrt{L_1L_2}$ approaches unity or is considerably less than unity.

Apparent self-inductance of mutually-inductive series circuits. If the two mutually-inductive circuits A, B considered above are connected in series, the joint, or apparent, inductance is given by $L = L_1 + L_2 \pm 2M$.

Thus the self-induced E.M.F. in A due to the current (i) is equal to $-L_1 di/dt$, and that in B is equal to $-L_2 di/dt$. Again, the E.M.F. induced in A due to mutual induction from B is equal to $\mp M di/dt$, and that induced in B due to mutual induction from A is equal to $\mp M\, di/dt$, the double sign denoting that two combinations of the circuits are possible, i.e. the connections may be such that the magneto-motive forces act either cumulatively or differentially.

Hence, the sum of the induced E.M.Fs. due to the variation of current is

$$-L_1\frac{di}{dt} - L_2\frac{di}{dt} \mp 2M\frac{di}{dt} = -\frac{di}{dt}(L_1 + L_2 \pm 2M) = -L\frac{di}{dt}$$

where $L = L_1 + L_2 \pm 2M$

If the mutual inductance is variable between the limits M_{max}, M_{min}, then the joint, or apparent, inductance of the circuits will be variable between the limits $2(M_{max} \pm M_{min})$.

This principle is applied extensively in practice to variable standards of self and mutual inductance, and to receiving apparatus for radio telegraphy and telephony.

Non-uniform distribution of current over cross-section of conductor (skin effect). The elementary theory of mutually-inductive circuits may be applied to show that the distribution of current over the cross-section of a non-magnetic conductor is not uniform.*

Thus, consider two concentric tubular elements A, B, in the cross-section of a circular conductor, and let the cross-sections of these elements be equal. Further, let i_A, i_B denote the currents in the elements, L_A, L_B, their self-inductances, M their mutual inductance, and R the resistance of each element for a given length of conductor. Then, if the potential difference across this length is denoted by e, we have

$$e = Ri_A + L_A \frac{di_A}{dt} + M \frac{di_B}{dt} = Ri_B + L_B \frac{di_B}{dt} + M \frac{di_A}{dt}$$

whence $i_A = i_B + \frac{di_B}{dt}\left(\frac{L_B - M}{R}\right) + \frac{di_A}{dt}\left(\frac{M - L_A}{R}\right)$.

Now if A is the outer element, $L_B > L_A$, and $M = L_A$.

Hence $i_A = i_B + \frac{di_B}{dt}\left(\frac{L_B - L_A}{R}\right)$

which shows that the current density in the outer element (A) is greater than that in the inner element (B).

Therefore, due to inductive effects, the current tends to concentrate towards the surface of the conductor. This phenomenon is called the *skin effect*. It results in the heating, or I^2R, loss in the conductor being greater than that when the current is uniformly distributed, and, in consequence, the "effective resistance" of a conductor when carrying alternating current is greater than the true resistance of the conductor. The skin effect becomes of considerable importance at high frequencies, and highly stranded conductors, laid up on a hemp core, must be employed for transmitting high-frequency currents, the stranding being necessary for reducing the loss due to eddy currents.

Moreover, on account of the non-uniform distribution of the current, the self-inductance of the conductor will be slightly lower than that calculated upon the assumption of uniform distribution of the current.

Additional theory relating to mutually-inductive circuits is deferred until the relationship between current and impressed E.M.F. for the simpler alternating-current circuits has been established.

Relation between current and E.M.F. for a simple circuit possessing resistance. Consider a non-inductive circuit, such as an incandescent lamp, in which an alternating current is passing. Let the resistance of the circuit be R ohms and let the current be represented

* The full treatment involves complex quantities and may be developed along similar lines to those employed in Chapter XI in connection with the calculation of the flux distribution in a magnetic core. In fact the distribution of current over the cross-section of a large conductor may, at high frequencies, be represented by a curve similar to that of Fig. 197.

by the equation $i = I_m \sin \omega t$. It is required to determine the impressed E.M.F. necessary to maintain this current.

Now in any electric circuit the resultant of all the E.M.Fs. (internal and external) acting in the circuit must be zero, i.e. the internal E.M.Fs. which come into existence with the current, and are due to the resistance or other properties of the circuit, must balance the external, or impressed, E.M.F.

If the circuit possesses only resistance we must have $(e - \Sigma ir) = 0$; or $e = \Sigma ir$, where e is the impressed, or applied, E.M.F., and Σir the internal E.M.Fs. produced by the passage of the current through the resistance of the circuit. It is important to observe that the internal E.M.Fs. (Σir) act in opposition to the current, and at any instant their phase difference with respect to the current is 180°.

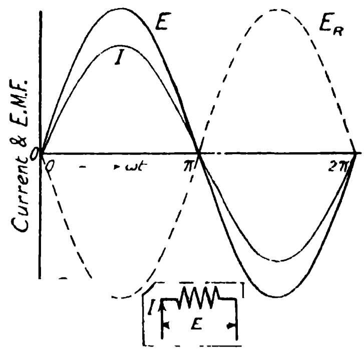


FIG. 20. Representation of Current and E.M.Fs. (External and Internal) for Non-inductive Circuit

Hence for the circuit under consideration the internal E.M.F. due to resistance is, at any instant, given by

$$e_R = -Ri = -RI_m \sin \omega t,$$

the minus sign indicating that e_R acts in opposition to the current.

Therefore the equation to the impressed E.M.F. is

$$e = -e_R = Ri = RI_m \sin \omega t \qquad . \quad . \quad (14)$$

Thus the impressed E.M.F. must be sinusoidal and of the same frequency as the current. It must also be in phase with the current. These conditions are shown graphically in Fig. 20, in which the sine curve I represents the current, the sine curve E the impressed E.M.F., and the sine curve E_R the internal E.M.F.

Conversely, if a circuit possessing pure resistance be connected to a source of sinusoidal E.M.F. the current in the circuit will be sinusoidal and will have the same frequency and phase as the impressed E.M.F. For example, if the impressed E.M.F. is given by the equation

$$e = E_m \sin \omega t$$

the equation to the current is

$$i = \frac{E_m}{R} \sin \omega t \qquad (15)$$

The maximum value of the current is

$$I_m = E_m/R$$

and the R.M.S. value is

$$I = E/R$$

Hence the relationship between current and E.M.F. is the same as for a continuous-current circuit, and the ratio E/I, or E_m/I_m, gives the true resistance of the circuit, such as would be obtained by a test with continuous current. This statement, however, holds true only in cases where (*a*) the current is uniformly distributed over the cross-section of the conductors, (*b*) the conductors are removed from the influence of external alternating magnetic fields.

Relation between current and E.M.F. for circuits possessing inductance only. Consider a purely inductive circuit of which the inductance is constant and equal to L. Although such a circuit cannot be realized absolutely in practice, it is approximated to by a non-magnetic torus with a low-resistance winding. If the torus is of laminated iron, or of iron wire, a closer approximation to a purely inductive circuit is obtained; but in this case the inductance will vary with the saturation of the iron core.

Let the current in the circuit be represented by the equation $i = I_m \sin \omega t$. Then the E.M.F. induced in the circuit by the alternations of the current is

$$\begin{aligned} e_L &= -L\frac{di}{dt} = -L\frac{d}{dt}(I_m \sin \omega t) \\ &= -L\,\omega\, I_m \cos \omega t \\ &= \omega L I_m \sin (\omega t - \tfrac{1}{2}\pi), \end{aligned}$$

which is of the same frequency as the current but lags behind it by an angle of $\frac{1}{2}\pi$ radians, or 90°.

Since the resistance of the circuit is zero the impressed E.M.F. (e) must balance the E.M.F. of self-induction. Hence the equation to the impressed E.M.F. is

$$\begin{aligned} e &= -e_L = -\omega L I_m \sin (\omega t - \tfrac{1}{2}\pi) \\ &= \omega L I_m \sin (\omega t + \tfrac{1}{2}\pi) \qquad . \qquad . \quad (16) \end{aligned}$$

Thus the impressed E.M.F. must be sinusoidal and of the same frequency as the current. Moreover, it must lead the current by an angle of 90°.

Conversely, if a purely inductive circuit be connected to a source of sinusoidal E.M.F. the phase difference between E.M.F. and

current will be 90° (lagging). A graphical representation of these conditions is shown in Fig. 21, in which the sine curve I represents the current, the sine curve E the impressed E.M.F., and the sine curve E_L the E.M.F. of self-induction.

From equation (16) the maximum value of the impressed E.M.F. is

$$E_m = \omega L I_m$$

and the R.M.S. value is

$$E = \omega L I = 2\pi f L I$$

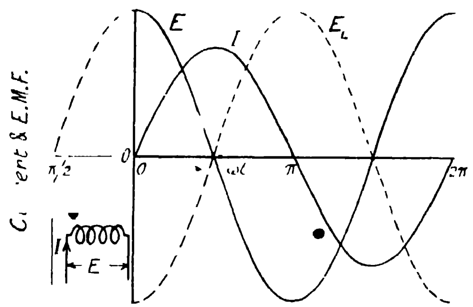


FIG. 21. Representation of Current and E.M.F.s (External and Internal) for Purely Inductive Circuit

The ratio $E/I = \omega L = 2\pi f L$ is called the *reactance* of the circuit, and is denoted by the symbol X. Reactance has the same dimensions as resistance (both having the dimension [L/T] in the electromagnetic system of units) and is accordingly expressed in ohms.

Example. A cylindrical wooden core, 4 cm. in diameter, is wound over uniformly with one layer of fine wire, there being 8 turns per cm. for a length of 100 cm. Around the middle of this helix is wound a search coil of 50 turns. What voltage would you expect to get at the terminals of the search coil when a current of 10 A. at 50 cycles per second is sent through the helix ?

If M is the mutual inductance of the coils, the instantaneous value (e_2) of the E.M.F. induced in the search coil is

$$e_2 = -M\,di_1/dt$$

where i_1 is the instantaneous value of the current in the primary coil. If this current is represented by the equation $i = I_m \sin \omega t$, we have

$$\begin{aligned} e_2 &= -M\frac{d}{dt}(I_m \sin \omega t) \\ &= -MI_m\omega \cos \omega t \\ &= \omega M I_m \sin(\omega t - \tfrac{1}{2}\pi) \end{aligned}$$

The maximum value of the induced E.M.F. is

$$E_{2m} = \omega M I_m$$

and the R.M.S. value is

$$E_2 = \omega M I$$

The value of M has been calculated on p. 41 and is 0·063 mH.

Hence $E_2 = 2\pi \times 50 \times 0\cdot063 \times 10^{-3} \times 10$
$= 0\cdot2$ V.

Relation between current and E.M.F. for circuits possessing resistance and inductance. Consider a circuit of resistance R ohms and an inductance (which is assumed to be constant) of L henries. Let the current in the circuit be represented by the equation $i = I_m \sin \omega t$.

The internal E.M.Fs. in the circuit are

(1) the E.M.F. due to resistance ($e_R = -Ri$) which has a phase difference of 180° with respect to the current,

(2) the E.M.F. of self-induction ($e_L = -L\,di/dt$) which has a phase difference of 90°, lagging, with respect to the current.

The impressed E.M.F. (e) must balance the internal E.M.Fs. Therefore

$$\begin{aligned} e &= -(e_R + e_L) = Ri + L\,di/dt \\ &= RI_m \sin \omega t + \omega L I_m \cos \omega t \qquad . \qquad . \qquad (17) \end{aligned}$$

But the sum of two sinusoidal quantities can be expressed as a single sinusoidal quantity. Thus, multiplying and dividing each term of equation (17) by $\sqrt{(R^2 + \omega^2 L^2)}$, we have

$$e = I_m \sqrt{(R^2 + \omega^2 L^2)} \left\{ \frac{R}{\sqrt{(R^2 + \omega^2 L^2)}} \sin \omega t + \frac{\omega L}{\sqrt{(R^2 + \omega^2 L^2)}} \cos \omega t \right\}$$

Now if $\tan \varphi = \omega L / R$

$$\cos \varphi = R / \sqrt{(R^2 + \omega^2 L^2)}$$

$$\sin \varphi = \omega L / \sqrt{(R^2 + \omega^2 L^2)}$$

we obtain on substituting these values in the above equation,

$$\begin{aligned} e &= I_m \sqrt{(R^2 + \omega^2 L^2)} \{\cos \varphi \sin \omega t + \sin \varphi \cos \omega t\} \\ &= I_m \sqrt{(R^2 + \omega^2 L^2)} \sin (\omega t + \varphi) \qquad . \qquad . \qquad . \qquad (18) \end{aligned}$$

This equation shows that the impressed E.M.F. is sinusoidal and leads the current by the angle φ, the tangent of which is equal to

$$\frac{\omega L}{R} = \frac{\textbf{reactance}}{\textbf{resistance}}$$

Conversely, if the equation to the current had been given as $i = I_m \sin(\omega t - \varphi)$, where $\varphi = \tan^{-1} \omega L/R$, we should have obtained for the impressed E.M.F. the equation

$$e = I_m \sqrt{(R^2 + \omega^2 L^2)} \sin \omega t$$

Hence, if a sinusoidal E.M.F. represented by the equation $e = E_m \sin \omega t$ be applied to the circuit, the current, when the steady or cyclic state* is reached, will be given by

$$i = \frac{E_m}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \varphi) \quad . \quad . \quad . \quad . \quad (19)$$

Thus in an inductive circuit, of constant resistance and inductance, supplied by a source of sinusoidal E.M.F., the current is sinusoidal and of the same frequency as the impressed E.M.F., but is lagging with respect to the latter.

The maximum value of the current, from equation (19), is

$$I_m = E_m/\sqrt{(R^2 + \omega^2 L^2)},$$

and the R.M.S. value is

$$I = E/\sqrt{(R^2 + \omega^2 L^2)}$$

The ratio $E/I = \sqrt{(R^2 + \omega^2 L^2)}$ is called the *impedance*, or "apparent resistance," of the circuit, and is usually denoted by the symbol Z. Impedance has the same dimensions as resistance, and is accordingly expressed in ohms.

The equation connecting E.M.F., current, and impedance in an alternating-current circuit is therefore similar to that connecting E.M.F., current, and resistance in a continuous-current circuit. Hence, by regarding impedance as "apparent resistance," Ohm's law is applicable to alternating-current circuits in which E.M.F. and current are sinusoidal. We shall see later that this law is applicable to all simple series and parallel circuits, even when electrostatic capacity in a concentrated form is present. It should be observed, however, that impedance is not necessarily a constant property of an alternating-current circuit, as this quantity includes resistance, inductance, and frequency. But with constant resistance, inductance, and frequency the impedance will be constant, and therefore the ratio E/I will be constant.

The relationship between the current, the two internal E.M.Fs., and the impressed E.M.F. is shown graphically in Fig. 22. The impressed E.M.F. is represented by the sine curve E, and the

* The general equation to the current must take into account the value of the impressed E.M.F. at the instant of closing the circuit. In general, the first few cycles of the current wave are irregular, but the waves become sinusoidal after a short time (see Chapter XVI).

current is represented by the sine curve I, lagging φ° behind the latter. The internal E.M.Fs due to resistance and inductance are represented by the sine curves E_R, E_L, respectively, the former having a phase difference of 180° (lagging), and the latter a phase difference of 90° (lagging), with respect to the current curve I.

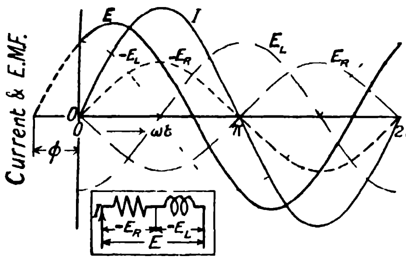


FIG. 22.—Representation of Current and E M Fs. (External and Internal) for an Inductive Circuit

The sum of curves E_R and E_L gives a sine curve which represents the resultant internal E.M.F. and balances the impressed E.M.F. Obviously the impressed E.M.F. curve E may be resolved into two components which balance the internal E.M.F. curves E_R, E_L. These components are represented by the sine curves $-E_R$, $-E_L$, respectively, the former being in phase with the current and of maximum value RI_m, the latter leading the current by 90° and having a maximum value equal to ωLI_m. The curves $-E_R$, $-E_L$ therefore represent the components of the impressed E.M.F. which are expended against resistance and inductance respectively.

Vector diagram for a series circuit containing resistance and inductance. The vector diagram for this circuit is shown in Fig. 23, in which the current vector OI is taken as the vector of reference. The internal E.M.Fs, E_R, E_L, are represented by the vectors OA, OB respectively, the lengths of which are proportional to RI and ωLI respectively. The components of the impressed E.M.F. which balance the internal E.M.Fs. are represented by the vectors OC, OD. The impressed E.M.F. is therefore represented by OE, which is the resultant of OC and OD, and leads the current vector by the angle φ.

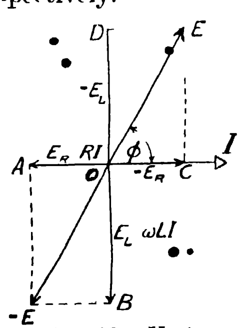


FIG. 23.—Vector Diagram for an Inductive Circuit

The length of OE, which is proportional to the R.M.S. value of the impressed E.M.F., is given by

$$E = \sqrt{(OC^2 + OD^2)} = \sqrt{[(RI)^2 + (\omega LI)^2]}$$
$$= I\sqrt{(R^2 + \omega^2 L^2)}$$

and $\quad \tan\varphi = OD/DE = \omega LI/RI = \omega L/R$

The vector triangle OCE is a triangle of E.M.Fs. for the circuit *referred to the external source of supply*. Thus OE represents the impressed E.M.F., OC the component which is expended against resistance, and CE the component which is expended against inductance.

Now, $$OC : CE : OE = RI : \omega LI : I\sqrt{(R^2 + \omega^2L^2)}$$
$$= R : \omega L : \sqrt{(R^2 + \omega^2L^2)}$$

Hence the sides OC, CE, OE of triangle OCE are proportional to the resistance, reactance, and impedance respectively. On account of this feature the triangle OCE, when drawn to an ohm scale, is called the *impedance triangle* of the circuit.

Impedance is therefore a complex quantity, i.e. it is only completely specified when its magnitude and inclination, or alternatively its two perpendicular components with respect to the current, are given. But impedance is not a vector quantity as it is a non-directive quantity.

In symbolic notation impedance is represented by the complex number

$$\underset{.}{Z} = r + jx$$

the absolute value of which is

$$Z = \sqrt{(R^2 + X^2)}$$

and its phase or argument with respect to the current axis is

$$\varphi = \tan^{-1} X/R$$

In the polar forms of representation, impedance is given by

$$\underset{.}{Z} = Z\varepsilon^{j\varphi}\ ;\quad \underset{.}{Z} = ZJ^{\varphi}\ ;\quad \underset{.}{Z} = Z_{/\underline{\varphi}}$$

Examples. (1) An inductive circuit has a resistance of 12 Ω, and an inductance of 0·2 H. What current will be taken when a sinusoidal E.M.F. of 100 V., at 50 frequency, is applied, and what will be the phase difference between E.M.F. and current?

The reactance of the circuit is

$$X = 2\pi fL = 2\pi \times 50 \times 0{\cdot}2 = 62{\cdot}8\ \Omega$$

and the impedance is

$$Z = \sqrt{(R^2 + \omega^2L^2)} = \sqrt{(R^2 + X^2)} = \sqrt{(12^2 + 62{\cdot}8^2)} = 63.9\ \Omega$$

Hence the current is

$$I = E/Z = 100/63{\cdot}9 = 1{\cdot}56\ \text{A.}$$

and the phase difference between E.M.F. and current is

$$\omega = \tan^{-1} X/R = \tan^{-1}(62{\cdot}8/12) = 79^\circ\ 10'$$

(2) A current passing through a choking coil having an inductance of 3 H. and a resistance of 2 Ω. varies according to the following law: At time 0

it is 0; it increases at the rate of 3 A. per second for 2 seconds; it then remains constant for 1 second; it then decreases at the rate of 2 A. per second for 3 seconds. Plot the current curve and also the voltage at the terminals of the choking coil. (C. and G., 1918.)

The solution is given in Fig. 24, in which the trapezoidal curve I represents the current and the stepped curve E represents the voltage at the terminals of the coil. The latter is obtained from its components, viz. (1) the E.M.F. (E_R) which is expended against resistance; (2) the E.M.F. ($-E_L$) which balances the E.M.F. of self-induction. The component $-E_R$ is in phase with the current and is equal to RI $2I$ volts. The component $-E_L$ is equal

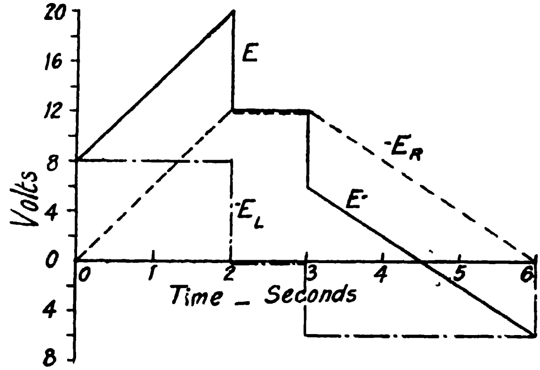


FIG. 24 Graphical Solution to Example

to $L\,di/dt$ (3 × amperes per second) volts; it is positive i.e. in the same direction as the current when the current is increasing (di/dt positive) and negative when the current is decreasing (di/dt negative).

ADDITIONAL THEORY RELATING TO MUTUALLY-INDUCTIVE CIRCUITS

Mutually-inductive circuits have numerous applications in practice, some of which have already been mentioned. Another example of mutually-inductive circuits, which has a very extensive application in practice, is the *alternating-current transformer* (called also the *static transformer*, or, shortly, the *transformer*), whereby electrical energy is transferred from one stationary electric circuit to another stationary electric circuit through the medium of an alternating magnetic field which interlinks both circuits. In order that the transference of energy between these circuits shall be effected as efficiently as possible, the electric circuits are wound upon a common magnetic circuit of laminated iron. The winding to which *energy is supplied* is called the *primary* and that from which *energy is taken* is called the *secondary*.

Elementary theory of ideal transformer. In an ideal transformer (i.e. one without losses or magnetic leakage) a common alternating flux, Φ, links both primary and secondary windings. (Fig. 25*a*.) Therefore the E.M.Fs. induced in these windings by this flux are $e_1 = -N_1\,d\Phi/dt$, and $e_2 = -N_2 d\Phi/dt$ respectively, where N_1, N_2, denote the number of turns in the primary and secondary windings respectively. Hence, $e_1/e_2 = N_1/N_2$, i.e. the ratio of the induced E.M.Fs. is equal to the ratio of the numbers of turns in the respective windings. The ratio N_1/N_2 is called the "turn ratio" of the transformer.

Now in the ideal case the induced E.M.Fs. are equal to the terminal voltages v_1, v_2, so that we have $v_1/v_2 = N_1/N_2$. But in an actual transformer,

due to losses and magnetic leakage, the terminal voltages will not be equal to the induced E.M.Fs., and the ratio of these voltages will vary with the currents in the windings. At no load, however, when the secondary current is zero, the ratio of terminal voltages will be approximately equal to the ratio of the numbers of turns in the windings.

If the E.M.F. impressed upon the primary winding is constant, the E.M.F. induced in this winding must also be constant—since, when there are no losses or magnetic leakage, the induced E.M.F. must balance the impressed E.M.F.—and, therefore, if the frequency is constant, the flux must remain constant. To maintain this flux in the magnetic circuit requires a definite number of ampere-turns, which, with constant impressed E.M.F. and frequency, must remain constant for all loads.

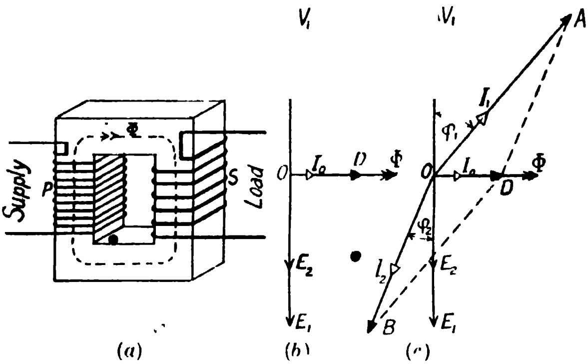


FIG. 25.—Magnetic Circuit and Vector Diagrams of Ideal Transformer

At no load (i.e. with zero current in the secondary winding) the current in the primary winding is just sufficient to supply the requisite magnetizing ampere-turns, and, if the magnetic circuit is unsaturated and free from hysteresis, this current will be in phase with the flux, and will lag 90° with respect to the impressed E.M.F. These conditions are represented in the vector diagram of Fig. 25*b*, in which the vector $O\Phi$ represents the flux, OD the magnetizing ampere-turns, OI_0 the primary current, OE_1, OE_2 the E.M.Fs. induced in primary and secondary windings, respectively, and OV_1 the impressed E.M.F.

When the secondary winding is loaded, the ampere-turns produced by the current in this winding act in opposition to the primary ampere-turns, and if the magnetizing ampere-turns are to remain constant, the primary ampere-turns must change, in both magnitude and phase, with respect to the magnetizing, or no-load, ampere-turns. Thus the magnetizing ampere-turns must be the resultant of the ampere-turns due to the currents in the primary and secondary windings, i.e. the primary ampere-turns must be equal to the vector difference of the magnetizing ampere-turns and the secondary ampere-turns.

These conditions are represented in the vector diagram of Fig. 25*c*, which contains the same vectors as the previous diagram (Fig. 25*b*), together with the vectors OA, OB, which represent the ampere-turns due to the currents, I_1, I_2, in the primary and secondary windings, respectively. Observe that OD is the diagonal of the parallelogram $OADB$, the sides (OA, OB) of which represent the ampere-turns due to the currents in the primary and secondary windings. Observe also that the phase difference, φ_1, between the impressed

E.M.F. and the primary current depends principally upon the corresponding phase difference, φ_2, between the terminal voltage and current of the secondary circuit. Observe, again, that if the magnetizing ampere-turns, *OD*, were zero, the phase difference between the primary and secondary ampere-turns (and also that between the primary and secondary currents) would be 180°. Under these conditions we should have $I_1N_1 = I_2N_2$, or $I_1/I_2 = N_2/N_1$, i.e. the ratio of currents would equal the inverse ratio of the E.M.Fs.

Elementary theory of practical transformer. In an actual transformer we have losses in the windings and magnetic core, magnetic leakage between the windings, and saturation in the core.

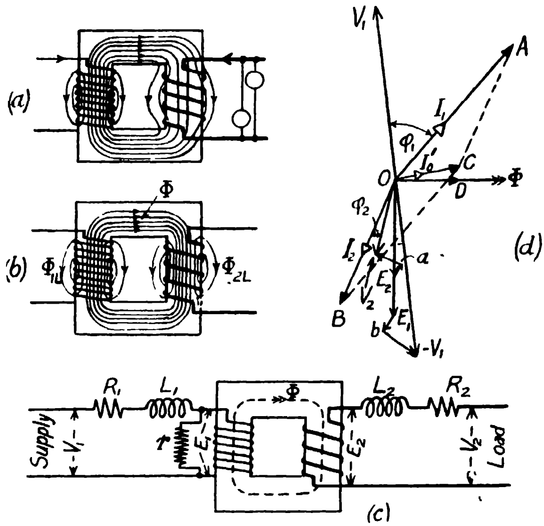


FIG. 26. Magnetic Leakage in Transformer

The effect of magnetic leakage is shown diagrammatically in Fig. 26*a* from which it is evident that the flux which links with the secondary winding is smaller than that which links with the primary winding. Moreover, since the leakage increases as the magneto-motive force increases, the flux which links with the secondary winding will decrease as the secondary current increases, assuming the E.M.F. and frequency applied to the primary winding to remain constant.

The theory of the transformer with magnetic leakage may be developed by considering fictitious leakage fluxes to be superimposed upon the ideal flux Φ which links both windings and remains constant. These conditions are represented in Fig. 26*b*.

The resultant internal E.M.F. in the primary winding is then equal to the vector sum of (1) the E.M.F. induced by the flux, Φ; (2) the E.M.F. induced by the leakage flux, Φ_{1L}; (3) the E.M.F. due to the resistance of the winding.

Similarly, the terminal E.M.F. of the secondary winding is compounded from (1) the E.M.F. induced by the flux, Φ; (2) the E.M.F. induced by the leakage flux, Φ_{2L}; (3) the E.M.F. due to the resistance of the winding.

Since a large portion of the paths of the leakage fluxes is through air and non-magnetic materials, these fluxes may be considered to be proportional to, and in phase with, the currents producing them, and therefore the E.M.Fs.

induced by the leakage fluxes will be proportional to, but will lag 90° with respect to, the currents in the primary and secondary windings. Therefore the conditions, so far as magnetic leakage and resistances of windings are concerned, are equivalent to those which would be obtained if inductive resistances L_1R_1, L_2R_2, were connected in series with the primary and secondary circuits, respectively, of the ideal transformer (Fig. 26*c*).

The effect of losses and magnetic saturation in the iron core is considered in detail in Chapter IX. For the present they may be taken into account by considering the no-load current to lead the flux. This is equivalent to a resistance r (Fig. 26*c*), being connected in parallel with the primary winding. The no-load ampere-turns are then represented by OC' (Fig. 26*d*), which must remain constant at all loads if the flux is constant.*

The modifications to the vector diagram of Fig. 25*c* to take into account magnetic leakage and losses are shown in Fig. 26*d*, in which OV_2 represents the secondary terminal voltage (which is compounded from the induced E.M.F., OE_2, the E.M.F. ($-R_2I_2$) due to resistance, E_2a, and the inductive E.M.F. (ωL_2I_2) aV_2), OV_1 represents the primary terminal voltage and $O-V_1$ represents the resultant internal E.M.F. in the primary winding, O V_1 being compounded from the induced E.M.F., OE_1, the E.M.F. (R_1I_1) due to resistance, E_1b, and the inductive E.M.F. (ωL_1I_1), b V_1.

The effect of magnetic leakage upon the performance of a transformer is to cause a decrease in the secondary terminal voltage with an increase of load (assuming constant primary voltage and frequency), and also an increase in phase difference between the secondary terminal voltage (OV_2) and the resultant internal E.M.F. of the primary winding ($O-V_1$). These effects may be minimized in practice by so arranging the windings that the magnetic leakage is small, e.g. with the type of core shown in Fig. 26, by placing primary and secondary windings on both limbs instead of only on separate limbs as shown and arranging these windings in the form of concentric cylinders.

* Actually, however, due to the resistance of the primary winding, the flux decreases slightly as the load increases, but in practice the decrease is so small that very little error is made in assuming the flux to remain constant.

CHAPTER IV

CAPACITY AND CONDENSERS

Electrostatic potential. The potential of a conductor is defined as the work done by, or against, electric forces in carrying unit positive charge from the conductor to the boundary of the electric field, which is considered to be at zero potential.

For example, consider an isolated spherical conductor, of radius r cm., surrounded by air and charged with Q positive units. Assuming this charge to be distributed uniformly over the surface, the force on unit positive charge at distance x from the centre of the sphere is $(Q \times 1)/x^2$. Hence the work done in bringing this unit charge from the boundary of the electric field (i.e. infinity) to the surface of the sphere is

$$V = \int_{\infty}^{r} \frac{Q}{x^2}\,dx = \left[\frac{Q}{x}\right]_{r}^{\infty} = \frac{Q}{r} \text{ ergs}$$

which, by definition, is the potential of the sphere.

Energy stored in an electric field. An electrically charged conductor is the seat of an electric (electrostatic) field, the lines of electric force being normal to the surface of the conductor. The work done in charging the conductor to a potential V is given by

$$W = \tfrac{1}{2}QV \text{ ergs,*}$$

where Q and V denote the charge and potential, respectively, in electrostatic units. This energy is stored in the electric field and is released when the conductor is discharged.

Capacity. The ratio of the charge (Q) on a conductor to its potential (V), when all other conductors in the electric field are at zero potential, is called its *capacity* (C), i.e. $Q/V = C$. With this relationship between Q and V we may express the energy (W)

* From the definition of potential it follows that the work done in carrying a charge dq from the boundary of the electric field to a conductor charged to potential v is $v\,dq$. Now the ratio of charge to potential is constant; hence if dv is the increment of potential due to a charge dq, $dq/dv = C$, a constant; i.e. $dq = C\,dv$. Therefore the total work done in charging the conductor to a potential V is

$$W = \int_0^V v\,dq = C\int_0^V v\,dv = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV \text{ ergs,}$$

where Q is the charge corresponding to a potential V.

stored in the electric field in terms of capacity and potential, thus

$$W = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 \text{ ergs} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

whence $$C = 2W/V^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

i.e. the capacity of a conductor is equal to twice the energy stored in the electric field when its potential is unity. Capacity may, therefore, be regarded as a property of a conductor, or a system of conductors, in virtue of which electrical energy can be stored in the surrounding electric field.

Unit of capacity. In equation (21) the capacity will be given in electrostatic units—i.e. centimetres—when W is expressed in ergs and V in electrostatic units of potential. To obtain C in practical units—farads—W and V must be expressed in practical units—i.e. joules or watt-seconds, and volts respectively. Now all practical electrical units are derived from the corresponding absolute electromagnetic units: hence, in deriving the practical unit of capacity from equation (21), we require the ratio between the electrostatic and electromagnetic units of energy and potential, as well as the ratio between the practical and electromagnetic units of these quantities. As both electrostatic and electromagnetic systems of units are derived from the centimetre-gramme-second fundamental units, the unit of work is the same (viz. the dyne-centimetre or erg) for both systems. But potential, or potential difference, is defined, in the electrostatic system, in terms of work and electric charge—i.e. potential = work/charge—and in the electromagnetic system, in terms of the rate at which work is expended and current—i.e. potential difference = work/(time × current).

According to these definitions the dimensions of potential in terms of length (L), mass (M), and time (T), and the electrostatic and electromagnetic constants κ, μ, respectively, are

$$\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}\kappa^{-\frac{1}{2}} \text{ in the electrostatic system,}$$

and $$\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-2}\mu^{\frac{1}{2}} \text{ in the electromagnetic system.}$$

Whence the ratio of the dimensions is

$$\frac{\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}\kappa^{-\frac{1}{2}}}{\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-2}\mu^{\frac{1}{2}}} = \frac{\mathrm{T}}{\mathrm{L}} \cdot \frac{1}{\sqrt{(\mu\kappa)}}$$

which, since μ and κ are non-dimensional quantities, is the reciprocal of a velocity. The value of this velocity, deduced from measurements of the same quantity (e.g. current or E.M.F.) in electrostatic and electromagnetic units, is 3×10^{10} cm. per second.

Hence, as the ratio between magnitudes of the same kind but in different units, is in the inverse ratio of their dimensions, we have

1 electrostatic unit of potential $= 3 \times 10^{10}$ electromagnetic units of potential.

Now 1 volt $= 10^8$ electromagnetic units of potential,
$= 10^8/(3 \times 10^{10}) = 1/300$ of electrostatic unit of potential,

and 1 joule $= 10^7$ ergs.

Therefore the quantities W, V, in equation (21) must be divided by 10^7 and $(1/300)$ respectively, to obtain the capacity in practical units.

$$\text{Hence,} \quad C\text{ (farads)} = \frac{2W/10^7}{V^2/(1/300)^2} = \frac{2W}{V^2 \times 9 \times 10^{11}}$$

Whence, 1 farad $= 9 \times 10^{11}$ electrostatic units of capacity.

The farad, however, is too large a unit for commercial purposes, and therefore the commercial unit is chosen equal to one-millionth of a farad —i.e. a microfarad— the symbol for which is μF., μ here being the prefix denoting one millionth.

Hence, $1\mu\text{F.} = 10^{-6}\text{F.}$
$= 9 \times 10^{11} \times 10^{-6} = 9 \times 10^5$ electrostatic units.

Capacity of isolated spherical conductor. Consider an insulated spherical conductor, of radius r cm., surrounded by air and isolated from other conductors. Assume the sphere to be originally uncharged, and let a charge Q be given to it. Then, provided that the charge distributes itself uniformly over the surface, the potential becomes $V = Q/r$. Whence charge/potential $= Q/(Q/r) = r$: i.e. the capacity of an isolated sphere is, in electrostatic units, equal to its radius in cm.

Hence to obtain a capacity of 1μ F., the radius of the sphere must be 9×10^5 cm.

Condenser. The capacity of a conductor depends on its size and geometrical form; its position relative to other bodies in the electric field; and the specific inductive capacity (see p. 64) of the surrounding insulating medium. With conductors of the forms and dimensions commonly used in practice the capacity of a single isolated conductor is extremely small. But the capacity can be increased by bringing the boundary of the field nearer to the conductor, e.g. by placing a second (earthed) conductor near to the charged one. For example, if an insulated conducting sphere of

radius r cm. is surrounded by an earthed concentric conducting shell of radius r_1 cm., the potential of the sphere due to a charge Q becomes $\int_r^{r_1} \frac{Q}{x} dx = Q\left(\frac{1}{r} - \frac{1}{r_1}\right)$, and its capacity is

$$C = Q/V = rr_1/(r_1 - r) = r\{1 + (r/\delta)\}$$
$$(r + \delta)r/\delta$$
$$= r(r/\delta), \text{ approximately,}$$

where $\delta = r_1 - r$. Hence when δ is small in comparison with r the capacity will be very much greater than that of the isolated sphere.

Under these conditions it is possible to obtain large charges on the conductors with only a moderate potential difference between them. Such a system of two conductors, insulated from each other and having large surfaces a small distance apart, is called a *condenser*.

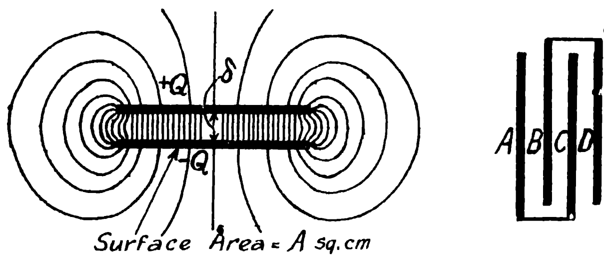
In all practical forms of condensers the conductors (called the *plates* of the condenser) consist of sheets of metal foil separated by a thin insulating medium called the *dielectric*. Alternate plates are electrically connected together, so that, by employing a large number of plates, it is possible to obtain the equivalent of a large surface area although the area of the individual plates may be relatively small. The plates are so close together that they always receive equal and opposite charges, and the latter are unaffected by the presence of neighbouring charged or uncharged conductors. In such cases the numerical value of the charge on either plate, when the potential difference between them is unity, is called the *capacity of the condenser*.

Practical uses of condensers. Condensers, in virtue of their property of storing electrical energy, have a number of practical uses: for example, the stored energy, when discharged under suitable conditions, may be utilized to set up electrical oscillations, the energy being radiated in the form of electric waves—as in radio-telegraphy and telephony—or again, the stored energy may be used to alter, or modify, the characteristics of electric circuits, as discussed in Chapter VI. Condensers are used extensively in telephony: they also form an essential part of magneto-ignition generators for internal-combustion engines. A further use is for the protection of electric circuits, supplied by overhead conductors, against high-voltage, high-frequency, surges.

Calculation of capacity of condensers. *General.* In calculating the capacity of a condenser, or a system of conductors, we first obtain an expression for the potential of the conductors, or plates,

of the condenser due to a given charge. The potential is usually calculated from the summation of the work done on a unit charge when moved from one conductor to the other, and this requires a knowledge of the magnitude and direction of the electric field at all points in the space between the conductors. With the potential known, the capacity follows directly from the ratio of potential to charge.

Capacity of parallel plate condenser. Let A denote the area (in square cm.) of each plate and δ the distance (in cm.) between the plates. Then if δ is small in comparison with A, the electric field



(a) (b)

FIG. 27. Pertaining to Theory of Parallel Plate Condenser

between the plates due to charges $+Q$, $-Q$, will be uniform and normal to their surfaces. Near the edges of the plates, however, the field is not uniform owing to the "fringing" of the lines of force (see Fig. 27*a*). Neglecting fringing effects, the electric force at any point in the air space between the plates is equal to $4\pi\sigma$ dynes, where σ $(= Q/A)$ is the surface density of the charges on the plates. Hence the work done in carrying unit charge from one plate to the other is equal to $4\pi\sigma\delta = 4\pi\delta Q/A$ ergs, which represents the potential difference between the plates. Therefore the capacity is given by

$$C = Q/(4\pi\delta Q/A) = A/4\pi\delta \text{ electrostatic units} \quad . \quad . \quad (22)$$

$$= A/(4\pi\delta\times 9\times 10^5) = A/(\delta\times 113\times 10^5) \text{ microfarads} \quad . \quad (22a)$$

NOTE.—If the dielectric is not air but a material having a dielectric constant equal to κ, the capacity will be κ times that given by the above equations.

Example. The plates of a parallel-plate condenser are each 30 cm. × 25 cm., and the dielectric is a sheet of paraffined paper 0·015 cm. thick, for which the dielectric constant is 3. Assuming the plates to be in intimate contact with the dielectric, the capacity of the condenser is

$$C = \frac{\kappa A}{4\pi\delta \times 9 \times 10^5} = \frac{30 \times 25 \times 3}{4\pi \times 0{\cdot}015 \times 9 \times 10^5} = 0{\cdot}013\mu\text{F}.$$

Multiple-plate condenser. If two additional plates, and sheets of dielectric, are added, and alternate plates are connected together, as shown in Fig. 27*b*, the effective area of each "plate" of the condenser will now be $3 \times 30 \times 25 = 2250$ cm.^{2}, as both sides of the intermediate plates *B*, *C*, Fig. 29*b*, are effective, but only one side of the end plates *A*, *D*, is effective. Hence the capacity will now be three times that due to a single pair of plates. Similarly, if two more plates and dielectric are added, the capacity will be five times that due to a single pair of plates. In general, if n is the number of pairs of similar plates in a multiple-plate condenser, and C is the capacity due to a single pair of plates, the capacity of the condenser $= (2n - 1)\,C$.

Hence to obtain a capacity of 1μF. with plates 30 cm. $\times$ 25 cm and a dielectric of paraffined paper 0·015 cm thick, the number of pairs of plates is

$$n = \tfrac{1}{2}\left(\frac{1}{0{\cdot}013} + 1\right) = \tfrac{1}{2}(75 + 1) = 38$$

Capacity of cylindrical condenser, or concentric cylinders. Assume the cylinders to be of indefinite length and let r = radius, in cm., of surface of inner cylinder and r_1 = radius, in cm., of the internal surface of the coaxial surrounding cylinder (see Fig. 28*a*). Further, let the charge per cm. length of the inner cylinder be $+Q$, and let the outer cylinder be earthed. Then the electric force in the air space between the cylinders is normal to their surfaces and acts radially outwards from the inner cylinder. In the case of long cylinders and an air dielectric the force at any point, P, in this space, distant x from the common axis, is equal to 4π × density of electric field at that point ; i.e. electric force $= 4\pi Q/2\pi x = 2Q/x$ dynes. Therefore, work done in moving unit positive charge from outer to inner cylinder

$$= \int_r^{r_1} \frac{2Q}{x}\,dx = 2Q\,\big[\log_e x\big]_r^{r_1} = 2Q \log_e \frac{r_1}{r} \text{ ergs,}$$

which is equal to the potential difference between the cylinders.

Whence the capacity per cm. length of the cylinders is

$$C = \frac{1}{2 \log_e (r_1/r)} \text{ electrostatic units} \quad . \quad . \quad . \quad . \quad (23)$$

$$= \frac{1}{2 \times 2{\cdot}3 \times 9 \times 10^5 \log_{10} (r_1/r)} \text{ microfarads}$$

$$= \frac{1}{41{\cdot}4 \times 10^5 \log_{10} (r_1/r)} \text{ microfarads} \quad . \quad . \quad . \quad . \quad (23a)$$

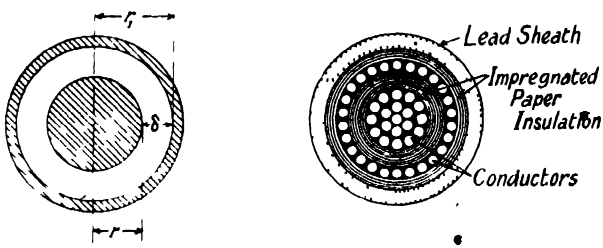
If $(r_1 - r) = \delta$, equation (23) becomes

$$C = \frac{1}{2 \log_\epsilon (1 + \delta/r)}$$

Now $\log_\epsilon (1 + \delta/r)$ may be represented by the series,

$$\log_\epsilon (1 + \delta/r) = \delta/r - \tfrac{1}{2}(\delta/r)^2 + \tfrac{1}{3}(\delta/r)^3 - \tfrac{1}{4}(\delta/r)^4 + \ldots$$

and when δ/r is small, the second and following terms may, for a first approximation, be neglected.



(a) (b)

FIG. 28.—(a) Pertaining to Theory of Concentric Cylindric Condenser. (b) Cross Section of Concentric Cable.

Hence when δ is small in comparison with r, the capacity of the concentric cylinders will be given by

$$C = \frac{1}{2\delta/r} = \frac{r}{2\delta} = \frac{2\pi r}{4\pi\delta} = \frac{A}{4\pi\delta}$$

where A is the surface of the inner cylinder per cm. of its length. Thus in this special case the capacity is approximately equal to that of a parallel-plate condenser of the same equivalent surface.

Applications of cylindric condensers and conductors. Condensers formed of concentric cylinders with air dielectric are occasionally employed as standards of capacity in electrical measurements, but this form of condenser is not used commercially.

Concentric conductors of the form shown in Fig. 28*b* have, however, a large practical application as distributing mains in single-phase alternating-current systems, as, with this arrangement of conductors, no inductive effects are produced by the alternations of the current; moreover, such cables cannot produce external magnetic fields.

Concentric cables, however, may possess an appreciable capacity, which may be calculated by means of equation (23*a*). Since in practice we usually require the capacity per 1000 yd., or per mile,

equation (23*a*) is modified so as to give the capacity for these lengths instead of unit (cm.) length. Thus

$$C = \frac{1000 \times 36 \times 2{\cdot}54 \times \kappa}{41{\cdot}4 \times 10^5 \log_{10}(r_1/r)} = \frac{0{\cdot}022 \times \kappa}{\log_{10}(r_1/r)}\ \mu\text{F. per 1000 yd.} \quad (23b)$$

$$= \frac{1760 \times 36 \times 2{\cdot}54 \times \kappa}{41{\cdot}4 \times 10^5 \log_{10}(r_1/r)} = \frac{0{\cdot}039 \times \kappa}{\log_{10}(r_1/r)}\ \mu\text{F. per mile} \quad . \ (23c)$$

Example. The inner conductor of a concentric cable, designed for a working pressure of 2200 volts, consists of 37 strands of 0·064 in. wire, the overall diameter being 0·45 in. The insulation consists of impregnated paper of a radial thickness of 0·12 in. The outer conductor consists of a singe layer of 29 wires, laid over the insulation.

[*Note*. In practice, the outer conductor of a concentric cable is earthed at the generating station.]

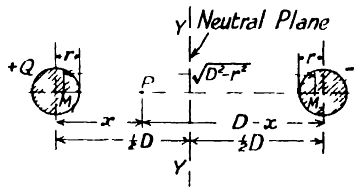


FIG. 29. Pertaining to Theory of Parallel Cylindric Condenser

The capacity of the cable is obtained by substituting in equations (23*b*), (23*c*). The ratio $(r_1/r) = (0{\cdot}225 + 0{\cdot}12)/0{\cdot}225 = 1{\cdot}533$ and $\log_{10} 1{\cdot}533 = 0{\cdot}1855$. Hence, assuming the dielectric constant of the impregnated paper to be 3·2, we have

$$C = (0{\cdot}022 \times 3{\cdot}2)/0{\cdot}1855 = 0{\cdot}38\ \mu\text{F. per 1000 yd.}$$

$$= (0{\cdot}0388 \times 3{\cdot}2)/0{\cdot}1855 = 0{\cdot}67\ \mu\text{F. per mile.}$$

Capacity of parallel cylinders. Consider two long, straight, and parallel conductors surrounded by air and removed from other conductors. Let r cm. be the radius of each conductor, D cm. the distance between the axes of the conductors, $+Q$, $-Q$ the charges per cm. length. Then, assuming r to be small in comparison with D, the charges may be considered to be concentrated at the axes of the cylinders. Hence the force acting on unit charge at a point P, distant x cm. from the axis of one cylinder, Fig. 29, is

$$F = \frac{4\pi Q}{2\pi x} + \frac{4\pi Q}{2\pi (D-x)} = \frac{2Q}{x} + \frac{2Q}{D-x}\ \text{dynes.}$$

This force is a minimum in the neutral plane YY, which bisects perpendicularly, the plane containing the axes of the conductors.

Therefore the work done in moving unit charge from the neutral plane to the surface of one conductor is equal to

$$\int_r^{\frac{1}{2}D} \left(\frac{2Q}{x}dx + \frac{2Q}{D-x}dx\right) = 2Q\Big[\log_\varepsilon x - \log_\varepsilon (D-x)\Big]_r^{\frac{1}{2}D}$$

$$= 2Q \log_\varepsilon \frac{D-r}{r}\ \text{ergs;}$$

and the total work done when unit charge is moved from one conductor to the other is

$$2\int_r^{\frac{1}{2}D}\left(\frac{2Q}{x}dx + \frac{2Q}{D-x}dx\right) = 4Q\log_\epsilon\frac{D-r}{r} = 4Q\log_\epsilon\left(\frac{D}{r}-1\right)\text{ ergs,}$$

which is equal to the potential difference between the conductors.
Hence the capacity per cm. length of the conductors is

$$C = \frac{1}{4\log_\epsilon\{(D/r)-1\}}\text{ electrostatic units} \quad . \quad . \quad . \quad (24)$$

$$= \frac{1}{4 \times 2{\cdot}3 \times 9 \times 10^5\log_{10}\{(D/r)-1\}}\text{ microfarads}$$

$$= \frac{1}{82{\cdot}8 \times 10^5\log_{10}\{(D/r)-1\}}\text{ microfarads} \quad . \quad (24a)$$

or approximately,

$$C = \frac{1}{82{\cdot}8 \times 10^5\log_{10}(D/r)}\text{ microfarads} \quad . \quad . \quad . \quad . \quad (24a_1)$$

The capacity per 1000 yd. of the (double) conductors is given by

$$C = \frac{0{\cdot}011}{\log_{10}\{(D/r)-1\}}\text{ microfarads} \quad . \quad . \quad . \quad . \quad . \quad (24b)$$

and the capacity per mile is

$$C = \frac{0{\cdot}0195}{\log_{10}\{(D/r)-1\}}\text{ microfarads} \quad . \quad . \quad . \quad . \quad (24c)$$

If the distance apart of the conductors is not large in comparison with their radii, the charges cannot be assumed to be concentrated at the axes of the conductors. The charges may, however, be considered to be concentrated along axes M_1, M_2, Fig. 29, which are contained by the plane containing the axes of the conductors and are a distance $\sqrt{(D^2-4r^2)}$ apart. In this case the capacity per cm. length is given by

$$C = \frac{1}{82{\cdot}8 \times 10^5\log_{10}\left\{\frac{D+\sqrt{(D^2-4r^2)}}{2r}\right\}}\text{ microfarads} \quad . \quad (24d)$$

which, when r is small in comparison with D, reduces to

$$C = \frac{1}{82{\cdot}8 \times 10^5\log_{10}(D/r)}$$

When the conductors are not isolated from other conductors and the earth, the effect of charges, if any, on the conductors, and the new boundaries of the electric field due to the earthed conductors, must be taken into account in calculating the capacity. For the simplest case, when the above conductors are suspended at a distance h above the earth's surface, or an earthed plane, the capacity per cm. length of the system is given by

$$C = \frac{1}{82{\cdot}8 \times 10^5\left[\log_{10}\left\{\dfrac{D + \sqrt{(D^2 - 4r^2)}}{2r}\right\} - \log_{10}\sqrt{\left\{1 + \left(\dfrac{D}{2h}\right)^2\right\}}\right]}$$

microfarads . (24*e*)

Examples. (*a*) Two conductors, each 0·5 in. in diameter, are stretched horizontally in space with their axes parallel and 2 ft. apart.

The capacity per 1000 yd. of the system is given with sufficient accuracy by

$$C = \frac{0{\cdot}011}{\log_{10}(D/r)} \text{ microfarads}$$

$$= \frac{0{\cdot}011}{\log_{10}(24/0{\cdot}25)} = \frac{0{\cdot}011}{1{\cdot}9823} = 0{\cdot}00[illegible]6\ \mu\text{F}.$$

(*b*) Two conductors, each 1 in. in diameter, are supported horizontally in space with their axes parallel and 4 in. apart.

The capacity per 1000 yd. of the system is now given by

$$C = \frac{0{\cdot}011}{\log_{10}[\{D + \sqrt{(D^2 - r^2)}\}/2r]} \text{ microfarads}$$

$$= \frac{0{\cdot}011}{\log_{10}[\{4 + \sqrt{(4^2 - 1^2)}\}/1]} = \frac{0{\cdot}011}{\log_{10} 7{\cdot}87} = 0{\cdot}01238\ \mu\text{F}.$$

(*c*) If the conductors in example (*b*) are each 6 in. above an earthed plane the capacity per 1000 yd. of the system becomes

$$\frac{0{\cdot}011}{\log_{10}\left[\left\{\dfrac{D + \sqrt{(D^2 - 4r^2)}}{2r}\right\}\right] - \log_{10}\sqrt{\left\{1 + \left(\dfrac{D}{2h}\right)^2\right\}}} \text{ microfarad}$$

$$= \frac{0{\cdot}011}{\log_{10}[\{4 + \sqrt{(4^2 - 1^2)}\}/1] - \log_{10}\sqrt{\{1 + (4/2 \times 6)^2\}}}$$

$$= \frac{0{\cdot}011}{\log_{10} 7{\cdot}87 - \log_{10} 1{\cdot}03} = 0{\cdot}01257\ \mu\text{F}.$$

Dielectric constant (specific inductive capacity). The effect of an insulating medium other than air between the plates of a condenser invariably leads to an increase in the capacity of the condenser. The ratio of the capacity of a given condenser, with a given substance as dielectric, to the capacity of the same condenser with air as dielectric, is called the *specific inductive capacity*, or the *dielectric constant*, of the substance, and is denoted by κ. With ordinary

gases κ differs little from unity, but with solids and liquids κ exceeds unity. Approximate values of κ for a number of dielectrics are

Air [0° C. 760 mm. Hg.	1·000	Paper, manilla (dry)	1·95
Glass, heavy flint	9·9	,, impregnated with resin oil	3·2
,, light flint	6·6	,, paraffin waxed	3·0
,, hard crown	6·9	Paraffin wax	2·36
Gutta-percha	2·8	Rubber, pure para	2·6
Mica	4·0	,, vulcanized	2·72
Oil, linseed	3·35	,, hard (ebonite)	3·15
,, petroleum	2·13	Sulphur	4·2
,, rape seed	2·85	Vacuum	0·99941
,, turpentine	2·23		

Notes. - The above values are based principally upon test results obtained by employing alternating electric forces, the frequency being approximately 1000 cycles per second and the temperature about 15 20° C. With certain substances, e.g. rubber, gutta-percha, the values of κ given above may differ appreciably from those obtained with steady electric force, but with other substances κ varies only slightly with frequency. In general κ is smaller for alternating than for steady electric forces.

Hygroscopic substances show large variations of κ according to the quantity of moisture present. With these substances the variation of κ due to frequency and temperature increases with increase of moisture.

With solid dielectrics the variation of κ with temperature is usually small. In general, an increase of temperature results in a decrease in κ, e.g. the dielectric constant of paraffin wax decreases 0·036 per cent per 1° C. over a range of 11–32° C., but those of ebonite and sulphur increase about 0·1 per cent per 1° C. rise of temperature over a range of 10–20° C.

When a conductor is surrounded by a dielectric other than air, the electric force at a given point in the dielectric, due to a given charge, is $1/\kappa$ of that at the same point when the conductor is surrounded by air, the charge and other conditions remaining unaltered. Hence the potential in the former case will be $1/\kappa$ of that in the latter case.

Commercial forms of condensers. When condensers of fixed capacity are required for commercial purposes the parallel-plate form is generally adopted. The manufacture of these condensers for telephone and lighting circuits is carried out by employing continuous sheets of foiled paper (i.e. paper coated with a very thin layer of metal powder and burnished to obtain continuity of the metal surface) for the " plates " and winding these, together with one or more sheets of interleaving paper, on a mandrel; the whole process being carried out by machinery. During winding, narrow strips of very thin copper foil are laid across the foiled surfaces at the centre of the winding, and also at other points when the condenser is required for lighting circuits. These strips project beyond the interleaving papers and are ultimately connected to the terminals of the condenser. When the requisite length of foiled

paper has been wound on the mandrel the roll is removed, dried in *vacuo*, impregnated with paraffin wax, pressed, and finally sealed hermetically in a metal case.* Such condensers of 2μF. capacity are used in large numbers for telephone circuits. Similar condensers of smaller and larger capacities are used in connection with electric lighting (see p. 88).

For medium-voltage power circuits (up to 600 V.) oil is used as the impregnating medium, and the foiled sheets are permanently immersed in oil.

Condensers for high-voltage power circuits require a dielectric which possesses a high specific disruptive strength; moreover, special precautions must be taken to avoid breakdown of the dielectric at the edges of the plates due to increased electric stress at these places. For the parallel-plate type of condenser, mica and glass are employed as dielectrics, but for the concentric cylindrical type, glass is employed. In the latter case the glass takes the form of a tube with one end closed and with thickened walls at the open end; the "plates" are formed by chemically depositing silver on the glass, the silver coating being electroplated in some cases with copper The edges of the coatings extend to the thickened walls of the glass.

Condensers, of fixed capacity, for high-frequency (radiotelegraphy and telephony) circuits are usually constructed with mica dielectric and tinfoil "plates," but in certain cases, e g for high-voltage transmitting apparatus, the cylindrical type with glass dielectric is employed.

Standard condensers for laboratory purposes have either mica or air dielectric, air being employed when a condenser without dielectric losses and dielectric absorption is required.

Variable-capacity condensers, with continuous adjustment of capacity between definite limits, are of the multiple-plate type, with either air or oil dielectric. One "plate" consists of a fixed stack of thin aluminium, or brass, vanes spaced uniformly a small distance apart and electrically connected together; the other "plate" consists of a similar set of vanes mounted on a spindle and so arranged that they can be rotated in the spaces between the fixed vanes so as to present to the latter a variable surface.

Charging current of a condenser. Assume the dielectric to be a perfect non-conductor and let the condenser receive a charge q during the time t. Then the mean rate of charge during this

* For further details of construction, see paper on "The manufacture of electrical condensers," by G. F. Mansbridge, *Journal I.E.E.*, 1908, lxi, 535. See also *Journal I.E.E.*, 1912, lxix, 704.

interval is q/t, and is called the *charging current*. To obtain the instantaneous value of this current t must be taken infinitely small, i.e. dt. Then if the corresponding charge which accumulates on the plates of the condenser is dq, the instantaneous charging current (i) is equal to dq/dt. Hence if C is the capacity of the condenser and e the potential difference between the plates when the charge is q,

$$dq = Cde,$$

whence $$i = Cde/dt$$

Thus *the charging current is directly proportional to the rate of change of the potential difference at the plates of the condenser.*

If the applied potential difference varies sinusoidally, and is given by $e = E_m \sin \omega t$, then the current will be given by

$$\begin{aligned} i &= C\frac{de}{dt} = C\frac{d}{dt}(E_m \sin \omega t) \\ &= \omega C E_m \cos \omega t \\ &= \omega C E_m \sin (\omega t + \tfrac{1}{2}\pi) \qquad . \quad . \quad . \quad (25) \end{aligned}$$

Thus the current varies sinusoidally with the same frequency as the applied E.M.F., but leads the latter by 90°.

The maximum value of the current is

$$I_m = \omega C E_m$$

and the R.M.S. value is

$$I = \omega C E = 2\pi f C E$$

Thus the charging current is directly proportional to the frequency and the impressed E.M.F. For example, the charging current of a 1μ F. condenser connected to a 100 V., 50-cycle, circuit is

$$I = 2\pi \times 50 \times 1 \times 10^{-6} \times 100 = 0{\cdot}0314 \text{ A}$$

For the same condenser connected to a 100 V., 25-cycle circuit the charging current is

$$I = 2\pi \times 25 \times 1 \times 10^{-6} \times 100 = 0{\cdot}0157 \text{ A}$$

and when it is connected to a 600 V., 50-cycle, circuit the charging current is

$$I = 2\pi \times 50 \times 1 \times 10^{-6} \times 600 = 0{\cdot}1884 \text{ A}$$

The ratio $E/I = 1/\omega C$ is called the *reactance* of the condenser, and is expressed in ohms. As the term reactance is employed in

connection with inductive circuits, it is necessary to distinguish between reactance due to inductance and that due to capacity. The former is usually called "inductive reactance," and the latter "capacitive reactance," or "capacitance."

The reactance of a condenser of given capacity is therefore inversely proportional to the frequency. Hence a condenser of relatively small capacity may have an extremely low reactance at exceptionally high frequencies, such as those of lightning discharges. This property of condensers is utilized in practice for protecting apparatus from high-voltage, high-frequency, surges; a condenser being connected across the circuit, or between the circuit and earth,

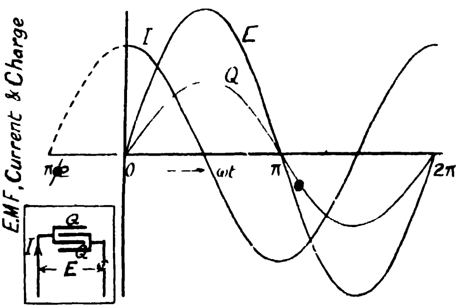


FIG. 30.—Graphic Representation of Current, Charge and Impressed E.M.F. for a Condenser

so that the high-frequency surge may discharge through the condenser. The dielectric of the condenser, however, must be capable of withstanding high voltage, and therefore the special form of construction mentioned on p. 65 is generally employed.

A *graphical representation* of the conditions expressed in equation (25) is shown in Fig. 30, in which the sine curve E represents the impressed E.M.F., and the sine curve I, 90° in advance of E, represents the charging current. The variation of the charge with respect to time is shown by the sine curve Q, which is the integral of the current curve, since $Q = \int i.dt = \int I_m \cos \omega t\, dt = (I_m/\omega) \sin \omega t = CE_m \sin \omega t$.

A condenser connected to an alternating supply is therefore charged and discharged periodically: it receives a charge during the first positive quarter-period of the E.M.F., discharges during the next quarter-period, and is again charged and discharged successively during the following half-period. During charge the current decreases from its positive maximum value to zero, and

during discharge the current rises from zero to its maximum value. The energy stored in the condenser during charging is given back to the circuit during discharge, so that there is a continual transference, or surge, of energy to and from the condenser. In practice, due to losses in the dielectric, the energy given back to the circuit during discharge is always slightly less than that stored in the condenser during charge. This, however, is not the case when the dielectric is dry air.

Relation between current and E.M.F. for circuits containing capacity and resistance in series. Let the current in the circuit be represented by $i = I_m \sin \omega t$. Also let e_1, e_2, denote, at any instant, t, the internal E.M.Fs. due to (1) the current passing through the resistance, and (2) the charging of the condenser. Then if R is the resistance of the circuit and C is the capacity of the condenser

$$e_1 = Ri = RI_m \sin \omega t$$

$$e_2 = (1/C)\int i\,dt = (I_m/C)\int \sin \omega t . dt = -(I_m/\omega C) \cos \omega t$$

The total internal E.M.F. acting in the circuit at this instant is therefore equal to $(e_1 + e_2)$, and must balance the impressed E.M.F. (e). Hence

$$e = (e_1 + e_2) = RI_m \sin \omega t - (I_m/\omega C) \cos \omega t$$

$$= I_m\sqrt{\{R^2 + (1/\omega C)^2\}}\left\{\frac{R}{\sqrt{\{R^2 + (1/\omega C^2)\}}}\sin \omega t - \frac{1/\omega C}{\sqrt{\{R^2 + (1/\omega C^2)\}}}\cos \omega t\right\}$$

$$= I_m\sqrt{\{R^2 + (1/\omega C)^2\}} \sin (\omega t - \varphi) = E_m \sin (\omega t - \varphi)\ ,$$

where $\tan \varphi = \dfrac{1/\omega C}{R} = \dfrac{1}{\omega CR}$

Thus the impressed E.M.F. is sinusoidal and lags behind the current by the angle φ.

Conversely, if a sinusoidal E.M.F., represented by the equation $e = E_m \sin \omega t$, be applied to the circuit, the current, when the steady. or cyclic, state is reached will be given by

$$i = \frac{E_m}{\sqrt{\{R^2 + (1/\omega C)^2\}}} \sin (\omega t + \varphi) \quad . \tag{26}$$

The maximum value of the current is

$$I_m = \frac{E_m}{\sqrt{\{R^2 + (1/\omega C)^2\}}}$$

and the R.M.S. value of the current is

$$I = \frac{E}{\sqrt{\{R^2 + (1/\omega C)^2\}}}$$

Hence the impedance of the circuit is given by

$$Z = \sqrt{\{R^2 + (1/\omega C)^2\}}$$

The *vector diagram* for this circuit is shown in Fig. 31, in which OE represents the impressed E.M.F. and OI the current. The vector OE may be resolved into two components: one, OA, in phase with the current vector, and the other, OB, perpendicular to the current vector. OA therefore represents the component of the impressed E.M.F. which is expended against the resistance of the circuit, and OB the potential difference at the terminals of the condenser. Now $OA = RI$; $OB = AE = I/\omega C$; $OE = I\sqrt{\{R^2 + (1/\omega C)^2\}}$. Hence $OA : AE : OE = R : 1/\omega C : \sqrt{\{R^2 + (1/\omega C)^2\}}$. Therefore triangle OEA is the impedance triangle for the circuit.

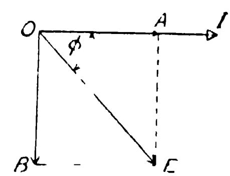


FIG. 31. Vector Diagram for Circuit Containing Resistance and Capacity in Series

Some practical applications and properties of this circuit are considered in Chapter VI.

Relation between current and E.M.F. for a series circuit containing resistance, inductance, and capacity. Let the current in the circuit be represented by the equation $i = I_m \sin \omega t$. Then if the resistance, inductance, and capacity are denoted by R, L, C respectively, the equation to the impressed E.M.F. is given by

$$e = RI_m \sin \omega t + \omega L I_m \cos \omega t - (I_m/\omega C) \cos \omega t$$
$$= I_m \{R \sin \omega t + (\omega L - 1/\omega C) \cos \omega t\}$$

which, when simplified by the method given on p. 46, becomes

$$e = I_m \sqrt{[R^2 + \{\omega L - (1/\omega C)\}^2]} \sin (\omega t + \varphi) = E_m \sin(\omega t + \varphi)$$

where $\tan \varphi = \{\omega L - (1/\omega C)\}/R$.

Conversely, if a sinusoidal E.M.F.—represented by the equation $e = E_m \sin \omega t$—be applied to the circuit, the current, when the steady, or cyclic, state is reached, will be given by

$$i = \frac{E_m}{\sqrt{\{R^2 + [\omega L - (1/\omega C)]^2\}}} \sin (\omega t - \varphi) \quad . \quad . \quad . \quad (27)$$

The maximum value of the current is

$$I_m = \frac{E_m}{\sqrt{\{R^2 + [\omega L - (1/\omega C)]^2\}}}$$

and the R.M.S. value of the current is

$$I = \frac{E}{\sqrt{\{R^2 + [\omega L - (1/\omega C)]^2\}}}$$

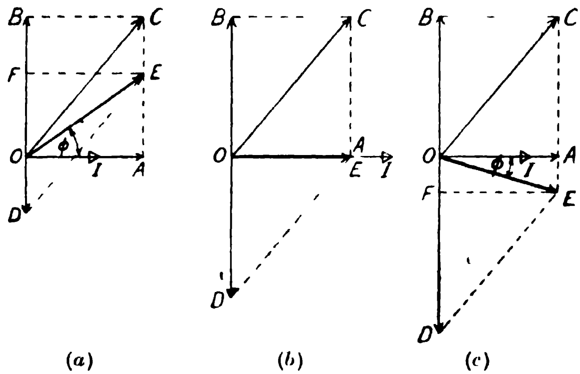


FIG. 32.—Vector Diagrams for a Series Circuit Containing Resistance, Inductance, and capacity ; (a) $\omega L > 1/\omega C$, (b) $\omega L = 1/\omega C$, (c) $\omega L < 1/\omega C$

Hence the impedance of the circuit is given by

$$Z = \sqrt{\{R^2 + [\omega L - (1/\omega C)]^2\}}$$

and the effective reactance by

$$X = \omega L - (1/\omega C)$$

$= \text{inductive reactance} - \text{capacitive reactance}.$

The phase difference, φ, between current and E.M.F. may be lagging, zero, or leading, according to whether $\omega L >=< 1/\omega C$. For example, when $\omega L > 1/\omega C$ the current is lagging with respect to the impressed E.M.F., but when $\omega L < 1/\omega C$ the current leads the impressed E.M.F.

The *vector diagrams* for these cases are shown in Fig. 32, diagram (*a*) referring to the case when $\omega L > 1/\omega C$ and φ is positive ; (*b*) referring to the case when $\omega L = 1/\omega C$ and φ is zero ; (*c*) referring to the case when $\omega L < 1/\omega C$ and φ is negative. In these diagrams the current vector, OI, is taken as the vector of reference, and the impressed E.M.F. is represented by OE. The component OA, which

is in phase with the current vector, represents the E.M.F. expended against the resistance of the circuit. The component OB, which leads the current vector by 90°, represents the E.M.F. which balances the E.M.F. of self-induction. The resultant, OC, of OA and OB represents the potential difference across the resistance and inductance; the angle AOC being the phase difference between this potential difference and the current. The potential difference between the terminals of the condenser is represented by OD, which lags 90° with respect to the current vector. Obviously, the vector sum of OC and OD must equal the impressed E.M.F. OE. The component, OF, which is perpendicular to the current vector is equal to the vector difference of OB and OD. Now $OA : OF (= AE) : OE = RI : I(\omega L - 1/\omega C) : I\sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}} = R : (\omega L - 1/\omega C) : \sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}}$. Therefore triangle OFE is the impedance triangle for the circuit. Observe that when $\omega L > 1/\omega C$ the effective reactance of the circuit is positive and the current is lagging, but when $\omega L < 1/\omega C$ the effective reactance is negative and the current is leading.

In the special case when $\omega L = 1/\omega C$, the current is in phase with the impressed E.M.F. and is equal to E/R. The circuit is therefore equivalent, so far as its impedance is concerned, to a non-inductive circuit of resistance R, and is said to be in a *condition of resonance*. Under these conditions the voltages across the condenser and inductance may each be much greater than the impressed E.M.F. (see examples in Chap VI).

CHAPTER V

POWER IN ALTERNATING-CURRENT CIRCUITS

Instantaneous power in an alternating-current circuit. In an alternating-current circuit the power (p) at any instant is equal to the product of the instantaneous values of current and E.M.F. Thus $p = ci$.

Since current and E.M.F vary with respect to time, the power will also vary from instant to instant, and may be positive, negative, or zero according to the signs and magnitudes of current and E.M.F. If, at a given instant, the current and E.M.F. have the same sign, the power is positive—indicating that power is being supplied *to* the circuit—but if these quantities have opposite signs the power is negative—indicating that power is being returned *from* the circuit to the generator—while if either, or both, of the quantities are zero, the power is zero.

Graphical representation. In the general case of sinusoidal current and E.M.F. differing in phase, the power has four zero values for each cycle of the current, or E.M.F., and the direction of the power reverses four times in each cycle, as shown graphically in Fig. 33, in which the curves E, I, represent the impressed E.M.F. and current respectively, and curve P represents the power. The shaded areas in this diagram represent energy; areas above the abscissa axis denote that energy is being supplied *to* the circuit, and those below the axis denote that energy is being returned *from* the circuit. For each cycle the difference between the areas above and below the abscissa axis represents the energy expended in the circuit, either in doing useful work or supplying losses.

Two *special cases* of the general case (Fig. 33) are of importance, viz. (1) when the phase difference between current and E.M.F. is zero, (2) when it is 90°. These cases are represented graphically in Figs. 34 and 35. In the case represented in Fig. 34—where the current and E.M.F. are in phase—the power pulsates between zero and a definite maximum value, but does not change sign throughout the cycle. Hence in this case power is transmitted always in one direction, viz. from generator to circuit.

In the cases represented in Fig. 35—in which E.M.F. and current have a phase difference of 90°, this being lagging in one case, Fig. 35*a*, and leading in the other, Fig. 35*b*—the power curve (P) alternates at twice the frequency of the current or E.M.F. Hence

for each half-cycle of current, or E.M.F., there are two alternations of power. Therefore during this interval a certain amount of energy is supplied to the circuit and an equal amount is returned to

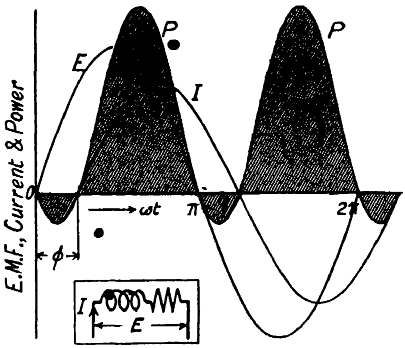


FIG. 33.—E.M.F., Current and Power in Circuit Containing Resistance and Inductance

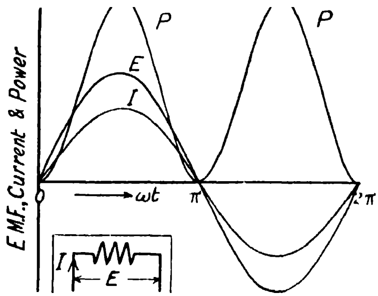
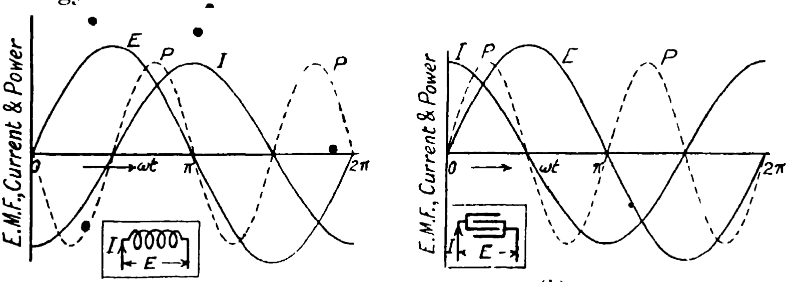


FIG. 34.—E.M.F., Current and Power in Non-inductive Circuit

the generator. Thus, although energy is continually surging between generator and circuit, no energy is actually expended in the latter.

During the time that the current and E.M.F. have the same sign energy is stored in the circuit—in either the electrostatic or the



(a) (b)

FIG. 35.—E.M.F., Current and Power in Purely Reactive Circuits

electromagnetic form, according to whether the circuit is purely capacitive or inductive—and this energy is returned to the generator during the time that the current and E.M.F. have opposite signs.

An extension of the special case, in which the phase difference between current and E.M.F. is 90°, occurs in a series circuit possessing pure inductance and capacity in which the resistance and losses

are both zero. In such a circuit, although E.M.F. and current have a phase difference of 90°, energy may exist simultaneously in both electrostatic and electromagnetic forms. For example, when the magnetic field, associated with the inductive part of the circuit, is being established and energy is being stored electromagnetically, the condenser is discharging Conversely, when the condenser is

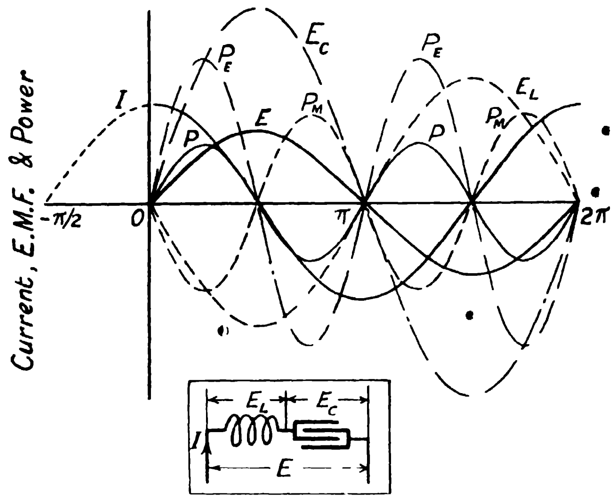


FIG. 36 E M F, Current and Power in a Reactive Circuit Containing Inductance and Capacity

being charged the magnetic field is decreasing and releasing its stored energy. Hence there is a continual transference of energy between the inductive and capacitive portions of the circuit.

If the energy required to charge the condenser exceeds that released from the magnetic field the difference must be obtained from the supply system. But if the energy released from the magnetic field exceeds that necessary to charge the condenser, the difference is returned to the supply system.

Thus at any instant the power supplied to, or returned from, the circuit is equal to the difference between the rate at which energy is being stored, at that instant, in one part of the circuit and the rate at which it is being released from the other part.

Components of power curve. A graphical representation of the variation of power in this reactive circuit is shown in Fig. 36, in which the curves *E*, *I*, represent the impressed E.M.F. and current respectively, and the curve *P* represents the power surging

between generator and circuit. The curve P may be obtained in the same manner as in the previous diagrams (Figs. 33, 34, 35) by plotting as ordinates the products of the instantaneous values of current and E.M.F., but in the present case it is interesting to obtain the curve from the component power curves for the two parts of the circuit. To obtain the latter we determine the curves E_L, E_C, for the potential differences across the inductance and condenser respectively. The product of instantaneous values of these curves and the current curve gives the power curves P_M, P_E, respectively, and the difference between P_M, P_E, gives the power curve P.

In the *special case* when $E_L = E_C$ (i.e. when $\omega L = 1/\omega C$) the ordinates of curves P_M, P_E, are, at any particular instant, equal and therefore P is zero. Under these conditions, which cannot, however, be realized in practice, no power is supplied to, or returned from the circuit at any instant, although energy may be surging between the inductive and capacitive portions of the circuit. Hence electrical oscillations, when once started, will continue indefinitely provided that the impressed E.M.F. is maintained across the circuit. This phenomena is called electrical resonance, and the frequency of the oscillations, which is equal to that of the generator, is called the "natural frequency" of the circuit

In the *general case* of a circuit containing resistance, inductance and capacity, the power curve is of the form shown in Fig. 33. This curve may also be resolved into components representing the power in the several parts of the circuit. To separate out these components it is necessary to determine the curves for the potential differences across the several parts of the circuit. Now the potential difference across the resistance is in phase with the current, that across the inductance leads the current by 90°, and that across the condenser lags 90° with respect to the current. The curves for these quantities, together with those for the impressed E.M.F. and current, are shown in Fig. 37; curves E, I, denoting the impressed E.M.F. and current, respectively, and curves E_R, E_L, E_C, denoting the components of the impressed E.M.F. which are expended against resistance, inductance, and capacity respectively. The instantaneous power in the several parts of the circuit is given by the product of the current and the appropriate E.M.F. curve. The power component curves so obtained are marked P_R, P_M, P_E. P_R therefore represents the power expended in supplying the losses in the resistance; P_M, P_E, represent the power supplied to the inductive and capacitive portions, respectively, of the circuit. The difference between the curves P_M, P_E, represents the component of the power which surges between the generator and the circuit

Analytical expressions for instantaneous power and power components. Let the equation to the impressed E.M.F. be $e = E_m \sin \omega t$, and that to the current be $i = I_m \sin (\omega t - \varphi)$. Then the instantaneous power is given by

$$p = ei = E_m I_m \sin \omega t \,.\, \sin (\omega t - \varphi) \quad . \quad . \quad . \qquad (28)$$

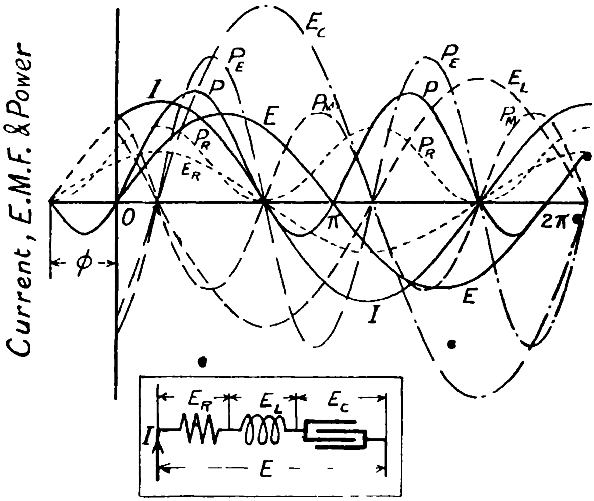


FIG. 37.—E.M.F., Current and Power in Circuit Containing Resistance, Inductance, and Capacity.

The equations to the components of the impressed E.M.F. for the several portions of the circuit are

$$e_R = Ri = RI_m \sin (\omega t - \varphi)$$

$$e_L = L\, di/dt = \omega L I_m \cos (\omega t - \varphi)$$

$$e_C = (1/C) \int i.dt = -(I_m/\omega C) \cos (\omega t - \varphi)$$

The components of the power for the several parts of the circuit are therefore,

$$p_R = ie_R = RI_m^2 \sin^2(\omega t - \varphi) = \tfrac{1}{2} RI_m^2 \left\{ 1 - \cos 2 (\omega t - \varphi) \right.$$

$$= \tfrac{1}{2} RI_m^2 \left\{ 1 + \sin[2(\omega t - \varphi) - \tfrac{1}{2}\pi] \right\}$$

$$p_M = ie_L = \omega L I_m^2 \sin (\omega t - \varphi) \cos (\omega t - \varphi)$$

$$= \tfrac{1}{2} \omega L I_m^2 \sin 2(\omega t - \varphi)$$

$$p_E = ie_C = -\tfrac{1}{2} (I_m^2/\omega C) \sin (\omega t - \varphi) \cos (\omega t - \varphi)$$

$$= -\tfrac{1}{2} (I_m^2/\omega C) \sin 2(\omega t - \varphi).$$

The first expression represents a pulsating quantity; the second and third expressions represent alternating quantities of a frequency twice that of the current. The resultant of the two alternating quantities p_M, p_E, is, except in the special case when $p_M = p_E$, another alternating quantity, viz.—

$$\begin{aligned} p_X &= p_M + p_E \\ &= \tfrac{1}{2}I_m^2[\omega L - (1/\omega C)]\sin 2(\omega t - \varphi). \end{aligned}$$

Hence in the general case (Fig. 33) the instantaneous power consists of a pulsating component equal to

$$p_R = RI_m^2 \sin^2(\omega t - \varphi) = \tfrac{1}{2}RI_m^2\{1 + \sin[2(\omega t - \varphi) - \tfrac{1}{2}\pi]\} \quad (29)$$

and a double-frequency alternating component equal to

$$p_X = \tfrac{1}{2}I_m^2\{\omega L - (1/\omega C)\}\sin 2(\omega t - \varphi) \quad . \quad . \quad . \quad (30)$$

The pulsating component is in phase with the current, i.e. its zero and maximum values occur at the same instants as the corresponding values of the current. The alternating component has a phase difference of 90° with respect to the current, being lagging when $\omega L > 1/\omega C$, and leading when $\omega L < 1/\omega C$.

Equation to power curve. This equation may be obtained from equations (29) and (30), since $p = p_R + p_X$, but it is best obtained by expanding equation (28). Thus

$$\begin{aligned} p &= E_m I_m \sin \omega t \,.\, \sin(\omega t - \varphi) \\ &= E_m I_m \times \tfrac{1}{2}\{\cos[\omega t - (\omega t - \varphi)] - \cos[\omega t + (\omega t - \varphi)]\} \\ &= \tfrac{1}{2}E_m I_m \{\cos\varphi - \cos(2\omega t - \varphi)\} \\ &= \tfrac{1}{2}E_m I_m \cos\varphi + \tfrac{1}{2}E_m I_m \sin[2\omega t - (\tfrac{1}{2}\pi + \varphi)] \quad . \quad (31) \end{aligned}$$

The first term represents a constant quantity of value $\frac{1}{2}E_m I_m \cos\varphi$; the second term represents a sinusoidal quantity of maximum value $\frac{1}{2}E_m I_m$, and of twice the frequency of the current or E.M.F., lagging $(\frac{1}{2}\pi + \varphi)$ with respect to the impressed E.M.F.

Hence the power curve (P, Fig. 33) may be represented by a double-frequency sine curve—of maximum value $\frac{1}{2}E_m I_m$ and lagging $(\frac{1}{2}\pi + \varphi)$ with respect to the impressed E.M.F.—superimposed upon a horizontal axis at a distance $\frac{1}{2}E_m I_m \cos\varphi$ above the abscissa axis, as shown in Fig. 38. When $\varphi = 0$, the axis of the double-frequency curve is at a distance $\frac{1}{2}E_m I_m$ above the abscissa axis, and since the maximum value of this curve is $\frac{1}{2}E_m I_m$, it does not, therefore, cross the abscissa axis. Hence we have the conditions shown in Fig. 34. When $\varphi = \frac{1}{2}\pi$, $\cos\varphi = 0$, and the axis of the double-frequency curve coincides with the abscissa axis. We then have the conditions shown in Fig. 35.

Mean, or true, power. The mean, or true, power in an alternating current circuit is defined as the arithmetic mean of the instantaneous values of the power taken over a period, i.e. $P = \int p \, . \, dt$. In the general case, when the instantaneous power is given by

$$p = \tfrac{1}{2}E_m I_m \cos \varphi + \tfrac{1}{2}E_m I_m \sin [2\omega t - (\tfrac{1}{2}\pi + \varphi)],$$

the mean power is

$$P = \tfrac{1}{2}E_m I_m \cos \varphi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

since the mean value of the double-frequency term, taken over a period is zero.

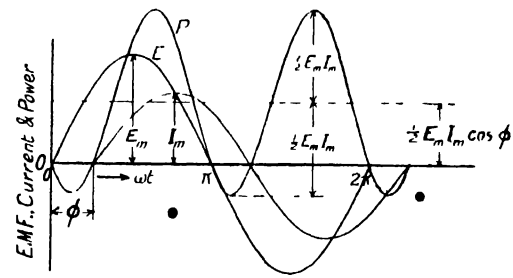


FIG. 38.—Graphical Representation of Equation to Power Curve of Fig. 33

The same result could have been obtained analytically from equation (28) by determining the mean value of this expression during a period. Thus

$$p = E_m I_m \sin \omega t . \sin (\omega t - \varphi)$$

$$P = \frac{1}{T}\int_0^T p \, . \, dt = \frac{1}{T}\int_0^T E_m I_m \sin \omega t \, . \sin (\omega t - \varphi) \, . \, dt$$

$$= \frac{1}{T} E_m I_m \int_0^T (\sin^2 \omega t \, . \cos \varphi - \sin \omega t \, . \cos \omega t \, . \sin \varphi) dt$$

$$= \frac{1}{T} E_m I_m \, \tfrac{1}{2} T \cos \varphi$$

$$= \tfrac{1}{2} E_m I_m \cos \varphi,$$

since $\int_0^T (\sin \omega t \, . \cos \omega t \, . \sin \varphi) dt$ is zero, and $\int_0^T \sin^2 \omega t . dt = \frac{1}{2}T$.

Substituting R.M.S. values for current and E.M.F., we have

$$P = \tfrac{1}{2}(\sqrt{2}E \, . \, \sqrt{2}I) \cos \varphi$$

$$= EI \cos \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32a)$$

Therefore, *the mean power in an alternating-current circuit is given by the product of the impressed E.M.F. and the component of the current which is in phase with the impressed E.M.F.*

Apparent power. The product $E\,I$ is called the *apparent power*, but in practice it is usually called the "volt-amperes." This quantity is important in connection with the rating of machines and cables, as the temperature rise of machines depends principally upon the volt-amperes output, and the temperature rise of extra-high-voltage cables depends entirely upon this quantity.

Power factor. The term $\cos\varphi$, in the expression for power, is called the "power factor" of the circuit, as it is the factor by which the apparent power must be multiplied to obtain the true power.

In the case of a non-inductive circuit, $\varphi = 0$, $\cos\varphi = 1$, and the power factor is unity. The true power is then given by the product of the impressed E.M.F. and current.

In the case of circuits containing pure inductance or capacity, $\varphi = 90°$, $\cos\varphi = 0$, and the power factor is zero.

Reactive power. The product of the impressed E.M.F. and the component of the current which is perpendicular to it, i.e. the product $EI\sin\varphi$, is called the "reactive power"; it represents the power which surges between the generator and the circuit, and *vice versa*, without doing work. In commercial circuits the reactive power supplies the magnetic and electrostatic fields, and the true power supplies the losses and performs useful, or mechanical, work.

Measurement of power. The true power in any circuit may be measured either indirectly by measuring each of the quantities E, I, $\cos\varphi$, separately by means of suitable instruments as discussed in Chapter XV, or directly by means of a wattmeter (see p. 394). The latter method is usually adopted in practice on account of its superior accuracy, but special precautions are necessary in order to obtain accurate readings at low power factors.

Power component and wattless component of current. In a circuit supplied at constant voltage the true power is proportional to $I\cos\varphi$, and the reactive power is proportional to $I\sin\varphi$. Now $I\cos\varphi$ is the component of the current in phase with the impressed E.M.F., and $I\sin\varphi$ is the component at right angles to the impressed E.M.F. The component $I\cos\varphi$ is therefore called the "power component" of the current, and $I\sin\varphi$ is called the "wattless component" (sometimes the terms "idle current," "quadrature component," and "reactive component" are employed).

This method of resolving the current into power and wattless

components is useful in the solution of problems and its applications are discussed in Chapter VI. The method is particularly useful in connection with parallel circuits. For example, if a number of circuits are connected in parallel and if I_p, I_w, denote respectively the sums of the power and wattless components of the currents in the several branch circuits, the total, or "line," current supplied to the circuits is given by $I = \sqrt{(I_p^2 + I_w^2)}$, and the phase difference between the impressed E.M.F. and line current is given by $\varphi = \tan^{-1} (I_w/I_p)$.

Example. Three circuits A, B, C, connected in parallel are supplied with power from 220 V. mains. Circuit A consists of a bank of incandescent lamps taking a current of 15 A. at unity power factor; B consists of an inductive resistance taking a current of 20 A. at a power factor of 0·85 lagging; C consists of an apparatus taking a current of 10 A. at a power factor of 0·95 leading. Determine the current and power supplied by the mains; also the power factor.

Let the currents in the several circuits be denoted by I_A, I_B, I_C, respectively, and the phase differences by φ_A, φ_B, φ_C. Let the current in the mains be denoted by I, and its phase difference with respect to the impressed E.M.F. by φ.

$$
\begin{array}{llllll}
\text{Then } I_A = 15 & \cos\varphi_A = 1{\cdot}0 & \sin\varphi_A = 0 \\
I_B = 20 & \cos\varphi_B = 0{\cdot}85 & \sin\varphi_B = -\sqrt{(1-0{\cdot}85^2)} = -0{\cdot}28 \\
I_C = 10 & \cos\varphi_C = 0{\cdot}95 & \sin\varphi_C = \sqrt{(1-0{\cdot}95^2)} = 0{\cdot}1
\end{array}
$$

Resolving each current into its power and wattless components, we have

$$
\begin{array}{ll}
\text{Power components.} & \text{Wattless components.} \\
I_A \cos\varphi_A = 15 \times 1{\cdot}0 = 15 & I_A \sin\varphi_A = 15 \times 0 = 0 \\
I_B \cos\varphi_B = 20 \times 0{\cdot}85 = 17 & I_B \sin\varphi_B = 20 \times (-0{\cdot}28) = -5{\cdot}6 \\
I_C \cos\varphi_C = 10 \times 0{\cdot}95 = 9{\cdot}5 & I_C \sin\varphi_C = 10 \times 0{\cdot}1 = 1{\cdot}0 \\
\therefore I\cos\varphi = (15 + 17 + 9{\cdot}5) = 41{\cdot}5 & \therefore I\sin\varphi = (-5{\cdot}6 + 1{\cdot}0) = -4{\cdot}6
\end{array}
$$

$$\text{Hence} \quad I = \sqrt{[(I\cos\varphi)^2 + (I\sin\varphi)^2]} = \sqrt{(41{\cdot}5^2 + 4{\cdot}6^2)} = 41{\cdot}8 \text{ A.}$$

$$\cos\varphi = (I\cos\varphi)/I = 41{\cdot}5/41{\cdot}8 = 0{\cdot}993 \text{ (lagging)}$$

$$P = 220 \times 41{\cdot}8 \times 0{\cdot}993 = 9130 \text{ watts}$$

CHAPTER VI

SERIES AND PARALLEL CIRCUITS

I.—SERIES CIRCUITS

Series circuits of constant impedance. The impedance of a simple series circuit containing resistance and reactance is given by

$$Z = \sqrt{(R^2 + X^2)},$$

and the phase difference between impressed E.M F. and current is

$$\varphi = \tan^{-1}(X/R).$$

In these equations X is the effective reactance of the circuit : it is equal to the algebraic difference between the inductive reactance (ωL) and the capacitive reactance ($1/\omega C$), i e. $X = \omega L - (1/\omega C)$.

In the case of a complex series circuit, such as is represented in Fig. 39, the joint impedance is given by

$$Z = \sqrt{\{(R_1 + R_2 + R_3 + \ldots)^2 + (X_1 + X_2 + X_3 \ldots)^2\}} \quad (33)$$

where R_1, R_2, R_3 . . . , are the resistances of the several parts of the circuit, and X_1, X_2, X_3 . . . . , are the effective reactances. The phase difference between impressed E M.F. and current is given by

$$\varphi = \tan^{-1}(X_1 + X_2 + X_3 + \ldots)/(R_1 + R_2 + R_3 + \ldots),$$

or by $\varphi = \cos^{-1}(R_1 + R_2 + R_3 + \ldots)/Z.$

These expressions follow directly from the vector diagram for the circuit. Thus, in Fig. 40 the current vector OI is taken as the vector of reference, and the potential differences across the several parts of the circuit are represented by the vectors OA, OB, OC. The geometric sum (OD) of these vectors represents the terminal voltage (E) of the circuit. From the diagram

$$E^2 = (E_1\cos\varphi_1 + E_2\cos\varphi_2 + E_3\cos\varphi_3)^2 + (E_1\sin\varphi_1 + E_2\sin\varphi_2 + E_3\sin\varphi_3)^2$$
$$= (IR_1 + IR_2 + IR_3)^2 + (IX_1 + IX_2 + IX_3)^2,$$

Whence,

$$Z = E/I$$
$$= \sqrt{\{(R_1 + R_2 + R_3)^2 + (X_1 + X_2 + X_3)^2\}},$$

and

$$\tan\varphi = (X_1 + X_2 + X_3)/(R_1 + R_2 + R_3).$$

Observe that the joint impedance is equal to the *geometric* sum of the separate impedances. In symbolic notation

$$\begin{aligned} Z &= Z_1 + Z_2 + Z_3 \qquad \qquad (34) \\ &= (R_1 + jX_1) + (R_2 + jX_2) + (R_3 + jX_3) \\ &= (R_1 + R_2 + R_3) + j(X_1 + X_2 + X_3) \end{aligned}$$

Graphical construction for obtaining the joint impedance of a series circuit. The construction is similar to that of the E.M.F. vector diagram of Fig. 40. In fact, by a suitable change of scale,

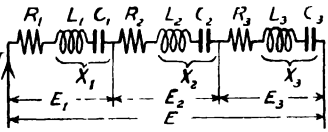


FIG. 39. Diagrammatic Representation of Complex Series Circuit

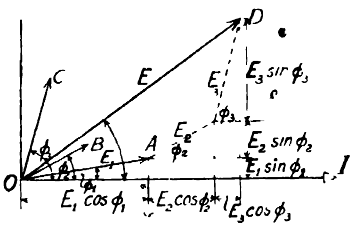


FIG. 40.—Vector Diagram for Circuit Represented in Fig. 39

the diagram becomes an impedance diagram, the vector OD now representing the impedance of the circuit. Instead of setting off the magnitudes Z_1, Z_2, Z_3 of the impedances and their phase angles φ_1, φ_2, φ_3, it is usually more convenient to set off their rectangular co-ordinates—resistance as abscissæ, and effective reactance as ordinates. In this case the only calculations required are those for obtaining the reactances of the several parts of the circuit.

Special cases. When the reactance of the entire circuit is zero, the impedance becomes

$$Z = R_1 + R_2 + R_3 + \ldots = R,$$

where R is the joint resistance, of the circuit.

For circuits containing only reactance we have

$$Z = X_1 + X_2 + X_3 \ldots = X,$$

where X is the joint reactance of the circuit.

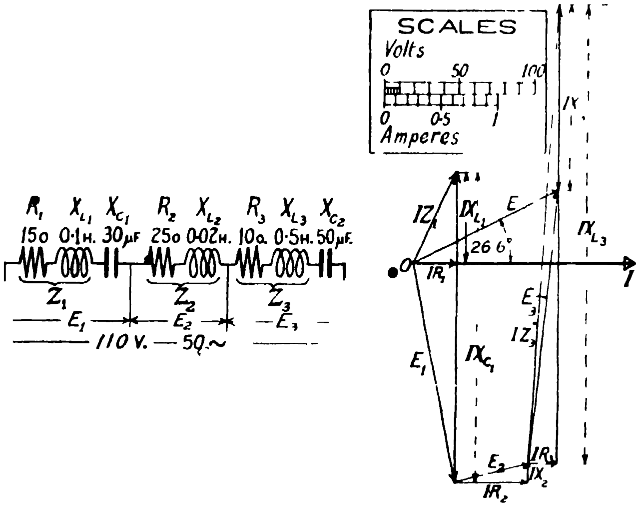
Hence if a number of condensers of capacities C_1, C_2, C_3, . . . are connected in series, the joint reactance will be given by

$$\begin{aligned} X &= X_1 + X_2 + X_3 + \ldots \\ &= \frac{1}{\omega C_1} + \frac{1}{\omega C_2} + \frac{1}{\omega C_3} \end{aligned}$$

Whence the joint capacity is

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

Thus the series connection of condensers gives a joint capacity smaller than the lowest individual capacity of the group. For example, if two condensers of 10 μF. and 20 μF. are connected in series the joint capacity is $1/(\frac{1}{10} + \frac{1}{20}) = 6{\cdot}66$ μF.



FIGS. 41, 42.—Circuit and Vector Diagram for Worked Example

In the case of a circuit containing a number of condensers connected in series, the terminal potential difference of the circuit is divided across the condensers in the inverse ratio of their capacities, and the potential difference across the smallest condenser may be very much higher than that across the largest condenser. For example, if condensers of 5 μF. and 50 μF. are connected in series across 200 V. mains, the potential difference across the 5 μF. condenser will be $\{200 \times 50/(5 + 50)\} = 181{\cdot}8$ V., and that across the 50 μF. condenser will be $\{200 \times 5/(5 + 50)\} = 18{\cdot}2$ V.

Example. Determine, for the circuit shown in Fig. 41, (1) the joint impedance, (2) the current, (3) the phase difference between current and terminal E.M.F., (4) the potential differences across each part of the circuit. The circuit is supplied at a pressure of 110 V. and a frequency of 50 cycles per second.

The total resistance $= 15 + 25 + 10 = 50$ O.

The inductive reactance $= 2\pi \times 50(0{\cdot}1 + 0{\cdot}02 + 0{\cdot}5) = 195$ O.

The capacitive reactance $= \dfrac{10^6}{2\pi \times 50}\left(\dfrac{1}{30} + \dfrac{1}{50}\right) = 170$ O.

Hence the effective reactance $= 195 - 170 = 25$ O.

Whence the joint impedance $= \sqrt{(50^2 + 25^2)} = 55{\cdot}9$ O.

Therefore the current $= 110/55{\cdot}9 = 1{\cdot}97$ A.

The phase difference between impressed E.M.F. and current $= \cos^{-1} 50/55{\cdot}9$
$= 26{\cdot}6°$

The potential differences across the several parts of the circuit are

$$
\begin{aligned}
IR_1 &= 1{\cdot}97 \times 15 = 29{\cdot}5 \text{ V}.\\
IX_1 &= 1{\cdot}97 \times 2\pi \times 50 \times 0{\cdot}1 = 61{\cdot}8 \text{ V}.\\
IZ_1 &= \sqrt{(29{\cdot}5^2 + 61{\cdot}8^2)} = 68{\cdot}5 \text{V}.\\
I/\omega C_1 &= 1{\cdot}97 \times 10^6/(2\pi \times 50 \times 30) = 209 \text{ V}\\
IR_2 &= 1{\cdot}97 \times 25 = 49{\cdot}2 \text{ V}.\\
IX_2 &= 1{\cdot}97 \times 2\pi \times 50 \times 0{\cdot}02 = 12{\cdot}35 \text{V}\\
IZ_2 &= \sqrt{(49{\cdot}2^2 + 12{\cdot}35^2)} = 50{\cdot}8 \text{V}.\\
IR_3 &= 1{\cdot}97 \times 10 = 19{\cdot}7 \text{ V}.\\
IX_3 &= 1{\cdot}97 \times 2\pi \times 50 \times 0{\cdot}5 = 309 \text{ V}.\\
IZ_3 &= \sqrt{(19{\cdot}7^2 + 309^2)} = 309{\cdot}5 \text{ V}.\\
I/\omega C_2 &= 1{\cdot}97 \times 10^6/(2\pi \times 50 \times 50) = 125{\cdot}3 \text{ V}.
\end{aligned}
$$

A vector diagram drawn to scale is given in Fig. 42.

Series circuits of variable impedance. (1) *Constant reactance, variable resistance.* Two cases are important, viz. (1) circuits in which the resistance is variable and the inductance is constant; (2) circuits in which the resistance is variable and the capacity is constant. In both cases the phase difference between the E.M.Fs. across the two parts of each circuit is 90°. Hence for any particular value of resistance the vector diagram for the E.M.Fs. may be drawn as a right-angled triangle, of which the hypotenuse represents the supply, or terminal, E M.F.

With variable resistance and constant reactance the E.M.F. vector diagrams for the varying conditions may, for constant terminal E.M.F., be represented by a series of right-angled triangles having a common hypotenuse, as shown in Fig. 43. The locus of the apex of the vector triangle is therefore a semicircle described on the hypotenuse. The semicircle for the R–L circuit is on one side of the hypotenuse, and that for the R–C circuit is on the opposite side as shown in Fig. 43, in which the semicircle OAE refers to the R–L circuit and the semicircle OBE refers to the R–C circuit.

The locii of the vectors of the currents in the circuits are also semicircles as shown in Fig. 43, but their centres lie on the opposite sides of, and in an axis perpendicular to, the vector (OE) representing the terminal E M.F.

Proof. Let, for the R–L circuit, the co-ordinates of the current vector with respect to the origin O, Fig. 44, be

$$y = I\cos\varphi = E\frac{R}{R^2+X^2}; \quad x = I\sin\varphi = E\frac{X}{R^2+X^2}$$

Then, squaring and adding, we have

$$x^2+y^2 = \left(\frac{ER}{R^2+X^2}\right)^2 + \left(\frac{EX}{R^2+X^2}\right)^2 = \frac{E^2(R^2+X^2)}{(R^2+X^2)^2} = \frac{E^2}{R^2+X^2}$$

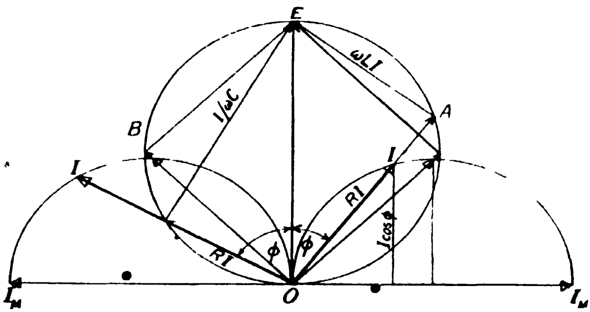


FIG. 43.—Vector Diagrams for Series Circuits of Variable Impedance (Constant Reactance, Variable Resistance)

Now $\quad R^2 + X^2 = EX/x.$

Hence $\quad x^2 + y^2 = x(E/X).$

This equation may be expressed in the form

$$y^2 + \left(x - \frac{E}{2X}\right)^2 = \frac{E^2}{4X^2}$$

which is the equation to a circle; the co-ordinates of the centre being $x = E/2X$, $y = 0$, and the radius being $E/2X$.

For the R–C circuit the radius of the semicircle is $E/2X$, and the co-ordinates of its centre are $x = -E/2X$, $y = 0$, where X is the reactance of the condenser.

If, however, X be variable and R constant, the current semicircles have their centres in the E.M.F. vector OE. Thus the sum of the squares of the co-ordinates of any current vector is now written as

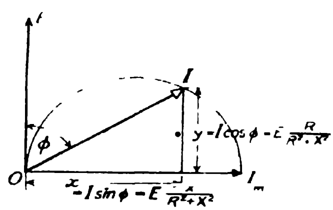


FIG. 44.—Vector Diagram for Circuit of Variable Resistance and Constant Inductance

$$x^2 + y^2 = y(E/R),$$

whence $x^2 + \left(y - \dfrac{E}{2R}\right)^2 = \dfrac{E^2}{4R^2}$

The co-ordinates of the centre are therefore $x = 0$, $y = E/2R$, and the radius is $E/2R$.

Properties of series circuits containing variable resistance and constant reactance. The circle diagram of Fig. 43 shows that

these circuits, when supplied at constant voltage and frequency, possess a number of important properties. Thus— (1) the current has a limiting value; (2) the power supplied to the circuit has a limiting value; (3) the power factor when maximum power is being supplied is 0·707.

The maximum current in the circuit is obtained when the resistance is zero: its value for the R–L circuit is $E/\omega L = I_M$

The power supplied to the circuit is equal to $E\,I \cos\varphi$, and if E is constant the power will be proportional to $I \cos\varphi$. Now ordinates in the current semicircles (Fig. 43) are proportional to this quantity. Hence the maximum ordinate in either semicircle represents the maximum power which can be supplied to the circuit. This ordinate passes through the centre to the semicircle, and, therefore, the current vector makes an angle of 45° to the diameter of the semicircle, and also to the vector of the impressed E.M.F. Thus the power factor corresponding to maximum power is $\cos 45° = 0{\cdot}707$. The maximum power is, therefore, given by

$$P_M = 0{\cdot}707\,EI = 0{\cdot}707\,EI_M \sin 45° = \tfrac{1}{2}EI_M \quad . \quad . \quad (35)$$

For the R–L circuit

$$P_M = \tfrac{1}{2}E^2/\omega L,$$

and for the R–C circuit

$$P_M = \tfrac{1}{2}\omega CE^2.*$$

At maximum power ($\varphi = 45°$) the vector triangle of E.M.Fs. is an isosceles triangle, and therefore the voltages across the resistance and reactance are each equal to $0{\cdot}707 \times$ supply E.M.F. Hence the *condition for maximum power* is that the resistance of

* These expressions may also be obtained by determining the maximum value of $EI \cos\varphi$. Thus, substituting for I and $\cos\varphi$ in terms of resistance and reactance, differentiating, and equating to zero, we have

$$\frac{dP}{dR} = \frac{d}{dR}\left\{\frac{E^2}{\sqrt{(R^2+\omega^2L^2)}} \cdot \frac{R}{\sqrt{(R^2+\omega^2L^2)}}\right\} = \frac{d}{dR}\left(\frac{E^2R}{R^2+\omega^2L^2}\right)$$

$$= \frac{E^2(R^2+\omega^2L^2) - E^2R \,.\, 2R}{(R^2+\omega^2L^2)^2}$$

$$= 0.$$

$$\therefore \quad E^2(R^2+\omega^2L^2) = 2E^2R^2$$

or $$R = \omega L = X.$$

Substituting ωL for R in the power expression for we obtain

$$P_M = \frac{E^2\omega L}{\omega^2L^2+\omega^2L^2} = \frac{E^2}{2\omega L} = \frac{E^2}{2X}$$

the circuit must equal the reactance of the circuit, i.e. $R = \omega L$, or $R = 1/\omega C$. The expressions for maximum power may therefore be written $P_M = \frac{1}{2}E^2/R$.

Practical applications. The current and power-limiting properties of series circuits containing variable resistance and constant inductive reactance are of considerable practical value in connection with *electric smelting furnaces* of the arc type. Thus, by inserting reactance in series with the furnace, (1) the power input to the furnace cannot exceed a predetermined limit ; (2) the current

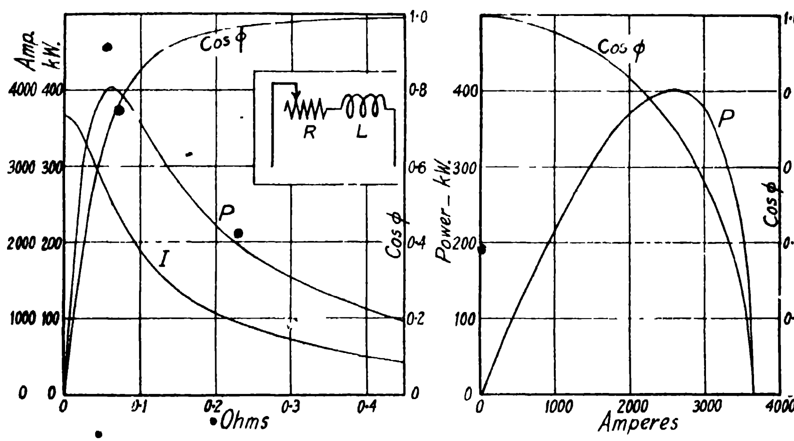


FIG 45 – Characteristic Curves for Series Circuit Containing Variable Resistance and Constant Inductance.

Representative of Arc-type Electric Smelting Furnace

taken when "striking" the arc cannot exceed a predetermined value ; (3) the furnace is more economical in power consumption than one controlled by a series resistance. These advantages, however, are obtained at the expense of the power factor.

The characteristic curves for a series circuit containing variable resistance and constant reactance are shown in Fig. 45, and refer to a 400 kW. arc-type furnace supplied at 220 V., 50 frequency ; the reactance of the furnace circuit being 0·06 ohm. The curves of Fig. 45 (*a*) show the manner in which the power and current input vary as the resistance varies, due to variations in the length of the arc, temperature, etc. The maximum power input occurs when the resistance is equal to 0·06 ohm. The curves of Fig. 45 (*b*) show the variation of power and power factor with the current input. The calculations for these curves are given in Table I.

TABLE I

Calculations for Fig. 45. Series circuit : reactance = 0·06 Ω. (constant), resistance variable. Supply pressure = 220 V. (constant), frequency = 50 (constant)

Resistance. ohm.	Impedance. ohm.	Current Input. amp.	Power Input. kW.	Power Factor.
0	0·06	3666	0	0
0·01	0·0608	3620	131	0·1645
0·02	0·0632	3480	242	0·3163
0·04	0·0721	3050	373	0·555
0·06	0·0848	2590	402	0·707
0·08	0·1	2200	387	0·8
0·1	0·1166	1886	356	0·858
0·15	0·1615	1362	278	0·929
0·2	0·209	1052	222	0·957
0·3	0·306	719	155	0·98
0·5	0·504	436	95·2	0·993
1·0	1·018	219	48·5	0·99

Maximum power $P_M = \frac{E^2}{2X} = \frac{220^2}{2 \times 0{\cdot}06 \times 10^3} = 402$ kW.

The series circuit containing variable resistance and fixed capacity has a practical application in *electric lighting* schemes for artisan dwellings and cottages where the average demand is small and would be unprofitable to the supply company if house service meters had to be installed. In these cases a number of low-voltage lamps,* all rated at the same current, are connected in series, and a condenser of suitable capacity is connected in the circuit. Each lamp is controlled by a short-circuiting switch. The capacity of the condenser is so chosen that with the normal demand the lamps receive their rated current, and the current in the circuit is only slightly affected by the change in resistance due to switching in and out these lamps. But if the normal demand be exceeded by switching in additional lamps, the current in the circuit is reduced appreciably, and a considerable diminution in the candle power of all the lamps in circuit occurs. Thus the circuit possesses the property of acting as its own "demand limiter," and consumers are restrained from exceeding the normal demand.

Example A lighting installation on the series-condenser system consists of seven 20 W., and one 10 W., 1 A. lamps and a condenser of 16·5 μF.

* The lamps must be so chosen that the aggregate of the lamp voltages, with normal demand, does not exceed 40 per cent of the supply voltage For further particulars, see "Condensers in series with metal filament lamps," by A. W. Ashton. *Journal, I.E.E.* (1912), xlix, 703.

capacity. The normal demand is equivalent to 50 W., and the supply is at 200 V., 50 frequency.

The capacity of the condenser for this installation is determined from the condition that the lamps must receive their rated current when the demand is normal. Thus, at normal demand, the aggregate lamp voltage is $50 \times 1 = 50$V. and the voltage across the condenser is equal to $\sqrt{(200^2 - 50^2)} = 193{\cdot}6$ V. The capacity of the condenser is therefore equal to $10^6/(2\pi \times 50 \times 193{\cdot}6) = 16{\cdot}45\mu$F. With this condenser the charging current at 200 V. is $(2\pi \times 50 \times 16{\cdot}5 \times 200)$

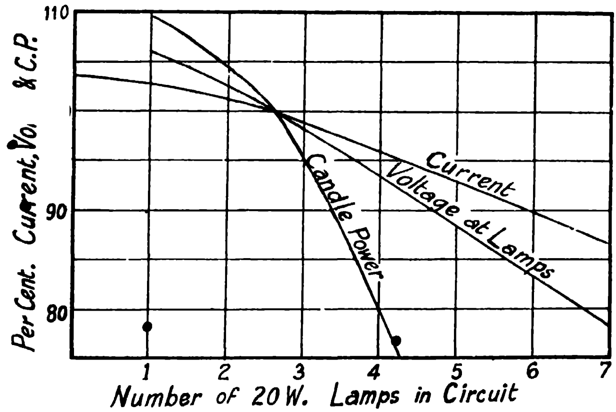


FIG. 46.—Variation of Current, Voltage, and Candle-power of Lamps for Electric Lighting Installation supplied on the Series-condenser System

$\times 10^{-6} = 1{\cdot}036$ A., which is the maximum current obtainable in the circuit. As the lamps are switched into circuit the current decreases as shown in Fig. 46, and in Table II.

TABLE II

Calculations for electric lighting installation on the series-condenser system. 7 – 20W., 1 – 10 W., 1 A. lamps. 16·5 μF. condenser. Supply pressure 200 V., frequency 50.

Resistance in Circuit (R).	Impedance $= \sqrt{\{R^2 + (1/\omega C)^2\}}$	Current.	Number of 20W. lamps in Circuit.	Corrected Value for resistance of Lamps.*	Current in Circuit.	Approximate Candle Power of each lamp (as % of normal c.p.).
Ohms.	Ohms.	Amp.		Ohms.	Amp.	
10	193·2	1·035				
20	194	1·03	1	20·5	1·029	110
30	195·4	1·024				
40	197	1·015	2	40·4	1·013	104
50	199·4	1·002				
60	202	0·99	3	59·6	0·989	95
80	209	0·956	4	77·8	0·96	80
100	217·4	0·92	5	95	0·93	65
120	227·3	0·88	6	111	0·897	
140	238·3	0·84	7	126·5	0·866	

* The values for the corrected resistance are obtained by trial from the current/resistance characteristic of the lamps and the current/resistance curve.

Series circuits of variable impedance. (2) *Constant resistance and variable reactance.* Four cases of these circuits are possible, viz. (1) constant resistance, variable inductance; (2) constant resistance, variable capacity; (3) constant resistance and inductance, variable capacity; (4) constant resistance and capacity, variable inductance. The first two cases have little application in practice, but are of academic interest: the last two cases have a large application in radio-telegraphy and telephony.

Vector diagram for circuits of constant resistance and variable inductance or capacity. The combined vector diagram for the R–L and R–C circuits, in which the resistance is constant, is shown in Fig. 47. In this diagram the vector OE, representing the impressed E.M.F., is taken as the vector of reference: OAE is the vector triangle of E.M.Fs. for the inductive circuit for a particular value of inductance; OBE is the corresponding triangle for the capacitive circuit. The loci of the apexes of these triangles are the semicircles $OCAE$, $ODBE$, respectively, their common centre being at the mid-point of OE.

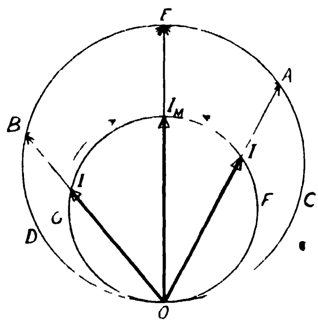


FIG. 47.—Vector Diagrams for Series Circuits of Variable Impedance (Constant Resistance, Variable Reactance)

The maximum current in either circuit is obtained when the reactance is zero: its value is E/R, and it is in phase with the impressed E.M.F. The vector OI_M, in phase with OE, represents this current, on the assumption of equal resistance and impressed E.M F. in the two cases.

for the circuit as given in columns I and III. For example, with 4 20W. lamps in circuit the resistance of the lamps will be slightly less than 80 O., owing to the current being less than normal. The current corresponding to a circuit resistance of 80 O. is 0·956 A. Assume the corrected current to be 0·96 A, which corresponds to a circuit resistance of 77·8 O. From the current/resistance characteristic of lamps the resistance at a current 96 per cent of normal is 97·2 per cent of normal resistance. Hence the resistance of four lamps in series is $4 \times 20 \times 0{\cdot}972 = 77{\cdot}8$ O Thus the value assumed for the corrected current is correct. Other values are calculated in a similar manner.

The current/resistance characteristic of the lamps is given by—

Current (% of normal)	105	102·5	100	97·5	95	92·5	90
Resistance (% of normal)	103·7	102	100	98·5	96·5	94·7	92·8

When the inductance is varied the current vector lags with respect to the OE and its locus is the semicircle OFI_M. Similarly, when the capacity is varied the current vector leads OE, and its locus is the semi-circle OGI_M. The centres of these semicircles are in OE and their radii are equal to $E/2R$ (see p. 85).

In each circuit the power is proportional to the projection of the current vector on the impressed E.M.F. vector: the power is therefore a maximum when the reactance is zero.

Vector diagram for the general circuit containing resistance and variable reactance. Series circuits containing resistance, inductance and capacity, in which either the inductance or the capacity is variable, have a large application in radio-telegraphy and telephony, the variable inductance or capacity being employed to adjust the circuit to resonance at a particular frequency, this adjustment being called "tuning."

The vector diagrams for these circuits are shown in Figs. 48 and 49, the former referring to the case in which the capacity is variable and the latter to the case when the inductance is variable. In both cases the vector of the impressed E.M.F. is taken as the vector of reference.

The current in either circuit is given, for any particular values of R, L, C, by $I = E/\sqrt{\{R^2 + [\omega L - 1/\omega C]^2\}}$, and its phase difference with respect to the impressed E.M.F. by $\tan \varphi = (\omega L - 1/\omega C)/R$. When the effective reactance is zero, i.e. when $\omega L = 1/\omega C$, the current is in phase with the impressed E.M.F. and its value is equal to E/R, which is the maximum value of the current for the circuit.

When the variable reactance is zero the current is given by $E/\sqrt{(R^2 + \omega^2 L^2)}$ for the circuit with fixed R and L; and by $E/\sqrt{[R^2 + (1/\omega C)^2]}$ for the circuit with fixed R and C. The phase difference between the impressed E.M.F. and the current is $\varphi = \tan^{-1}\omega L/R$, lagging, in the former case, and $\varphi = \tan^{-1}(1/\omega CR)$, leading, in the latter case. These currents are shown by the vectors OA in Figs. 48 and 49.

When the variable reactance is infinite the current in either circuit is zero.

The locus of the current vector when either the inductance in one circuit, or the capacity in the other circuit, is varied is the arc OAI_MB, the centre of which lies in OE at a distance $\frac{1}{2}I_M = E/2R$ from O. The portion AI_M of this arc corresponds to the condition when the fixed inductive reactance exceeds the variable capacitive reactance, i.e. $\omega L > 1/\omega C$; and the semicircle I_MBO corresponds to the condition when the latter exceeds the former, i.e. $(1/\omega C) > \omega L$.

If a circle be described upon *OE* as diameter, and any current vector, such as *OB*, be produced so as to cut this circle at *D*, then triangle *ODE* is the triangle of E.M.Fs. for the circuit; *OD* representing the component of the impressed E.M.F. expended against resistance, and *DE* that expended against the effective reactance, i.e. *DE* represents the vector difference between the E.M.Fs. across the inductance and capacity, these E.M.Fs. being represented by *OF* (leading the current vector by 90°) and *OG* (lagging 90° with

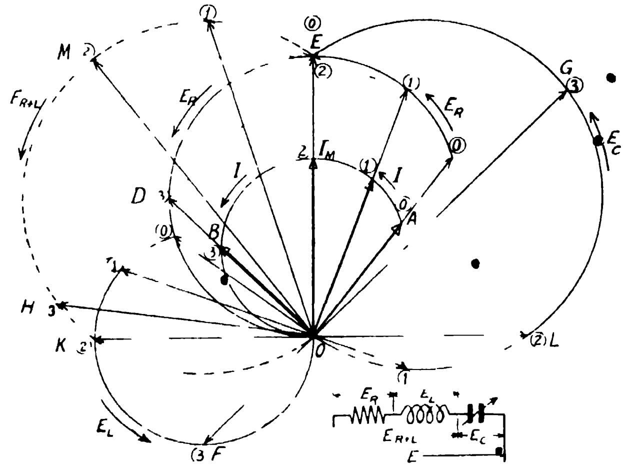


FIG. 48.—Vector Diagrams for Series Circuit of Variable Impedance (Constant Resistance and Inductance, Variable Capacity)

respect to the current vector) respectively. The voltage across the fixed portion of each circuit is represented in both figures by *OH*, which is the vector sum of *OD* and *OF*.

When the variable portion of the circuits is varied from zero reactance to infinite reactance the loci of the vectors *OD*, *OF*, *OG*, *OH*, are portions of circles, which are marked E_R, E_L, E_C, E_{R+L}, respectively, in Figs. 48 and 49. The centres of these circles may be determined very simply by considering the conditions occurring at resonance. Thus the effective reactance is zero, the current is a maximum and is in phase with the impressed E.M.F., the E.M.F. across the inductance is balanced by that across the capacity. These latter E.M.Fs. are represented by *OK* and *OL*, respectively. The voltage across the resistance is, at resonance, represented by *OE*—the vector of the impressed E.M.F.—and the vector sum of *OE* and *OK* is represented by *OM*.

For the circuit containing variable capacity, Fig. 48, the centre

of circle E_L lies at the mid-point of OK; the centre of the circle E_{R+L} lies at the mid-point of OM, and the centre of circle E_C is correspondingly situated on the opposite side of OE. Observe that the diameter of circle E_{R+L} is equal to that of circle E_C.

For the circuit containing variable inductance, Fig. 49, the centre of circle E_C lies at the mid-point of OL, the centre of the circle E_L lies at the mid-point of OM, and the centre of circle E_{R+L} lies at the mid-point of EM.

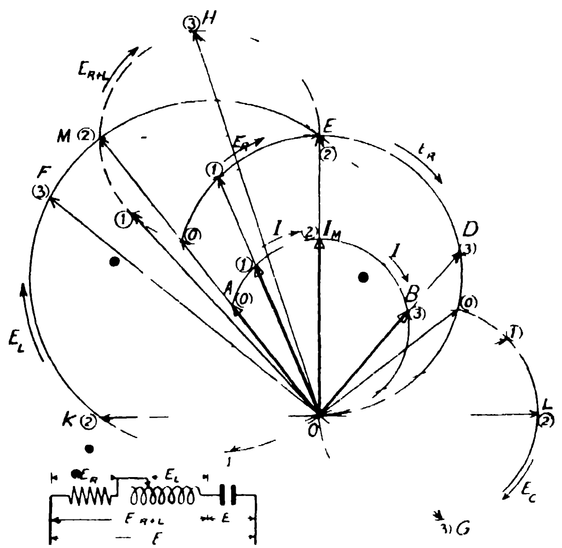


Fig. 49.—Vector Diagrams for Series Circuit of Variable Impedance (Constant Resistance and Capacity, Variable Inductance)

Resonance conditions. With both circuits the condition for obtaining resonance is that the inductive reactance must equal the capacitive reactance, i.e. $\omega L = 1/\omega C$. Whence $LC = 1/\omega^2 = 1/(2\pi f)^2$ or $\sqrt{(LC)} = 1/(2\pi f)$, where f is the frequency in cycles per second, and L, C, are the inductance and capacity, respectively, in practical units (i.e. henries and farads).

If instead of the frequency we are given the wave-length (λ) of the electrical oscillations to which the circuit must be tuned, then $\lambda_{metres} = 3 \times 10^8/f$, where 3×10^8 is the velocity of propagation, in metres per second, of the oscillations. Hence

$$\sqrt{(LC)} = \lambda/(2\pi \times 3 \times 10^8) = \lambda/(1885 \times 10^6).$$

But if L is expressed in micro-henries and C in micro-farads, the relationship between L, C, λ becomes

$$\sqrt{(LC)} = \lambda/1885 \tag{36}$$

This, then, is the condition which must be satisfied in a "tuned" radio circuit when the tuning is effected by the use of series capacity or inductance.

The curves of Fig. 50 show, for a particular circuit (for which $R = 25$ ohms, $L = 0{\cdot}4$ henries (constant), C (variable), the variation of current, voltage, and power factor when the capacity is varied and the impressed E M F and frequency are maintained constant.

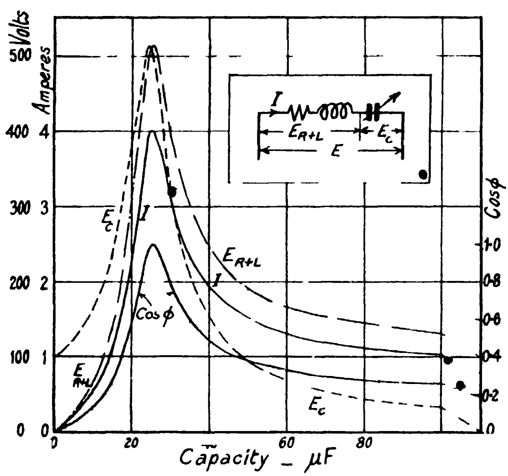


FIG. 50.—Characteristic Curves for Series Circuit of Variable Impedance (Constant Resistance and Inductance, Variable Capacity)

It will be observed that the peaks of the curves for the voltages across the condenser and inductive resistance are not coincident the peak of the condenser voltage occurring when the capacity has a value which is smaller than that which gives the peak of the voltage across the inductive resistance. This non-coincidence of the voltage peaks is caused by the resistance of the circuit. For a circuit of zero resistance the peaks of voltage are coincident and occur when the current in the circuit is a maximum. Data for these curves are given in Table III, and the method of calculating the quantities may be shown best by calculating one or two points. Thus at resonance we must have $\omega L = 1/\omega C$, or $C = 1/\omega^2 L$. Now for a frequency of 50, $\omega = 314$. Hence the capacity, in μF.,

required to give resonance is $C = 10^6/(314^2 \times 0{\cdot}4) = 25{\cdot}35\ \mu$F. With an impressed E.M.F. of 100 V. the current is $I_M = 100/25 = 4$ A. The voltage across the condenser is $E_C = I/_M\omega C = 4/(314 \times 25{\cdot}35 \times 10^6) = 502{\cdot}4$ V., and the voltage across the inductive resistance is $E_{R+L} = I_M(R^2 + \omega^2 L^2) = 4 \times 128 = 512$ V.

[NOTE.—$R^2 + \omega^2 L^2 = 25^2 + (314^2 \times 0{\cdot}4^2) = 128$ O.]

For the points corresponding to a capacity of 20 μF. we have—

Effective reactance $= (\omega L - 1/\omega C) = 314 \times 0{\cdot}4 - [10^6/(314 \times 20)]$
$= -33{\cdot}5$ O.

Impedance $(Z) = \sqrt{[R^2 + (\omega L - 1/\omega C)^2]} = \sqrt{(25^2 + 33{\cdot}5^2)}$
$= 41{\cdot}7$ O.

Current $= E/Z = 100/41{\cdot}7 = 2{\cdot}4$ A.

Voltage across condenser $= I/\omega C = 2{\cdot}4 \times 10^6/314 \times 20 = 382$ V.

Voltage across inductive resistance $= I\sqrt{(R^2 + \omega^2 L^2)} = 2{\cdot}4 \times 128$
$= 307$ V.

Power factor $= R/Z = 25/41{\cdot}7 = 0{\cdot}6$.

TABLE III

Calculations for Fig. 50. Series circuit : $R = 25$ O., $L = 0{\cdot}4$ H. (constant), C variable. Supply pressure $= 100$ V. (constant), frequency $= 50$ (constant).

C (μF.)	$1/\omega C$	$\omega L - 1/\omega C$	$Z = \sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}}$	$I = E/Z$	$\text{Cos}\ \varphi = R/Z$	$E_C = I/\omega C$	$E_{R+L} = I\sqrt{(R^2 + \omega^2 L^2)}$.
0	∞	∞	∞	0	0	100	0
5	636	−511	511	0·196	-	124·5	25
10	318	−193	194·6	0·514	0·1285	163·5	66
15	212	−87	90·4	1·106	0·2763	234	141
20	159	−33·5	41·7	2·4	0·6	382	307
23	138·4	−13	28·2	3·45	0·886	477	454
24	132·5	7	26	3·85	0·964	511	493
24·5	129·8	4·2	25·35	3·945	0·986	512	505
25	127·2	1·6	25·01	3·99	0·998	508	511
25·35	125·6	0	25	4	1·0	502·4	512
27	117·8	8·8	26·5	3·77	0·943	445	483
30	106·1	19·5	31·7	3·15	0·788	331	404
35	91	34·6	42·7	2·34	0·585	213	300
40	79·6	46	52·4	1·91	0·477	152	244
50	63·6	62	67·6	1·48	0·37	94	189
100	31·8	94	97·3	1·026	0·257	32·6	131
∞	0	125·6	128	0·781	0·195	0	100

Series circuits of variable impedance. (3) *Constant resistance, inductance and capacity, variable frequency.* This circuit is of practical importance on account of its possessing a "resonance frequency" which is given by

$$f_r = \frac{1}{2\pi\sqrt{(LC)}}$$

in which L and C are in practical units (i.e. henries and farads, respectively) and f_r is in cycles per second.

At resonance frequency the current is a maximum and is given by $I_M = E/R$. This current is in phase with the impressed E.M.F. The vector diagram for these conditions is given in Fig. 32 (*b*), p. 70.

Under resonance conditions, and with suitable values of inductance and capacity, the voltage across the condenser, and that across the

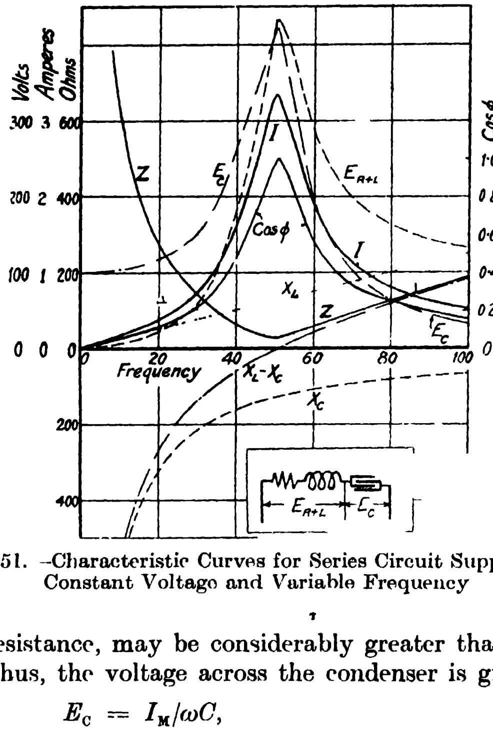


FIG. 51. –Characteristic Curves for Series Circuit Supplied at Constant Voltage and Variable Frequency

inductive resistance, may be considerably greater than the supply voltage. Thus, the voltage across the condenser is given by

$$E_C = I_M/\omega C,$$

which, when $1/\sqrt{(LC)}$ is substituted for ω, and E/R for I_M, becomes

$$E_C = \frac{E}{R}\sqrt{\frac{L}{C}}$$

$$= E\sqrt{\frac{L}{CR^2}}$$

Hence when $L > CR^2$ the voltage across the condenser will be greater than the supply voltage. The ratio of these voltages is given simply by $\sqrt{(L/CR^2)}$.

Similarly the voltage across the inductance and resistance is, at resonance frequency, given by

$$E_{L+R} = I_M \sqrt{(R^2 + \omega^2 L^2)}$$
$$= \frac{E}{R}\sqrt{\left(R^2 + \frac{L}{C}\right)} = E\sqrt{\left(1 + \frac{L}{CR^2}\right)}$$

which is greater than the supply voltage for all values of L and C except zero.

The manner in which these voltages vary with the frequency in the case of a particular circuit, for which $R = 30\ \Omega$., $L = 0{\cdot}4$ H., $C = 25\ \mu$F., is shown in Fig. 51. Other curves in this figure show the manner in which the current, power factor, inductive reactance, capacitive reactance, and effective reactance vary with the frequency when the supply pressure is maintained constant at 100 V. The calculations for these curves are given in Table IV.

TABLE IV

Calculations for Fig. 51. Series circuit: $R = 30\ \Omega$., $L = 0{\cdot}4$ H., C = 25 μF. Supply pressure = 100 V. (constant) Frequency variable.

f	$\omega = 2\pi f$	ωL	$10^6/\omega C$	$\omega L - 1/\omega C$	$Z = \sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}}$	$I = 100/Z$	$\cos\varphi = R/Z$	$E_C = I/\omega C$	$E_{L+R} = \sqrt{(R^2 + \omega^2 L^2)}$
0	0	0	∞	∞	∞	—	—	100	0
10	62·8	25 1	637	−612	612	0·1635	0 049	104 2	6 4
20	125·6	50 2	319	−260	270 8	0 369	lead 0·11	117 8	21
30	188·4	75 3	212	−137	140 1	0·713	0 211	151	57 8
40	251	105	159 5	− 54 5	62 2	1·608	0 483	256 5	175 5
45	283	113·2	141 2	− 28	41	2·41	0 732	345	286
50·35	316	126 5	126·5	0	30	3·33	1 0	422	433
55	345	138	116	22	37·2	2 69	0 807	312	380
60	377	151	106	45	54	1 85	0 556	196	285
70	440	176	90·9	85	90·2	1·11	lag 0 333	100	198·5
100	628	251	63 7	187·9	190	0 526	0 158	33 5	133

Resonance in Practice. The equivalent of the above circuit occurs in practice when an alternator is supplying unloaded cables, the inductance and resistance being represented by the armature winding of the alternator, and the condenser by the capacity between the cores of the cables. But the "constants" of commercial distributing systems are such that the resonance frequency is much higher than the normal frequency. Hence, with a pure sine wave of E.M.F. resonance cannot occur. If, however, the wave-form is non-sinusoidal resonance, due to one of the higher harmonics of the E.M.F., may occur either at normal speed, or at a speed lower than normal if the alternator is run up to speed with its field excited and its armature connected to unloaded cables.

Example. An 11,000-V., 50-cycle, alternator is connected to concentric cables having a total capacity of 3 μF. The inductance of the armature winding is 0·02 H., and the resistance of the armature winding is 1·3 Ω.

The resonance frequency of the system when the cables are unloaded is

$$f_r = \frac{1}{2\pi\sqrt{(0{\cdot}02 \times 3 \times 10^{-6})}} = 649 \text{ cycles per second}$$

If the wave-form of the no-load E.M.F. contains a thirteenth harmonic (the frequency of which = $13 \times 50 = 650$), resonance due to this harmonic will occur at normal speed. If the amplitude of this harmonic is 1 per cent of the fundamental, i.e. $\frac{1}{100} \times 11000 \times 1{\cdot}414 = 155$ V., the current due to it at resonance is $155/1{\cdot}3 = 119$ A. (maximum value). Hence if the peak of the E.M.F. wave of this harmonic occurs at the same instant as that of the fundamental, the voltage impressed on the cables is

$$11000 \times 1{\cdot}414 + 119/(2\pi \times 650 \times 3 \times 10^{-6}) = 25{,}250 \text{ V}.$$

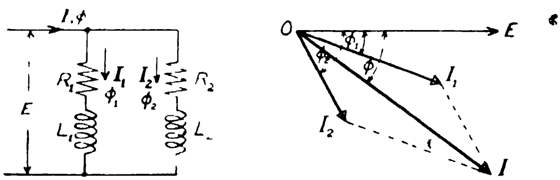


FIG. 52. -Circuit and Vector Diagrams for Parallel Circuit

II.—PARALLEL CIRCUITS

Parallel circuits of constant impedance. The vector diagram for a parallel circuit, consisting of two inductive resistances, is given in Fig. 52. The total, or " line " current, I, is equal to the geometric sum of the branch currents I_1, I_2 : its magnitude and phase may be calculated from Fig. 52 as follows

$$I = \sqrt{\{(I_1 \cos\varphi_1 + I_2 \cos\varphi_2)^2 + (I_1 \sin\varphi_1 + I_2 \sin\varphi_2)^2\}} \qquad (37)$$

$$\varphi = \tan^{-1}(I_1 \sin\varphi_1 + I_2 \sin\varphi_2)/(I_1 \cos\varphi_1 + I_2 \cos\varphi_2)$$

The joint impedance, Z, of the circuit may be obtained from Equation (37), and Fig. 52, as follows

$$I = E/Z = \sqrt{\{(I_1 \cos\varphi_1 + I_2 \cos\varphi_2)^2 + (I_1 \sin\varphi_1 + I_2 \sin\varphi_2)^2\}}$$

$$= \sqrt{\left\{\left(\frac{E}{Z_1}\cdot\frac{R_1}{Z_1} + \frac{E}{Z_2}\cdot\frac{R_2}{Z_2}\right)^2 + \left(\frac{E}{Z_1}\cdot\frac{X_1}{Z_1} + \frac{E}{Z_2}\cdot\frac{X_2}{Z_2}\right)^2\right\}}$$

$$\frac{1}{Z} = \sqrt{\left\{\left(\frac{R_1}{Z_1^2} + \frac{R_2}{Z_2^2}\right)^2 + \left(\frac{X_1}{Z_1^2} + \frac{X_2}{Z_2^2}\right)^2\right\}}$$

$$= \sqrt{\left\{\frac{1}{Z_1^2} + \frac{1}{Z_2^2} + \frac{2(R_1R_2 + X_1X_2)}{Z_1^2\,Z_2^2}\right\}} \quad . \quad . \quad . \quad . \quad (38)$$

Thus for the general case the reciprocal of the joint impedance is not equal to the sum of the reciprocals of the separate impedances : its calculation therefore requires a knowledge of the values of the separate impedances as well as the resistances and reactances.

Special Cases. In the special case when $X_1 = 0$, $X_2 = 0$, we have

$$\frac{1}{Z} = \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

and in the other case when $R_1 = 0$, $R_2 = 0$, we have

$$\frac{1}{Z} = \frac{1}{X} = \frac{1}{X_1} + \frac{1}{X_2}$$

When the reactances X_1, X_2, consist of condensers,

$$\frac{1}{X_1} = \omega C_1\,;\ \frac{1}{X_2} = \omega C_2\,;\ \frac{1}{X} = \omega C$$

Hence the joint capacity of the condensers is

$$C = C_1 + C_2.$$

Thus, when condensers are connected in parallel, the joint capacity is equal to the sum of the separate capacities.

Admittance. Reverting to the general case, Fig. 52, and employing symbolic notation, the equation to the line current may be written

$$\begin{aligned} I &= I_1 + I_2 \\ &= (E/Z_1) + (E/Z_2) \end{aligned}$$

whence

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

or

$$Y = Y_1 + Y_2$$

where Y is the reciprocal of the joint impedance and Y_1, Y_2, are the reciprocals of the separate impedances. These reciprocals are called *admittance*. Hence the joint admittance of a number of circuits connected in parallel is equal to the geometric sum of their separate admittances. The admittances of these circuits may therefore be compounded in a vector diagram in a manner similar to that adopted with the several impedances of a series circuit. For example, in Fig. 53 the vector OA represents the admittance Y_1, OB, the admittance Y_2. The joint admittance is represented by OC, which is the vector sum of OA and OB.

Inversion. The joint impedance Z, corresponding to the joint admittance. Y, may be obtained from the vector diagram by

determining the vector which is the reciprocal of the admittance vector, this process being called "inversion." For example, two reciprocal quantities, Y, Z $(= 1/Y)$, are represented in a vector diagram by vectors of different lengths—such as OY, OZ, Fig. 54—located on opposite sides of the axis of reference and making equal angles with this axis. The projection on OZ of the point Y gives the point Y', which is called the image of Y.

Two points such as Y' and Z, Fig. 54, are called "inverse points." The origin, O, is called the "centre of inversion," and the value of the product $OY'.OZ$ which corresponds to unit values of Y and Z, is called the "inversion constant," K_i.

If the admittance vector is drawn to such a scale that 1 cm. $= m$ mho (unit of admittance), and the impedance vector is drawn to

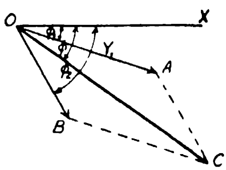


Fig. 53.—Vector Diagram Showing Addition of Admittances

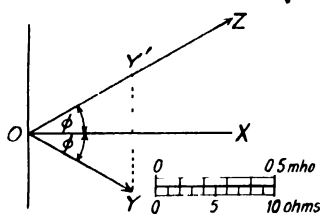


Fig. 54 -Vector Diagram of Reciprocal Vector Quantities

such a scale that 1 cm. $= n$ ohms then $OY' = Y/m$, $OZ = Z/n$. Whence $OY' \times OZ = YZ/mn = 1/mn$, since $Y.Z = 1$, i.e. the inversion constant is equal to the product of the reciprocals of the scales of the vectors. Hence the inversion of the image of an admittance vector—drawn to a scale of 1 cm. $= m$ mho—to an inversion constant K_i gives an impedance vector having a scale of 1 cm. $= (1/mK_i)$ ohms. Conversely, the inversion of the image of an impedance vector—drawn to a scale of 1 cm. $= n$ ohms—gives an admittance vector having a scale of 1 cm. $= (1/nK_i)$ mho. For example, if OY, Fig. 54, represents an admittance of 0·05 mho, and is drawn to a scale of 1 cm. = 0·025 mho, i.e. $OY = 0{\cdot}05/0{\cdot}025$ = 2 cm., then the length of the impedance vector OZ for inversion constant equal to 8, is $8/(0{\cdot}05/0{\cdot}025) = 4$ cm., and the scale of this vector is $1/(8 \times 0{\cdot}025)$, or 5 ohms = 1 cm. Therefore the magnitude of the impedance is equal to $4 \times 5 = 20$ ohms.

Instead of determining the impedance scale from the inversion constant and the admittance scale, it is more convenient in practice to *select* the impedance scale such that the longest impedance vector is of a convenient length.

The principle of inversion is of considerable value in the graphical solution of problems on parallel and series-parallel circuits, and further applications will be found in this and the following chapters.

Conductance and susceptance. Since admittance, like impedance, is a complex quantity it must be expressed, when the rectangular form of symbolic notation is employed, in terms of in-phase and quadrature components which are denoted by the symbols g and b, respectively. These components can be expressed in terms of the constants, R, X, of the circuit. For example,

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = \left(\frac{R}{R^2 + X^2}\right) - j\left(\frac{X}{R^2 + X^2}\right)$$
$$= g - jb$$

where $g = R/(R^2 + X^2) = R/Z^2$, and $b = X/(R^2 + X^2) = X/Z^2$.

g is called the *conductance** because, for a circuit of zero reactance, we have $g = R/R^2 = 1/R$. b is called the *susceptance* and, for circuits of zero resistance, is equal to the reciprocal of reactance, i.e. $b = X/X^2 = 1/X$.

Hence for the circuit of Fig. 52 we have

$$Y_1 = g_1 - jb_1\,; \quad Y_2 = g_2 - jb_2$$
$$Y = Y_1 + Y_2 = (g_1 - jb_1) + (g_2 - jb_2)$$
$$= (g_1 + g_2) - j(b_1 + b_2)$$

Whence,
$$Y = \sqrt{\{(g_1 + g_2)^2 + (b_1 + b_2)^2\}} \quad . \quad . \quad (40)$$
$$\varphi = \tan^{-1}(b_1 + b_2)/(g_1 + g_2).$$

Parallel circuits of variable impedance. (1) *Constant resistance, inductance, and capacity; variable frequency.* Let one branch of a parallel circuit consist of a fixed inductive resistance (R, L), and the other a condenser of fixed capacity (C). The current in the inductive resistance at a frequency f is $I_1 = E/\sqrt{(R^2 + \omega^2 L^2)}$, where $\omega = 2\pi f$. The power component of this current is $I_1 \cos \varphi_1 = ER/(R^2 + \omega^2 L^2)$, and the wattless component is $I_1 \sin \varphi_1 = E\omega L/(R^2 + \omega^2 L^2)$, which lags 90° with respect to the impressed E.M.F. The charging current of the condenser is $I_2 = \omega CE$, and leads the impressed E.M.F. by 90°. Therefore, the phase difference between the wattless component of the current in the

* The term conductance is employed in connection with continuous-current circuits to denote the reciprocal of resistance.

inductive resistance and the charging current of the condenser is 180°. Hence the wattless component of the line current is

$$I_W = \frac{E\omega L}{R^2 + \omega^2 L^2} - \omega C E$$

which, if the frequency is variable, becomes zero when

$$\frac{E\omega L}{R^2 + \omega^2 L^2} = \omega C E,$$

i.e. when

$$\frac{\omega L}{\omega C} = R^2 + \omega^2 L^2 = X_C X_L,$$

where X_L ($= \omega L$) is the inductive reactance and X_C ($= 1/\omega C$) is the capacitive reactance. Thus the condition for the wattless component of the line current to be zero is that the product of the inductive and capacitive reactances must equal the square of the impedance of the inductive branch.

The frequency corresponding to this condition is obtained by solving the equation for ω. Thus

$$\omega^2 = (2\pi f)^2 = \frac{1}{L^2}\left(\frac{L}{C} - R^2\right) = \left(\frac{1}{LC} - \frac{R^2}{L^2}\right)$$

whence $$f = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Resonance. Every parallel circuit consisting of a condenser, of fixed capacity, in one branch and a fixed inductive resistance in the other branch has, therefore, a particular frequency at which the charging current of the condenser balances the wattless current in the inductive resistance. This frequency, which is given by Equation (41), is called the "resonance frequency" of the (parallel) circuit. It is not, however, the natural frequency of the local circuit—consisting of the condenser and inductive resistance—as this frequency is given by

$$f_n = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

When $R = 0$ the resonance frequency becomes equal to the natural frequency of the circuit: moreover, the resonance frequency for the parallel circuit is then equal to that for the series circuit containing the same values of inductance and capacity. Under these hypothetical conditions—i.e. with zero resistance and no losses—*electric oscillations, when once started in the local circuit,*

may be maintained without the further supply of energy to this circuit.

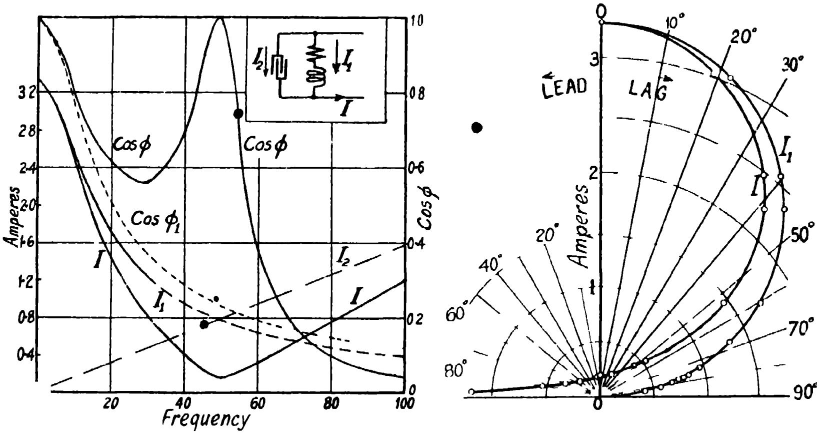
Variation of line current with frequency. Reverting to the practical case, in which the inductive branch contains resistance, the line current at a frequency f is given by

$$I = \sqrt{\{(I_1 \cos \varphi_1)^2 + (I_1 \sin \varphi_1 - \omega CE)^2\}}$$

$$= \sqrt{\left\{\left(\frac{ER}{Z^2}\right)^2 + \left[\omega E\left(\frac{L}{Z^2} - C\right)\right]^2\right\}}$$

$$= E\sqrt{\left\{\frac{1}{R^2 + \omega^2 L^2} - \omega^2\left(\frac{2LC}{R^2 + \omega^2 L^2} - C^2\right)\right\}} \quad . \quad . \quad (43)$$

where $\omega = 2\pi f$



(a) (b)

FIG. 55 (a) Characteristic Curves for Parallel Circuit Supplied at Constant Voltage and Variable Frequency. (b) Plot of Currents I, I_1, in Polar Co-ordinates

At zero frequency $\omega = 0$, and $I = E/R$. As the frequency is increased from zero the line current decreases in value and becomes a minimum at a frequency slightly above the resonance frequency: it then increases as the frequency increases Hence if line current be plotted against frequency a V curve is obtained as shown in Fig. 55 (a), which refers to a particular circuit for which $R = 30$ O, $L = 0{\cdot}4$ H, $C = 25\ \mu$F.

Other curves in this figure show the manner in which the resultant power factor, the branch-circuit currents, and the wattless and power components of the current in the inductive branch vary

when the frequency is varied and the supply pressure is maintained constant at 100 V. The calculations for these curves are given in Table V.

The line current and the current in the inductive branch are also plotted to polar co-ordinates in Fig. 55 (*b*), from which it will be observed that the curve for the latter is a semicircle.

It will be observed that as the frequency is increased from zero, the wattless component of the current in the inductive branch increases from zero to a definite maximum value and then decreases with increasing frequency. The frequency corresponding to the maximum wattless current in the inductive branch is obtained by differentiating, with respect to ω, the expression for this current, equating the result to zero and solving for ω. Thus

$$I_1 \sin \varphi_1 = E\omega L/(R^2 + \omega^2 L^2) = E\omega L/Z^2$$

$$\frac{d}{d\omega}(I_1 \sin \varphi_1) = \frac{ELZ^2 - 2E\omega^2 L^3}{Z^4}$$

$$\therefore \quad ELZ^2 = 2E\omega^2 L^2$$

or $$R^2 + \omega^2 L^2 = 2\omega^2 L^2$$

Whence $$\omega = R/L.$$

TABLE V

Calculations for Fig. 55. Parallel circuit : inductive branch, $R = 30$ Ω., $L = 0·4$ H. ; condensor branch, $C = 25$ μF. Supply pressure = 100 V (constant). Frequency variable.

	$\omega = 2\pi f$	ωL	$Z = \sqrt{(R^2 + \omega^2 L^2)}$	$I_1 = 100/Z$	$I_1 \sin \varphi_1$	$I_2 = \omega CE$	$I_1 \sin\varphi_1 - I_2 (= I \sin \varphi)$	$I_1 \cos\varphi_1 (= I \cos \varphi)$	$I = \sqrt{\{(I \sin \varphi)^2 + (I \cos \varphi)^2\}}$	Cos φ	Power Supplied $EI_1 \cos \varphi$
											Watts
0	0	0	30	3·33	0	—	—	3·33	3·33	1·0	333
5	31·4	12·5	32·5	3·07	1·19	0·08	1·11	2·84	3·05	0·938	284
10	62·8	25·1	39·1	2·56	1·65	0·16	1·49	1·96	2·[illegible]	0·8	196
12	75	30	42·4	2·36	1·67	0·19	1·48	1·67	2·25	0·747	167
20	125·6	50·2	58·5	1·71	1·46	0·31	1·15	0·88	1·44	0·61	88
30	188·4	75·3	81·1	1·23	1·14	0·47	0·67	0·46	0·51	0·56	46
40	251	100·4	105	0·95	0·91	0·63	0·28	0·294	0·42	0·7	29·4
45	283	113·2	117	0·85	0·82	0·71	0·11	0·218	0·244	0·894	21·8
48·9	307	122·8	126·4	0·791	0·768	0·768	0	0·188	0·188	1·0	18·8
50	314	125·6	129·3	0·773	0·751	0·786	−0·035	0·18	0·183	0·984	18
55	345	138	141·2	0·71	0·69	0·86	−0·17	0·15	0·23	0·654	15
60	377	151	154	0·65	0·64	0·94	−0·3	0·127	0·33	0·385	12·7
70	440	176	178·5	0·56	0·55	1·1	−0·55	0·094	0·55	0·171	9·4
100	628	993	252·8	0·395	0·393	1·57	−1·177	0·047	1·18	0·04	4·7

Analytical Investigation of Variation of Line Current with Frequency. The shape of the line-current/frequency curve for the general case may be ascertained analytically by investigating the law of variation, with respect

to frequency, of the several terms of Equation (43). Thus the value of the first term $[1/(R^2 + \omega^2 L^2)]$ is a maximum at zero frequency and decreases as the frequency increases: the rate of decrease being at first rapid, then diminishing gradually, and finally becoming zero at an infinite frequency. The value of the second term $[\omega^2(2LC/Z^2 - C^2)]$ is zero when the frequency is zero: it increases as the frequency increases, until it attains its maximum value at a frequency below the resonance frequency; it then decreases, becomes zero at a particular frequency which is higher than the resonance frequency — and then changes sign and increases continually as the frequency is still further increased. The variation is shown graphically in Fig. 56, the curves of which refer to the circuit calculated in Table V.

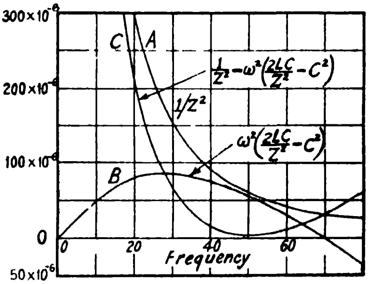


FIG. 56.—Variation of Quantities with Frequency, for Parallel Circuit ($R = 30$ Ω., $L = 0{\cdot}4$ H., $C = 25\ \mu$F.)

The general shape of these curves is the same for all circuits, but the frequencies at which the maximum and zero values of the second term occur depend upon the "constants" of the circuits. For example, the frequency corresponding to the maximum value of $[\omega^2(2LC/Z^2 - C^2)]$ is obtained by equating the first differential coefficient of this quantity to zero and solving for ω^2. Thus

$$\frac{d}{d\omega}\left\{\omega^2\left(\frac{2LC}{Z^2} - C^2\right)\right\} = \frac{Z^2(4\omega LC) - 2\omega L^2(2\omega^2 LC)}{Z^4} - 2\omega C^2 = 0$$

whence $\quad 2(Z^2 - \omega^2 L^2) = Z^4 C/L,$

which, when $(R^2 + \omega^2 L^2)$ is substituted for Z^2, reduces to

$$\omega^2 = \left(\frac{R}{L}\sqrt{\frac{2}{LC}}\right) - \frac{R^2}{L^2}$$

The quantity $[\omega^2(2LC/Z^2 - C^2)]$ is zero when $2LC/Z^2 = C^2$, and the frequency corresponding to this condition is obtained from the equation

$$\omega_1^2 = \frac{2}{LC} - \frac{R^2}{L^2}$$

Substituting numerical values for R, L, C, from Table V, we have

$$\omega^2 = \left(\frac{30}{0{\cdot}4}\sqrt{\frac{2 \times 10^6}{0{\cdot}4 \times 25}}\right) - \left(\frac{30}{0{\cdot}4}\right)^2 = 27{,}875,$$

whence $\quad f = \omega/2\pi = \sqrt{27875}/2\pi = 26{\cdot}6$ cycles per second.

Similarly, $\quad \omega_1^2 = \dfrac{2 \times 10^6}{0{\cdot}4 \times 25} - \left(\dfrac{30}{0{\cdot}4}\right)^2 = 194{,}375,$

whence $\quad f_1 = \omega_1/2\pi = \sqrt{194375}/2\pi = 71{\cdot}2$ cycles per second.

Now the difference between the quantities $[1/(R^2 + \omega^2 L^2)]$ and $[\omega^2(2LC/Z^2 - C^2)]$ is proportional to the square of the current [see Equation (43)].

Hence intercepts between ordinates of the curves in Fig. 56 are proportional to the current. As the frequency is increased from zero, these intercepts diminish on account of the curves approaching each other, but after a particular frequency is reached the curves diverge and the intercepts

increase. In the case of a hypothetical circuit without resistance, the curves touch at resonance frequency, but for practical circuits the minimum intercept occurs at a frequency slightly higher than the resonance frequency.

The frequency corresponding to the minimum value of the line current is determined by equating the first differential coefficient of Equation (43) to zero and solving for ω. Thus, writing Equation (43) in the form

$$I = E\sqrt{\left(\frac{1-2\omega^2 LC}{R^2+\omega^2L^2} + \omega^2C^2\right)}$$

and denoting the quantity in the bracket by u, we have

$$\frac{dI}{d\omega} = \frac{E}{2\sqrt{u}}\cdot\frac{du}{d\omega} = \frac{E}{2\sqrt{u}}\left\{\frac{-4\omega LC(R^2+\omega^2L^2)-2\omega L^2(1-2\omega^2LC)}{(R^2+\omega^2L^2)^2} + 2\omega C^2\right\} = 0$$

whence

$$\omega^2 = \left\{\frac{1}{LC}\sqrt{\left(2R^2\frac{C}{L}+1\right)}\right\} - \frac{R^2}{L^2}$$

This is greater than the value of ω^2 at resonance frequency—see Equation (41)—as the coefficient of the first term is greater than unity. The difference between these frequencies depends upon the values of R, L, C. For example, in the case of the circuit calculated in Table V the resonance frequency is equal to 48·9, and the frequency at which the line current is a minimum is equal to 50·3 cycles per second.

The minimum value of the line current may be obtained by substituting this value of ω^2 in Equation (43). For example, substituting the numerical value of 99875 [$= (2\pi \times 50{\cdot}3)^2$] and the values of R, L, C, from Table V, in Equation (43) we have

$$I_{\text{min.}} = 100\sqrt{\left(\frac{1-2\times 99875\times 0{\cdot}4\times 25\times 10^{-6}}{30^2+99875\times 0{\cdot}4^2} + 99875\times 25^2\times 10^{-12}\right)}$$

$$= 0{\cdot}1825 \text{ A.}$$

The value of the line current at resonance frequency is obtained by substituting the appropriate value of ω in Equation (43). Thus, at resonance frequency, $\omega^2 = \{(1/LC) - (R/L)^2\}$—see Equation (41)—and Equation (43) reduces to

$$I = ER(C/L).$$

Hence for the circuit calculated in Table V the line current at resonance frequency is

$$I = 100 \times 30 \times 25/(0{\cdot}4 \times 10^6)$$
$$= 0{\cdot}1875 \text{ A}$$

Joint impedance at resonance frequency. A parallel circuit consisting of an inductive resistance connected in parallel with a condenser possesses zero susceptance at resonance frequency, i.e. the expression for the joint impedance of the circuit contains no "reactance" term.

In the case of the circuit considered above, the line current at resonance frequency is given by

$$I = ER(C/L)$$

and therefore the joint impedance at this frequency is

$$Z = \frac{E}{I} = \frac{1}{R}\cdot\frac{L}{C}$$

i.e. the joint impedance at resonance frequency is equal to the product of the reciprocal of the resistance of the inductive branch and a numerical coefficient, the value of which is equal to the ratio L/C.

For the special case when $L = C$, we have

$$Z = 1/R.$$

Another case of interest is where the condenser branch contains resistance. In this case, let R_o denote the resistance in series with the condenser, and R the resistance in the inductive branch, as above. Then the wattless component of the current in the condenser branch is now given by

$$I_2 \sin \varphi_2 = \frac{E}{\omega C[R_o^2 + (1/\omega C)^2]}$$

The wattless component of the current in the inductive branch is given by

$$I_1 \sin \varphi_1 = \frac{E\omega L}{R^2 + \omega^2 L^2}$$

Hence the condition for resonance is

$$I_1 \sin \varphi_1 = I_2 \sin \varphi_2$$

i.e.
$$\frac{\omega L}{1/\omega C} = \frac{R^2 + \omega^2 L^2}{R_o^2 + (1/\omega C)^2} = \frac{Z^2}{Z_o^2}$$

Thus, for resonance, the ratio of the reactances of the two branches must equal the ratio of their impedances squared.

Solving for ω, we have

$$\omega^2 = \frac{R^2 - L/C}{CL(R_o^2 - L/C)}$$

The special cases are (1) when $R_o = R$; (2) when $R = \sqrt{(L/C)}$, $R_o = \sqrt{(L/C)}$; (3) when $R_o = R$, $L = C$. The values of ω corresponding to the resonance frequencies in these cases are (1) $\omega = \sqrt{(1/CL)}$; (3) $\omega = 1/L = 1/C$.

The line current at resonance frequency is equal to the sum of the power components of the branch-circuit currents, i.e.

$$I = I_1 \cos \varphi_1 + I_2 \cos \varphi_2$$

$$= \frac{ER}{R^2 + \omega^2 L^2} + \frac{ER_o}{R_o^2 + (1/\omega C)^2}$$

$$= E\left(\frac{R}{R^2 + \omega^2 L^2} + \frac{R_o}{R_o^2 + (1/\omega C)^2}\right)$$

Hence, substituting for the values of R_0 and ω corresponding to the above special cases, we have for the line currents at the resonance frequencies—

(1) $\omega^2 = 1/LC$ $\qquad I = E\,\{2R/[R^2 + (L/C)]\}$

(2) At all frequencies $\qquad I = E\,\sqrt{(C/L)}$

(3) $\omega^2 = 1/L^2 = 1/C^2$ $\qquad I = E\,[R/(R^2 + 1)]$

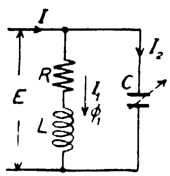


Fig 57

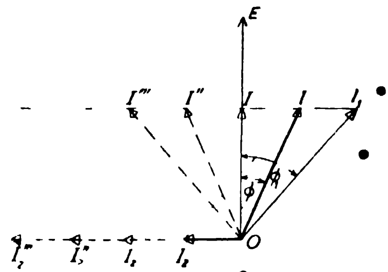


Fig. 58

Circuit and Vector Diagrams for Parallel Circuit of Variable Impedance (Variable Condenser in One Branch)

Whence the impedances at the resonance frequencies are

(1) $$Z = \frac{R}{2} + \frac{1}{R}\left(\frac{1}{2}\frac{L}{C}\right)$$

(3) $$Z = \frac{R}{2} + \frac{1}{2R}$$

For the second case—when $R = R_0 = \sqrt{(L/C)}$ the impedance at all frequencies is equal to $\sqrt{(L/C)}$.

In each of the cases (1), (3), the joint conductance of the circuits at resonance frequency is equal to twice the conductance of either of the branches. Thus the joint conductance, g, is equal to the sum of the separate conductances g_1, g_2. Now $g_1 = R/(R^2+\omega^2L^2)$, and $g_2 = R_0/[R_0^2+(1/\omega C)^2]$. Hence, when $R = R_0$, and $\omega^2=(1/LC)$; $g_1 = R/[R^2 + (L/C)]$, $g_2 = R/[R^2+(L/C)]$. Whence $g = 2g_1 = 2g_2$, and similarly for the case when $R_0 = R$, $L = C$, and $\omega = 1/L = 1/C$.

Parallel circuits of variable impedance. (2) *Constant resistance and inductance in one branch, variable reactance in the other branch.* The case which is of practical importance and which will be considered here is where the variable reactance consists of a condenser of adjustable capacity. The circuit diagram is shown in Fig. 57, and the vector diagram in Fig. 58. Assuming constant

supply voltage and frequency, the current, I_1, in the branch of constant impedance will be constant, while that, I_2, in the branch containing the condenser will be directly proportional to the capacity, e.g. $I_2 = \omega CE = kC$. The phase differences between these currents and the supply E.M.F. are constant: that between the supply E.M.F. and I_1 is $\varphi_1 = \tan^{-1}\omega L/R$ (lagging); that between the supply E.M.F. and I_2 is 90° (leading).

The line current, I, is equal to the vector sum of I_1 and I_2. Since I_1 is constant in magnitude and direction, and I_2 is of variable magnitude but fixed direction, the locus of the extremity of the line current vector will be a straight line drawn through I_1 parallel to OI_2 (Fig. 58). Thus, as the capacity is varied from zero, the line current is brought more into phase with the supply E.M.F., i.e. the effective reactance of the circuit as a whole is diminished and the power factor is improved. At a particular value of the capacity the line current is exactly in phase with the supply E.M.F., and has its minimum value*: at higher values of the capacity the line current increases in magnitude and leads the supply E.M.F.

If values of line current and capacity are plotted, we obtain a V curve, as shown in Fig. 59, and if the power factor is also plotted we obtain an inverted V curve (see Fig. 59).

The curves of Fig. 59 are calculated for a parallel circuit similar to that of Fig. 57, supplied at a constant E.M.F. of 200 V., and a constant frequency of 50 cycles per second. The inductive branch has a resistance of 40 ohms and a constant inductance of 0·54 henries: the capacitive branch contains a condenser, the capacity of which is adjustable between zero and 30 μF.

The method of calculating these curves is as follows—

The impedance of the inductive branch is

$$Z = \sqrt{\{40^2 + (2\pi \times 50 \times 0{\cdot}54)^2\}} = 174{\cdot}4 \text{ ohms.}$$

Whence

$$I_1 = 200/174{\cdot}4 = 1{\cdot}146 \text{ A.}$$

$$\cos \varphi_1 = R/Z = 40/174{\cdot}4 = 0{\cdot}23$$

The line current at unity power factor is

$$I = I_1 \cos \varphi_1$$
$$= 1{\cdot}146 \times 0{\cdot}23 = 0{\cdot}264 \text{ A.}$$

The capacity required to give unity power factor is determined from the condition that the charging current of the condenser must

* The condition which gives unity power factor is the same as that which gives the resonance frequency, for the parallel circuit, equal to the supply frequency. Under these conditions the stored energy of the circuits is transferred from the condenser to the inductance, and *vice versa*, during successive quarter periods.

be equal to the wattless current taken by the inductive resistance, i.e. $I_2 = I_1 \sin \varphi_1 = \omega CE$,

whence $$C = I_1 \sin \varphi_1/\omega E$$
$$= \{1{\cdot}146\sqrt{(1 - 0{\cdot}23^2)}\}/(2\pi \times 50 \times 200)$$
$$= 1{\cdot}116/62800 = 17{\cdot}75 \times 10^{-6} \text{ F.}$$

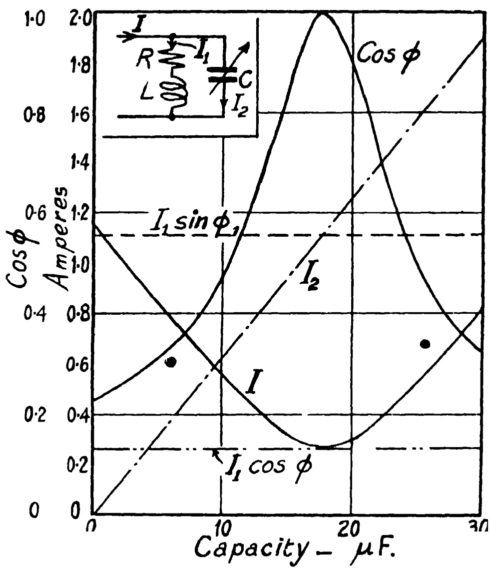


FIG. 59. Characteristic Curves for Parallel Circuit of Variable Impedance (R = 40 Ω., L = 0·51 H., C Variable)

For any other value of capacity, such as 10 μF., we have

Wattless component of line current $= I_1 \sin \varphi_1 - \omega CE$
$= 1{\cdot}116 - 62800 \times 10 \times 10^{-6}$
$= 0{\cdot}487$ A.

Power component of line current $= I_1 \cos \varphi_1$
$= 0{\cdot}2635$ A.

Line current $= \sqrt{(0{\cdot}2635^2 + 0{\cdot}487^2)}$
$= 0{\cdot}554$ A.

Power factor $=$ power component/line current
$= 0{\cdot}2635/0{\cdot}554$
$= 0{\cdot}475$ (lagging).

Other points are calculated in a similar manner, and are given in Table VI.

Practical applications. The curves of Fig. 59 show that a circuit consisting of a fixed inductance connected in parallel with a condenser possesses the property that if the capacity of the condenser be chosen suitably, the power factor of the combined circuit is higher than that of the inductive branch alone. This property is of considerable value in practice. For example, if a particular circuit possesses a low (lagging) power factor, the effects of this low power factor on the supply system may be avoided by connecting in parallel with the circuit a condenser of suitable capacity. Condensers used in this manner for improving the power factor of power circuits are generally of the oil-immersed, foiled-paper, type (see p. 64).

In the application of condensers to such circuits it is important to observe that the capacity required for corrective purposes is proportional to $\cos \varphi_1 (\tan \varphi_1 - \tan \varphi)$, where φ_1 is the phase difference for the inductive circuit alone and φ is the resultant phase difference for the combined circuits. Thus, if the resultant power factor is to be unity (i.e. $\varphi = 0$), the capacity required will be proportional to $\sin \varphi_1$, but if a lower resultant power factor is required, the capacity will be smaller. For example, if the original power factor is 0·707, corresponding to $\varphi_1 = 45°$, and the resultant power factor is to be unity, the capacity required will be proportional to $\cos 45° = 0{\cdot}707$. If, however, the resultant power factor is to be 0·95 (lagging), the capacity will be proportional to $\cos 45° \; (\tan 45° - \tan 18{\cdot}2°) = 0{\cdot}707 \; (1 - 0{\cdot}3288) = 0{\cdot}475$, which is about two-thirds of that required to obtain a power factor of unity. Hence, in cases where a resultant power factor of, say, 0·95 is satisfactory, the cost of the condenser required for the purpose will be considerably lower than that which would be necessary for correcting the power factor to unity.

Example. A 40 H.P., 550 V., 50 cycle, alternating current motor has a power factor at full load (which corresponds to an input of 34 kW.) of 0·85 (lagging). What capacity of condenser, connected in parallel with the motor, is required to obtain a resultant power factor of 0·98 (lagging)? If this condenser is permanently connected in parallel with the motor, what will be the resultant power factors when the motor is operating at half load and quarter load, assuming the power input to the motor at these loads to be 17·5 and 9·7 kW. respectively, and the corresponding power factors to be 0·78 and 0·63? Neglect losses in the condenser.

The current input to the motor at full load $= 34000/(550 \times 0{\cdot}85) = 72{\cdot}7$A.
,, ,, ,, ,, ,, half ,, $= 17500/(550 \times 0{\cdot}78) = 40{\cdot}7$A
,, ,, ,, ,, ,, quarter ,, $= 9700/(550 \times 0{\cdot}63) = 28$ A.

Wattless component of current input at full load $= 72{\cdot}7\sqrt{(1-0{\cdot}85^2)} = 38{\cdot}4$A.
,, ,, ,, ,, half ,, $= 40{\cdot}7\sqrt{(1-0{\cdot}78^2)} = 25{\cdot}5$A.
,, ,, ,, ,, quarter ,, = [illegible] $\sqrt{(1-0{\cdot}63^2)} = 21{\cdot}7$A.

Power component of full-load current $= 34000/550 = 61 \cdot 8$ A.

$\therefore$ Line current at 0·98 power factor and full load on motor $= 61 \cdot 8/0 \cdot 98 = 63 \cdot 1$ A.

Wattless component of this current $= 63 \cdot 1\sqrt{(1 - 0 \cdot 98^2)} = 12 \cdot 6$ A.

$\therefore$ Charging current of condenser = Difference between wattless components at power factors 0·85 and 0·98

$= 38 \cdot 4 - 12 \cdot 6$

$= 25 \cdot 8$ A.

Hence capacity of condenser $= 25 \cdot 8 \times 10^6/(2\pi \times 50 \times 550) = 149\ \mu$F.

NOTE —Capacity of condenser required to give a power factor of unity with full load on motor $= 38 \cdot 4 \times 10^6/(2\pi \times 50 \times 550) = 222\ \mu$F.

Assuming constant line voltage and frequency, the charging current of the condenser will remain constant.

Hence the wattless component of the line current when the motor is operating at half load $= 25 \cdot 5 - 25 \cdot 8 = -0 \cdot 3$ A.

[The minus sign indicates that this component is leading the impressed E.M.F.]

Power component of line current when the motor is operating at half load

$= 17500/550 = 31 \cdot 8$ A.

$\therefore$ Line current $= \sqrt{(31 \cdot 8^2 + 0 \cdot 3^2)} = 31 \cdot 8$ A.

Power factor $= 31 \cdot 8/31 \cdot 8 = 1 \cdot 0$

[NOTE.—Actually the phase difference is about 0·6° (leading).]

Wattless component of line current when the motor is operating at quarter load $21 \cdot 7 - 25 \cdot 8 = -4 \cdot 1$ A.

Power component of line current at this load

$9700/550 = 17 \cdot 65$ A·

$\therefore$ Line current $\sqrt{(17 \cdot 65^2 + 4 \cdot 1^2)} = 18 \cdot 1$ A

Power factor $= 17 \cdot 65/18 \cdot 1 = 0 \cdot 986$ (leading)

TABLE VI

Calculations for Fig. 59. Parallel circuit: inductive branch, $R = 40$ O., $L = 0 \cdot 54$ H.; condenser branch, C variable, 0 to 30 μF. Supply pressure = 200 V. (constant), frequency = 50 (constant).

$C(\mu$F)	$I_2 = \omega CE$	$I_1 \sin \varphi_1 - I_2$	$I_1 \cos \varphi_1$	$I = \sqrt{\{(I_1 \cos \varphi_1)^2 + (I_1 \sin \varphi_1)^2\}}$	$\cos \varphi$
0	0	1·116	0·2635	1·146	↑ 0·23
5	0·314	0·801		0·844	lag 0·312
10	0·628	0·487		0·554	0·475
15	0·942	0·173		0·315	↓ 0·836
17·75	1·116	0		0·2635	1·0
20	1·256	−0·141		0·29	↑ 0·909
25	1·57	−0·455		0·526	lead 0·5
30	1·885	−0·77	↓	0·814	0·323 ↓

Parallel circuits of variable impedance. (3) *Constant impedance in one branch, variable impedance in other branch.* Consider the

circuits represented in Fig. 60, in which one branch contains a variable resistance in series with a fixed reactance, X_2. Denoting the conductances, susceptances, and admittances of the circuits by g_1, g_2 ; b_1, b_2 ; Y_1, Y_2, respectively, we have

$$g_1 = R_1/(R_1^2 + X_1^2) \qquad g_2 = R_2/(R_2^2 + X_2^2)$$
$$b_1 = X_1/(R_1^2 + X_1^2) \qquad b_2 = X_2/(R_2^2 + X_2^2)$$
$$Y_1 = g_1 - b_1 \qquad Y_2 = g_2 - jb_2$$

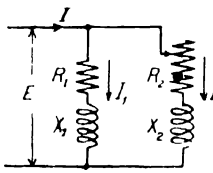


Fig. 60

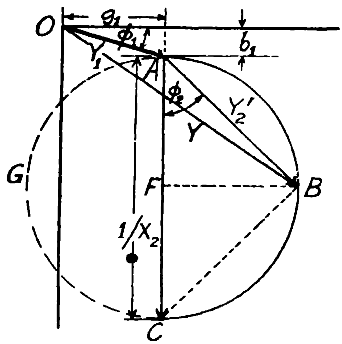


Fig. 61

Circuit and Vector Diagrams for Parallel Circuit of Variable Impedance (Variable Resistance in One Branch)

The joint admittance, Y, is equal to the sum of the separate admittances, i.e. $Y = Y_1 + Y_2$. Now Y_2 is variable, being zero when $R_2 = \infty$, and equal to $j(1/X_2)$ when $R_2 = 0$. The joint admittance, Y, may be obtained either by calculation or graphically, the graphical construction possessing the advantage that the magnitudes and phase differences of the branch and line currents may be obtained at the same time as the joint admittance.

The admittance diagram is shown in Fig. 61. The vector OA, in the fourth quadrant, represents the admittance $Y_1 = g_1 - jb_1$. The vector AC—parallel to the vertical axis and of length equal to $1/X_2$ on the admittance scale—represents the maximum value of the variable admittance Y_2, this value corresponding to $R_2 = 0$. A semicircle, ABC, described on AC, therefore, gives the locus of the admittance vector Y_2 when the resistance, R_2, is varied from 0 to ∞.

Proof. Let the vector AB, Fig. 61, represent any particular value, Y'_2, of the admittance Y_2, corresponding to a particular value, R'_2 of the variable,

resistance. Then $Y'_2 = g'_2 + jb'_2$, where $g'_2 = R'_2/(R'^2_2 + X_2^2)$, and $b'_2 = X_2/(R'^2_2 + X_2^2)$. Now if AF is the projection of AB on AC

$$AF = Y'_2 \cos \varphi_2 = b'_2 = \frac{X_2}{R'^2_2 + X_2^2}; \quad FB = Y'_2 \sin \varphi_2 = g'_2 = \frac{R'_2}{R'^2_2 + X_2^2}$$

$$Y'^2_2 = AB^2 = AF^2 + FB^2 = g'^2_2 + b'^2_2 = \left(\frac{R'_2}{R'^2_2 + X_2^2}\right)^2 + \left(\frac{X_2}{R'^2_2 + X_2^2}\right)^2 = \frac{1}{R'^2_2 + X_2^2}$$

But $Y'_2 \cos \varphi_2 = X_2/(R'^2_2 + X_2^2)$.

Hence $1/(R'^2_2 + X_2^2) = (Y'_2 \cos \varphi_2)/X_2$

i.e. $Y'^2_2 = (Y'_2 \cos \varphi_2)/X_2$

or $Y'_2 = (1/X_2) \cos \varphi_2$

Whence $Y'_2/\cos \varphi_2 = 1/X_2 =$ a constant.

If BC be drawn perpendicular to AB, then

$$AC = Y'_2/\cos \varphi_2 = 1/X_2.$$

Hence AC is of constant value, and since angle ABC is a right angle, point B lies on a semicircle described on AC as diameter.

The joint admittance, Y, is therefore represented by the vector OB, Fig. 61, and the locus of this vector when the resistance, R_2, is varied is the semicircle ABC referred to the pole O.

When the variable branch contains a condenser of fixed capacity, instead of a fixed inductance, the locus of the joint impedance vector is the semicircle AGC, the pole being at O, as before.

The vector, OB, Fig. 61, representing the joint admittance, also represents the line current to a scale E times the admittance scale, where E is the line voltage. Moreover, the vectors OA, AB, represent the branch currents, I_1, I_2, to this scale. Thus, *by a suitable change of scale, an admittance diagram is converted into a current diagram.*

Determination of the diagram for the joint impedance. Since the reciprocal of admittance is equal to impedance, the inversion of the locus of the joint admittance vector, in Fig. 61, will give the locus of the image of the vector of the joint impedance. Now the inversion of a circle with respect to a pole, or centre of inversion, external to the circle is another circle having its centre on the line joining the pole and the centre of the original circle. The position of the centre of this new circle is determined from the condition that the common tangent drawn from the pole to both circles must be divided at the points of contact such that the product of the lengths from the pole to the point of contact of each tangent must be equal to the inversion constant, K_i.

Proof. Let $ABCD$, Fig. 62, be a circle of which an inversion is required with respect to the pole, or inversion centre, O. From O draw the tangent OA and divide it at A_1 such that $OA.OA_1$ is equal to the inversion constant, K_i. At A_1 draw a perpendicular, A_1Q_1, to meet the line joining the centre of the circle $ABCD$ and the pole O. Then the point of intersection, Q_1, is the centre of the circle which is the inversion of circle $ABCD$.

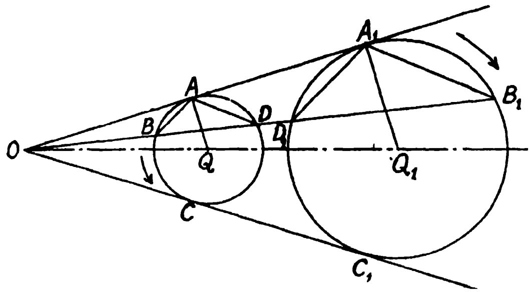


FIG. 62.—Inverse Circles

From O draw any line, such as $OBDB_1D_1$, to cut both circles at points D, B, B_1, D_1. Join AD, AB, A_1B_1, A_1D_1. Then AD is parallel to A_1B_1; AB is parallel to A_1D_1. Therefore triangles OAD, OA_1B_1 are similar; also triangles OAB, OA_1D_1 are similar. Hence $OD : OA = OB_1 : OA_1$; $OB : OA = OD_1 : OA_1$. Whence $OB.OB_1 = OD.OD_1 = OA.OA_1$. Points A, A_1; B, B_1; D, D_1; are therefore inverse points, and the arc ABC is inverse to the arc $A_1B_1C_1$. Similarly the arc $A_1D_1C_1$ is inverse to the arc ADC.

To construct the impedance diagram corresponding to the admittance diagram of Fig. 61, which is reproduced with the same lettering in Fig. 63, draw the line OQ, joining the centre, Q, of the semicircle ACB and the pole O: draw the tangent OD and determine the point D_1 such that $OD.OD_1 = K_i = 1/mn$, where K_i is the inversion constant and m, n, are the scales for admittance and impedance respectively. From D_1 draw a perpendicular to meet the line OQ, produced, if necessary, at Q_1, which is the centre of the inverse circle.

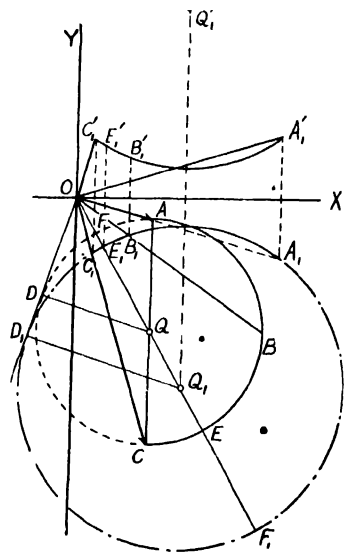


FIG. 63.—Impedance Diagram Corresponding to Admittance Diagram of Fig. 61

An alternative method of construction, which does not require a

knowledge of the inversion constant and which permits the scale for the inverse circle to be selected according to the space available and to the diameter required for this circle, is as follows: Join OQ and produce to the circumference of the admittance circle $ABECF$. Measure the lengths of the intercepts OF, OE, for the points where this line intersects the admittance circle, and calculate from the scale for admittance, the values, in mhos, of the admittances thus represented. Then the reciprocal of the admittance represented by OF gives the maximum value, in ohms, of the joint impedance. Hence, select the scale for impedance such that this quantity is represented by a vector, OF_1, of convenient length. Point F_1 is therefore inverse to point F. Determine similarly, to the impedance scale, the point E_1 (on OQ) representing the minimum value of the joint impedance, corresponding to the maximum value of the joint admittance OE. Hence, points F_1, E_1, are on the diameter of the inverse circle, the centre of which, Q_1, is obtained by bisecting the line joining F_1E_1.

The arc $A_1B_1E_1C_1$ is the inversion, with respect to the pole O, of the semicircle $ABEC$; the points $A_1B_1E_1C_1$ being the inverse points to $ABEC$ respectively. Thus, as the joint admittance vector moves along the semicircle $ABEC$ from A to C, the image of the joint impedance vector moves along the arc $A_1B_1E_1C_1$ from A_1 to C_1. Finally, by determining the image of this arc in the first quadrant we have the locus of the joint impedance vector. This locus is shown in Fig. 63 by the arc $A_1'B_1'E_1'C_1'$. Thus the vector OC_1' represents the joint impedance when the variable resistance, R_2, is zero, and the vector OA_1' represents the impedance when the variable resistance is infinite.

The diagram of Fig. 61 has a more extensive application than that of obtaining the joint admittance and joint impedance of the circuits represented in Fig. 60. We shall show in the following chapter that all quantities relating to the performance of the circuit when the resistance R_2 is varied may be obtained from this diagram.

WORKED EXAMPLES ON SERIES AND PARALLEL CIRCUITS

General remarks concerning the solution of problems on series and parallel circuits. *Series Circuits.* In problems on series circuits it is necessary to determine the impedance of the circuit for the purpose of calculating the current. The calculations are of a simple nature and are easily effected by algebraic methods.

Thus the impedance of the circuit is given by

$$Z = \sqrt{\{R^2 + (\omega L - 1/\omega C)^2\}}$$

where R is the total resistance of the circuit, L the total inductance, and C the total capacity.

The current is given by

$$I = E/Z,$$

and the phase difference between terminal E.M.F. and current is given by

$$\varphi = \tan^{-1}(\omega L - 1/\omega C)/R,$$

or by

$$\varphi = \cos^{-1}R/Z.$$

In connection with the first expression for the phase difference, it should be observed that a plus (+) sign for φ, corresponding to $\omega L > 1/\omega C$, denotes "lag," and a minus (–) sign, corresponding to $\omega L < 1/\omega C$, denotes "lead," of the current with respect to the impressed E.M.F.

When the second expression, or cosine, is used for obtaining φ, no indication is given as to whether the current is leading or lagging with respect to the impressed E.M.F. In this case it will be necessary to determine whether $\omega L \gtrless 1/\omega C$. Hence when the actual angle φ is required it should be calculated from the "tangent" expression, but when only the power factor of the circuit is required, the "cosine" expression should be used.

For the special cases of series circuits the equations—some of which were deducted in Chapter III—follow directly from the general equation and are tabulated below—

Special Case.	Impedance.	Current.	Tan φ	φ
R only	R	E/R	0	0
L only	ωL	$E/\omega L$	∞	90° (lag)
C only	$1/\omega C$	ωCE	$-\infty$	90° (lead)
R and L in series	$\sqrt{(R^2 + \omega^2 L^2)}$	$E/\sqrt{(R^2 + \omega^2 L^2)}$	$\omega L/R$	0 – 90° (lag)
R and C in series	$\sqrt{\{R^2 + (1/\omega C)^2\}}$	$E/\sqrt{\{R^2 + (1/\omega C)^2\}}$	$-1/\omega CR$	0 – 90° (lead)
L and C in series	$\sqrt{(\omega L - 1/\omega C)^2}$	$E/\sqrt{(\omega L - 1/\omega C)^2}$	∞ or 0	± 90° or 0, according to whether $\omega L \gtrless < 1/\omega C$

Parallel circuits. In the majority of problems on parallel circuits it is generally simpler to obtain the line current from the joint admittance of the circuits rather than from their joint impedance, and in these cases the results are best calculated by the symbolic method.

The numerical value of the joint admittance is given by

$$Y = \sqrt{\{(g_1 + g_2 + g_3 + \ldots)^2 + (b_1 + b_2 + b_3 + \ldots)^2\}}$$

where g_1, g_2, g_3 . . ., b_1, b_2, b_3, . . . are the conductances and admittances, respectively, of the several branch circuits.

The line current is given by

$$I = EY,$$

and its phase difference with respect to the impressed E.M.F. is given by

$$\varphi = \tan^{-1}(b_1 + b_2 + b_3 + \ldots)/(g_1 + g_2 + g_3 + \ldots).$$

Series-parallel circuits. In all problems on series-parallel circuits the parallel portions of the circuit are treated separately and the joint impedances determined. The series-parallel circuit is then replaced by an equivalent series circuit and the total impedance is readily determined.

Example 1. A coil having a resistance of 5 ohms and an inductance of 0·02 henry is arranged in parallel with another coil having a resistance of 1 ohm and an inductance of 0·08 H. Calculate the current flowing through each coil when a pressure of 100 volts at 50 cycles is applied to them. Find the total current passing, and estimate the resistance of a single coil which will take the same current at the same power factor. [*L.U.*, 1922.]

Solution by the trigonometric method

The steps in this method of solution are –

1. Calculate the current in each branch of the parallel circuit, and the phase difference between each of these currents and the impressed E.M.F.
2. Compound these currents to obtain the line current, determining also its power and energy components.
3. Calculate the joint impedance of the parallel circuits.
4. Thence, from the impedance and the power and wattless components of the line current, determine the resistance and reactance of the single coil which will take the same current, at the same power factor, as the parallel circuit.

Thus, the impedance of the 5 O., 0·02 H. coil is

$$Z_1 = \sqrt{[5^2 + (314 \times 0{\cdot}02)^2]} = 8{\cdot}02 \text{ O.}$$

Whence, $I_1 = 100/8{\cdot}02 = 12{\cdot}46$ A.

$$\cos\varphi_1 = R_1/Z_1 = 5/8{\cdot}02 = 0{\cdot}624$$

$$\sin\varphi_1 = X/Z_1 = 314 \times 0{\cdot}02/8{\cdot}02 = 0{\cdot}783$$

The impedance of the 1 O. 0·08 H. coil is

$$Z_2 = \sqrt{[1^2 + (314 \times 0{\cdot}08)^2]} = 25{\cdot}12 \text{ O.}$$

Whence $I_2 = 100/25{\cdot}12 = 3{\cdot}98$ A.

$$\cos\varphi_2 = R_2/Z_2 = 1/25{\cdot}12 = 0{\cdot}0398$$

$$\sin\varphi_2 = X_2/Z_2 = 314 \times 0{\cdot}08/25{\cdot}12 = 1{\cdot}0 \text{ approx.}$$

The line current is

$$I = \sqrt{(I_1^2 + I_2^2 + 2I_1I_2\cos(\varphi_2 - \varphi_1)} = 15{\cdot}85 \text{ A.},$$

and the joint impedance of the parallel circuit is

$$Z = 100/15{\cdot}85 = 6{\cdot}31 \text{ O.}$$

Power component of line current $= I\cos\varphi = I_1\cos\varphi_1 + I_2\cos\varphi_2$
$= 12{\cdot}46 \times 0{\cdot}624 + 3{\cdot}98 \times 0{\cdot}0398$
$= 7{\cdot}94$ A.

Wattless component of line current $= I\sin\varphi = I_1\sin\varphi_1 + I_2\sin\varphi_2$
$= 12{\cdot}46 \times 0{\cdot}783 + 3{\cdot}98 \times 1{\cdot}0$
$= 13{\cdot}74$ A.

$\therefore \quad \sin\varphi = I\sin\varphi/I = 13{\cdot}74/15{\cdot}85 = 0{\cdot}866,$

and $\quad \cos\varphi = I\cos\varphi/I = 7{\cdot}94/15{\cdot}85 = 0{\cdot}5.$

Now if R and X are the resistance and reactance, respectively, of the single coil, we have

$$R = Z\cos\varphi = 6{\cdot}31 \times 0{\cdot}5 = 3{\cdot}16\ \Omega.$$

$$X = Z\sin\varphi = 6{\cdot}31 \times 0{\cdot}866 = 5{\cdot}46\ \Omega.$$

Whence, $\quad L = 5{\cdot}46/(2\pi \times 50) = 0{\cdot}0174$ H.

Solution by the symbolic method.

Taking $\omega = 2\pi \times 50 = 314$, the admittance of the 5 Ω. 0·02 H. coil is

$$Y_1 = \frac{5}{5^2 + (314 \times 0{\cdot}02)^2} - j\left(\frac{314 \times 0{\cdot}02}{5^2 + (314 \times 0{\cdot}02)^2}\right)$$
$$= 0{\cdot}0777 - j\,0{\cdot}0975$$

Whence, $\quad Y_1 = \sqrt{(0{\cdot}0777^2 + 0{\cdot}0975^2)} = 0{\cdot}1246$ mho.

The admittance of the 1 Ω, 0·08 H. coil is

$$Y_2 = \frac{1}{1^2 + (314 \times 0{\cdot}08)^2} - j\left(\frac{314 \times 0{\cdot}08}{1^2 + (314 \times 0{\cdot}08)^2}\right)$$
$$= 0{\cdot}00158 - j\,0{\cdot}0397$$

Whence, $\quad Y_2 = \sqrt{(0{\cdot}00158^2 + 0{\cdot}0397^2)} = 0{\cdot}0398$ mho.

Hence, the joint admittance of the two coils connected in parallel is

$$Y = Y_1 + Y_2$$
$$= (0{\cdot}0777 + 0{\cdot}00158) - j(0{\cdot}0975 + 0{\cdot}0397)$$
$$= 0{\cdot}0793 - j\,0{\cdot}1372$$

Whence, $\quad Y = \sqrt{(0{\cdot}0793^2 + 0{\cdot}1372^2)} = 0{\cdot}1585$ mho.

The branch currents are

$$I_1 = 100\,Y_1 = 12{\cdot}46\text{ A.},$$
$$I_2 = 100\,Y_2 = 3{\cdot}98\text{ A},$$

and the line current is

$$I = 100\,Y = 15{\cdot}85\text{ A.}$$

The admittance of a single coil which will take this (line) current at the same power factor must equal the joint admittance of the above coils connected in parallel.

Hence if g and b denote the conductance and susceptance, respectively, of the coil,

$$Y = g - jb = 0{\cdot}0793 - j\,0{\cdot}1372$$

Whence, $\quad g = 0{\cdot}0793$

$b = 0{\cdot}1372$

$Y = \sqrt{(0{\cdot}0793^2 + 0{\cdot}1372^2)} = 0{\cdot}1584$

Now $g = R/(R^2 + X^2)$; $b = X/(R^2 + X^2)$

and $g^2 + b^2 = Y^2 = \dfrac{R^2}{(R^2 + X^2)^2} + \dfrac{X^2}{(R^2 + X^2)^2} = \dfrac{1}{R^2 + X^2}$

From these equations we obtain

$$R^2 + X^2 = 1/Y^2 = R/g = X/b.$$

$$\begin{aligned} \therefore \quad R &= g/Y^2 \\ &= 0{\cdot}0793/0{\cdot}1584^2 \\ &= 3{\cdot}16 \text{ O.} \\ X &= b/Y^2 \\ &= 0{\cdot}1372/0{\cdot}1584^2 \\ &= 5{\cdot}46 \text{ O.} \\ L &= 5{\cdot}46/(2\pi \times 50) \\ &= 0{\cdot}0174 \text{ H.} \end{aligned}$$

Example 2. If an inductive resistance, for which $R = 2$ ohms, $L = 0{\cdot}005$ henry, is connected in series with the circuit of the inductive coils in Example (1) above, to what value must the line voltage be raised in order that the currents through these coils may be the same as in the above case, the frequency being remaining unchanged? What will be the resultant power factor under these conditions?

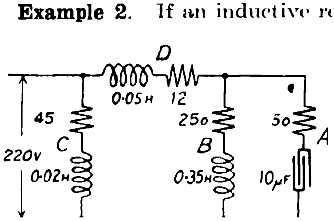


FIG. 64. Circuit Diagram for Worked Example No 3

Adopting the values of 3·16 O. and 5·46 O. obtained above for the joint resistance and reactance, respectively, of the parallel-connected inductive coils, the series-parallel circuit of this example may therefore be replaced by an equivalent series circuit having a total resistance of

$$3{\cdot}16 + 2 = 5{\cdot}16 \text{ O.},$$

and a total reactance of

$$5{\cdot}46 + (2\pi \times 50 \times 0{\cdot}005) = 7{\cdot}03 \text{ O.}$$

Hence the total impedance $= \sqrt{(5{\cdot}16^2 + 7{\cdot}03^2)} = 8{\cdot}72$ O.

Therefore the impressed E.M.F. $= 15{\cdot}85 \times 8{\cdot}72 = 138{\cdot}2$ V.

Resultant power factor $= 8{\cdot}72/\sqrt{(5{\cdot}16^2 + 7{\cdot}03^2)}$

$= 5{\cdot}16/8{\cdot}72 = 0{\cdot}592.$

Example 3. The circuit shown in Fig. 64 consists of three parallel branches A, B, C, the branches A and B being connected to branch C through an inductive resistance D. Determine the current in each branch circuit, the line current, the resultant power factor, and the power supplied to the circuits. Branch A consists of a condenser of 10 μF. capacity in series with a resistance of 50 O.; branch B consists of an inductive resistance, for which $R = 25$ O., $L = 0{\cdot}35$ H.; branch C consists of an inductive resistance, for which $R = 45$ O., $L = 0{\cdot}02$ H. The series inductive resistance D has a resistance of 12 O. and an inductance of 0·05 H. The supply pressure is 220 V. and the frequency 50 cycles per second.

Denoting the admittances of the branches A, B, C, by Y_A, Y_B, Y_C, respectively, and the conductances and susceptances by g_A, g_B, g_C, and b_A, b_B, b_C, respectively, we have

$$g_A = \frac{50}{50^2 + (10^6/314 \times 10)^2} = 0.482 \times 10^{-3}$$

$$g_B = \frac{25}{25^2 + (314 \times 0.35)^2} = 1.965 \times 10^{-3}$$

$$g_C = \frac{45}{45^2 + (314 \times 0.02)^2} = 21.8 \times 10^{-3}$$

$$b_A = -\frac{10^6/(314 \times 10)}{50^2 + (10^6/(314 \times 10))^2} = -3.07 \times 10^{-3}$$

$$b_B = \frac{314 \times 0.35}{25^2 + (314 \times 0.35)^2} = 8.65 \times 10^{-3}$$

$$b_C = \frac{314 \times 0.02}{45^2 + (314 \times 0.02)^2} = 3.04 \times 10^{-3}$$

Whence,
$$Y_A = g_A - jb_A = 0.482 \times 10^{-3} + j3.07 \times 10^{-3}$$
$$Y_B = g_B - jb_B = 1.965 \times 10^{-3} - j8.65 \times 10^{-3}$$
$$Y_{A+B} = (0.482 + 1.965)10^{-3} - j(8.65 - 3.07)10^{-3}$$
$$= 2.447 \times 10^{-3} - j5.58 \times 10^{-3}$$

The joint impedance, Y_1, of the series parallel branch ABD is therefore

$$Y_1 = \frac{1}{Z_1} = \frac{1}{R_D + jX_D + 1/(Y_A + Y_B)} = \frac{1}{R_D + jX_D + 1/(g - jb)}$$

$$= \frac{1}{R_D + \dfrac{g}{g^2 + b^2} + j\left(X_D + \dfrac{b}{g^2 + b^2}\right)}$$

where g and b are the joint conductances and susceptances, respectively, of the parallel branches A and B, and R_D, X_D, are the resistance and reactance, respectively, of the series portion, D.

Substituting numerical values for these quantities, we obtain

$$\frac{g}{g^2 + b^2} = 65.9 \qquad \frac{b}{g^2 + b^2} = 150.3$$

$$\therefore \; Y_1 = \frac{1}{(12 + 65.9) + j(314 \times 0.05 + 150.3)} = \frac{1}{77.9 + j166}$$

$$= \frac{77.9}{77.9^2 + 166^2} - j\left(\frac{166}{77.9^2 + 166^2}\right)$$

$$= 2.315 \times 10^{-3} - j4.93 \times 10^{-3}$$

Whence the joint admittance of the complete circuit shown in Fig 64 is

$$Y = Y_1 + Y_C = (2.315 + 21.8)10^{-3} - j(4.93 + 3.04)10^{-3}$$
$$= 24.115 \times 10^{-3} - j7.97 \times 10^{-3}$$

Therefore the line current $= 220 . Y$

$= 220\sqrt{(0.0241^2 + 0.00797^2)} = 5.59$ A

Power factor $= 0.0241/\sqrt{(0.0241^2 + 0.00797^2)} = 0.95$ (lagging)

Power supplied $= 220 \times 5.59 \times 0.95 = 1169$ W.

Current in branch C $\quad 220.Y_C$

$= 220\sqrt{(0{\cdot}0218^2 + 0{\cdot}00304^2)} = 4{\cdot}84$ A.

Current (I_D) in the series-parallel branch ABD

$= 220.Y_1$

$= 220\sqrt{(0{\cdot}0023^2 + 0{\cdot}00493^2)} = 1{\cdot}2$ A.

Potential difference across the parallel-connected branches A, B

$= 220 - I_D Z_D$

$= 220 - [\{220 \times 10^{-3}(2{\cdot}315 - j4{\cdot}93)\}\{12 + j(314 \times 0{\cdot}05)\}]$

$= 220 - (23{\cdot}13 - j5)$

$= 197 + j5$

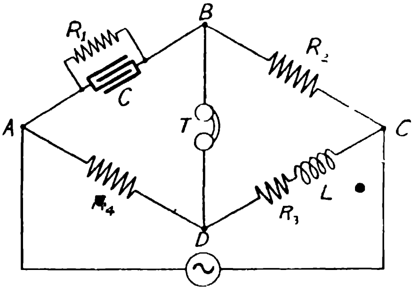


FIG. 65.—Circuit Diagram of Wheatstone Bridge for Alternating Current Measurements

Current in branch A—I_A $= Y_A$

$= (0{\cdot}482 \times 10^{-3} + j3{\cdot}07 \times 10^{-3})(197 + j5)$

$= 0{\cdot}079 + j0{\cdot}607$

$I_A = \sqrt{(0{\cdot}079^2 + 0{\cdot}607^2)} = 0{\cdot}612$ A.

Current in branch B—$I_B = Y_B(197 + j5)$

$= (1{\cdot}965 \times 10^{-3} - j8{\cdot}65 \times 10^{-3})(197 + j5)$

$= 0{\cdot}43 - j1{\cdot}695$

$\therefore \quad I_B = \sqrt{(0{\cdot}43^2 + 1{\cdot}695^2)} = 1{\cdot}75$ A.

[Note.—All currents are referred to the line voltage which is considered as the vector of reference.]

Example 4 The arms of a Wheatstone bridge, taken in order, consist of (i) a condenser shunted by a non-inductive resistance, (ii) a non-inductive resistance, (iii) an inductive resistance, (iv) a second non-inductive resistance. The bridge is supplied with alternating current of sine wave-form and a telephone is used to indicate balance. Find the condition for which there will be no sound in the telephone. (*L.U.*, 1921.)

[Note.—The Wheatstone bridge adapted for alternating currents is employed in practice for measuring inductance, capacity, and dielectric losses at commercial and higher frequencies. A telephone may be used as a detector for balance at audio-frequencies (800 to 2000 cycles), but for other frequencies a vibration galvanometer must be employed. Alternative forms of the bridge are discussed in Chapter XV.]

The circuits of the bridge are represented in Fig. 65.

In balancing the bridge the currents in the adjacent arms must be adjusted to equality of phase as well as magnitude. Therefore when the bridge is balanced the ratio of the impedances of the several arms must be the same as that of the resistances of the corresponding arms in a bridge supplied with steady currents. Hence if Z_1, Z_2, Z_3, Z_4 denote the impedances of the arms AB, BC, CD, DA, respectively (Fig. 65), the condition for balance is

$$Z_1 Z_3 = Z_2 Z_4.$$

Now $Z_1 = \dfrac{1}{\dfrac{1}{R_1} + j\omega C}$; $Z_2 = R_2$; $Z_3 = R_3 + j\omega L$; $Z_4 = R_4$.

$$\therefore \qquad \left(\frac{1}{\dfrac{1}{R_1} + j\omega C}\right)(R_3 + j\omega L) = R_2 R_4$$

i.e.
$$R_3 + j\omega L = R_2 R_4 \left(\frac{1}{R_1} + j\omega C\right)$$
$$= \frac{R_2 R_4}{R_1} + j\omega C R_2 R_4$$

For this equation to be true we must have equality between both the in-phase and quadrature parts. Thus

$$R_3 = R_2 R_4 / R_1$$

or
$$\frac{R_1}{R_2} = \frac{R_4}{R_3}$$

which is the condition for balance with steady currents.

Also
$$\omega L = \omega C R_2 R_4 = \omega C R_1 R_3$$
$$\frac{\omega L}{R_3} = \omega C R_1$$

Now $\tan^{-1}\omega L/R_3$ is the phase difference, when the bridge is balanced, between the impressed E.M.F. and the current in the branches CDA; and $\tan^{-1}\omega C R_1$ is the phase difference between the impressed E.M.F and the current in the branch ABC. Therefore the bridge is balanced for magnitude and phase

Example 5. A parallel circuit consists of two branches; one branch contains a fixed impedance (24 O./$\underline{30°}$); the other consists of a variable inductive resistance, of which the reactance is constant and equal to 10 O., and the resistance is variable between 1 and 50 O. Determine (1) the variation of line current, (2) the variation of the joint impedance, when the resistance is varied and the circuit is supplied at a constant pressure of 100 V., 50 frequency.

This problem will be solved graphically in order to illustrate the application of the principle of inversion. A reproduction, on a considerably reduced scale, of the diagram for the complete graphical solution is shown in Fig. 66, and the student is advised to re-draw this diagram, step by step, as explained below, on a sheet of drawing paper (half imperial size).

The steps in the construction are—

Draw rectangular axes XOX, YOY. Select the scale for impedance as 1 cm. = 1 ohm, and draw in the first quadrant the vector OA having a length of 24 cm and inclined at an angle of 30° to the horizontal axis. OA therefore represents the impedance of the non-variable branch. The admittance of this branch = 1/24 ≑ 0·0417 mho, and therefore a convenient scale for

admittance will be 1 cm. = 0·005 mho. Divide OA at A' such that OA' = 0·0417/0·005 = 8·34 cm., and determine the image, B, of A'. Hence, OB will represent, to a scale of 1 cm. = 0·005 mho, the admittance vector corresponding to the impedance vector OA.

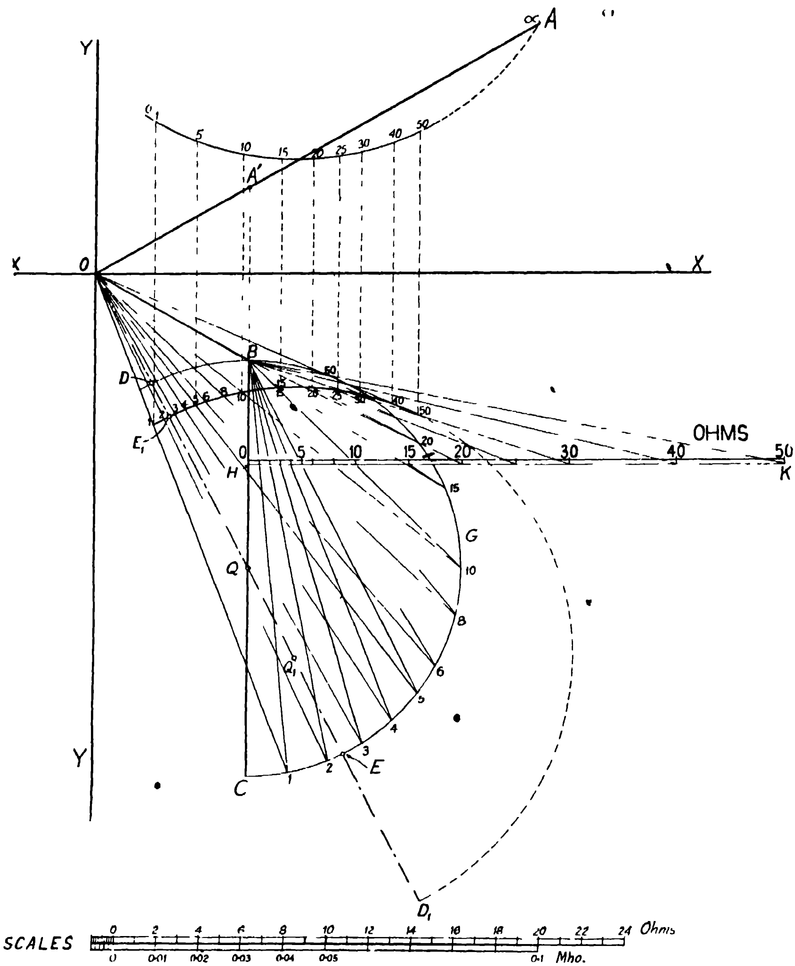


FIG. 66.—Graphical Solution to Example No. 5

Neglecting for the moment the limits of the resistance of the variable branch, the admittance of this branch is 1/10 = 0·1 mho when R = 0, and is zero when $R = \infty$. Hence from B draw BC parallel to the vertical axis, and of length = (0·1/0·005 =) 20 cm., to represent the admittance of the

variable branch when $R = 0$. Upon BC describe a semicircle, BCG, the centre of which is at Q. Then this semicircle is the locus of the admittance vector, drawn from B, for the variable branch circuit when the resistance is varied from zero to infinity.

To determine the positions of the vector for different values of the variable resistance we must determine the inversion of semicircle BCG with respect to the pole B. Now the inverse point corresponding to C lies along BC', or BC' produced, the actual position of the point depending upon the scale adopted for resistance. For the present purpose adopt a scale such that 1 cm. = 2 ohms. Then the inverse point corresponding to C' is at H, a distance of $10/2 = 5$ cm. from B. The inverse point corresponding to B lies at an infinite distance along a perpendicular drawn through H. Draw this perpendicular, HK, of length equal to $50/2 = 25$ cm. (representing the maximum value of the variable resistance) and construct upon it a scale to represent values of resistance from 1 to 50 Ω. Joint point 1 on this scale to B

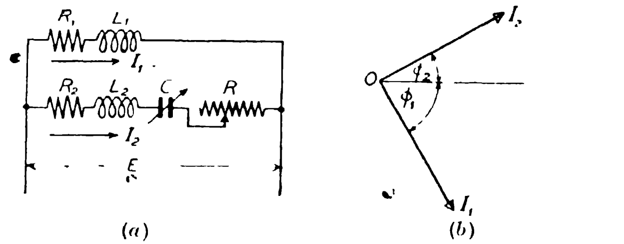


FIG. 67.—Circuit and Vector Diagrams for Worked Example No. 6

and produce so as to cut the semicircle BGC' at 1. Similarly, join point 50 on the resistance scale to B and obtain the point 50 where this line intersects the semicircle. Then the arc 1 G 50 is the locus of the admittance vector for the variable branch when the resistance of this branch is varied between 1 and 50 ohms. Vectors drawn from O to the points 1 and 50 represent the joint admittance of the parallel circuit when the resistance of the variable branch has values of 1 and 50 Ω. respectively.

The joint admittance corresponding to any other value of resistance, say 10 Ω., is obtained by first joining the appropriate point (10) on the resistance scale to B, determining the intersection of this line with the semicircle BGC', and then joining the latter point (10 on semicircle) to O. The line O–10 represents the joint admittance, and the value of this quantity is obtained by multiplying the length of this line by the appropriate scale. The lengths of the joint admittance vectors corresponding to various values of the variable resistance are given in tabular form below.

The line current is obtained by multiplying the joint admittance by the line pressure (100 V.).

To obtain the joint impedance graphically we must determine the inversion of the arc 1 G 50 with respect to the pole O. Thus, join OQ and produce so as to cut the semicircle CGB, produced, at D and E. Measure the lengths OD and OE, which should be 5·91 cm. and 25·9 cm. respectively. Now OD represents an admittance of $5{\cdot}91 \times 0{\cdot}005 = 0{\cdot}0295$ mho, and OE represents an admittance of $25{\cdot}9 \times 0{\cdot}005 = 0{\cdot}1295$ mho. The values of the impedances corresponding are therefore $1/0{\cdot}0295 = 33{\cdot}83$ ohms and $1/0{\cdot}1295 = 7{\cdot}72$ ohms respectively. These impedances are represented on the original impedance scale (viz. 1 cm. = 1 ohm) by OD_1 (= 33·83 cm.) and OE_1 (= 7·72) cm., both points D_1 and E_1 lying in OQ or its extension. The diameter of the inverse circle is therefore equal to D_1E_1, and its centre is at Q_1. In Fig. 66 only a portion of this circle is shown, and the arc corresponding to the inversion of

the arc 1 G 50 is shown in full line. This arc is the image of the locus of the impedance vector when the resistance of the variable branch is varied between 1 and 50 ohms. By transferring this arc to the first quadrant the impedance vectors and their locii are shown in the correct position with respect to the vector of reference.

The lengths of the impedance vectors corresponding to values of the variable resistance between 1 and 50 ohms inclusive, and the values of impedance deduced therefrom are given below—

R_2 (ohms)	1	2	3	4	5	6	8	10	15	20	30	40	50
Length of joint admittance vector (cm.)	25·68	25·84	25·86	25·6	25·14	24·0	23·55	22·28	19·4	17·25	14·55	13·05	12·05
Joint Admittance (mho.)	0·1284	0·1292	0·1293	0·128	0·1257	0·1245	0·1177	0·1114	0·097	0·0862	0·0727	0·652	0·602
Line current (amp.)	12·84	12·92	12·93	12·8	12·57	12·45	11·77	11·14	9·7	8·62	7·27	6·52	6·02
Length of joint impedance vector (cm.)	7·78	7·74	7·73	7·8	7·97	8·03	8·48	8·98	10·3	11·6	13·8	15·3	16·6
Joint impedance (ohms)	7·78	7·74	7·73	7·8	7·97	8·03	8·48	8·98	10·3	11·6	13·8	15·3	16·6

Example 6. One branch of a parallel circuit contains an inductive coil; the other branch also contains an inductive coil in series with which is connected a condenser of adjustable capacity and an adjustable non-inductive resistance. Determine the values of capacity and resistance such that the currents in the inductive coils are equal and have a phase difference of 90° with respect to each other. The circuits are supplied at constant voltage and frequency, and the wave-form of the E.M.F. is sinusoidal.

A diagram for the circuit is shown in Fig. 67 (a), and a vector diagram showing the required conditions is shown in Fig. 67 (b).

Adopting the symbols shown in Fig. 67 (a), we must have, for the above conditions to be satisfied,

(i) $\quad I_1 = I_2$, or $Z_1 = Z_2$.

(ii) $\quad I_1 = jI_2$, or $-\tan\varphi_1 = \cot\varphi_2$

Now $\quad Z_1^2 = R_1^2 + \omega^2 L_1^2$

and $\quad Z_2^2 = (R_2 + R)^2 + (\omega L_2 - 1/\omega C)^2$

Hence, from (i),

$$R_1^2 + \omega^2 L_1^2 = (R_2 + R)^2 + (\omega L_2 - 1/\omega C)^2$$

i.e. $$\omega^2 L_1^2 - \left(\omega^2 L_2^2 - \frac{2\omega L_2}{\omega C} + \frac{1}{\omega^2 C^2}\right) = (R_2 + R)^2 - R_1^2$$

whence $$C^2\{\omega^2(L_1^2 - L_2^2) + R_1^2 - (R_2 + R)^2\} + 2L_2 C - (1/\omega^2) = 0 \qquad (a)$$

From (ii), we have $\quad E(g_1 - jb_1) = jE(g_2 - jb_2)$

where g_1, g_2, b_1, b_2, are the conductances and susceptances respectively, of the branch circuits.

Hence, $$g_1 - jb_1 = g_2 + b_2$$

i.e. $\quad g_1 = b_2; \qquad g_2 = -b_1$

or $$\frac{R_1}{Z_1^2} = \frac{X_2}{Z_2^2} \qquad \frac{R_2 + R}{Z_2^2} = -\frac{X_1}{Z_1^2}$$

Whence $$\frac{R_1}{X_2} = -\frac{X_1}{R_2 + R}$$

i.e. $R_1(R_2 + R) = -X_1X_2 = \omega L_1(-\omega L_2 + 1/\omega C)$

from which we obtain

$$(R_2 + R) = \frac{\omega L_1}{R_1}\left(\frac{1}{\omega C} - \omega L_2\right) \quad . \quad . \quad . \quad . \quad (b)$$

Substituting this value of $(R_2 + R)$ in equation (a) and re-arranging terms, we have

$$C^2\left\{\omega^2(L_1^2 - L_2^2) - \frac{\omega^4 L_1^2 L_2^2}{R_1^2} + R_1^2\right\} + C\left\{2L_2\left(\frac{\omega^2 L_1^2}{R_1^2} + 1\right)\right\} - \frac{1}{\omega^2} - \frac{L_1^2}{R_1^2} = 0$$

from which C can be calculated when the constants of the inductive coils are known. When C is determined, the value of R is obtained by substituting in equation (b).

Whence, $$C = \frac{R_1/\omega + \omega L_1(L_1/R_1) - L_2[1 + (\omega L_1/R_1)^2]}{\omega^2\{(L_1^2 - L_2^2) - L_2^2(\omega L_1/R_1)^2\} + R_1^2} \quad . \quad . \quad (c)$$

If $L_1 = L_2 = 0{\cdot}2$ H., $R_1 = R_2 = 10$ O., and $\omega = 314$; then, on substituting these values in equations (c) and (b), we obtain $C = 43{\cdot}7\ \mu$F., $R = 52{\cdot}8$ O.

A check against these values for C and R is obtained by calculating the impedances of the branch circuits. Thus

$$Z_1 = \sqrt{(R_1^2 + \omega^2 L_1^2)} = \sqrt{[10^2 + (314 \times 0{\cdot}2)^2]} = \sqrt{(10^2 + 62{\cdot}8^2)} = 70{\cdot}3 \text{ O.}$$

$$Z_2 = \sqrt{\{(R_2 + R)^2 + (\omega L_2 - 1/\omega C)^2\}} = \sqrt{\{(10 + 52{\cdot}8)^2 + (314 \times 0{\cdot}2 - 10^6/314 \times 43{\cdot}7)^2\}}$$

$$= \sqrt{(62{\cdot}8^2 + 10^2)} = 76{\cdot}3 \text{ O.}$$

$$\varphi_1 = \tan^{-1}\omega L_1/R_1 = \tan^{-1}(314 \times 0{\cdot}2/10) = 80{\cdot}9^\circ$$

$$\varphi_2 = \tan^{-1}\frac{\omega L_2 - 1/\omega C}{R_2 + R} = \tan^{-1}\frac{(314 \times 0{\cdot}2) - 10^6/(314 \times 43{\cdot}7)}{10 + 52{\cdot}8}$$

$$= \tan^{-1} - 10/62{\cdot}8 = -9{\cdot}1^\circ.$$

$$\varphi_1 - \varphi_2 = 80{\cdot}9^\circ + 9{\cdot}1^\circ = 90^\circ.$$

[Note.—This method of obtaining a phase difference of 90° between currents in the two branches of a parallel circuit is called "phase splitting"; it has a number of practical applications, an important application being in the phase-shifting transformer for use with the alternating-current potentiometer (see Chapter XV).]

CHAPTER VII

THE LOAD DIAGRAM FOR THE ELECTRIC CIRCUIT

THE load diagram is an extended vector diagram and is so called because all quantities relating to the circuit (such as current, pressure, power, power factor), as well as the performance of the circuit (i.e. voltage regulation, efficiency, and losses) can be obtained by graphical construction.

The load diagram for a circuit is constructed from the no-load and short-circuit diagram for that circuit, and the latter is obtained from the admittance diagram by a suitable change of scale (see p. 114). In order that the final diagram may appear in a convenient position the admittance diagram is drawn in the first quadrant, instead of in the fourth quadrant, as hitherto, and the vector of reference is vertical. Impedances, or, more correctly, their images, therefore appear in the same quadrant as admittances.

In this chapter we shall deduce load diagrams for series, parallel, and series-parallel circuits, but as the no-load and short-circuit diagram for the parallel circuit has already been deduced (p. 113), the load diagram for this circuit will be obtained before considering series and series-parallel circuits. Moreover, the load diagram for the parallel circuit is more easily deduced than the diagrams for the other types of circuits.

PARALLEL CIRCUITS

Load diagram for a parallel circuit containing variable impedance in one branch. Let the variable impedance consist of a constant reactance and a variable resistance as represented in Fig. 60, p. 113, and discussed in the previous chapter. The admittance diagram for this circuit is shown in Fig. 61, p. 113, and was constructed with the vector of reference (i.e. the vector of the impressed E.M.F.) horizontal. If the diagram be reconstructed as a current diagram, with the vector of reference vertical, we obtain the diagram shown in Fig. 68. In this diagram the point I_0, corresponding to point A in Fig. 61 [i.e. R_2 (Fig. 60) $= \infty$], is called the *no-load point*: the point I_s, corresponding to point C in Fig. 61 [i.e. R_2 (Fig. 60) $= 0$], is called the *short-circuit point*. The line currents corresponding to these points are represented by OI_0 and OI_s, and are called the no-load and short circuit currents respectively. Any intermediate point G on the semicircle I_0GI_s corresponds to a particular value

of the variable resistance R_2 (Fig. 60) intermediate between zero and ∞, and the vector OG represents the line current under these conditions.

The diagram of Fig. 68 shows, therefore, the variation of the line current in magnitude and phase between no-load (i.e. $R_2 = \infty$) and short circuit (i.e. $R_2 = 0$), and for this reason it is called the *no-load and short-circuit diagram* for the circuit.

The complete performance of the circuit (e.g. the current, power input, power factor, and losses) may be obtained graphically as follows—

The *line current*, I, as already shown, is represented by the vector drawn from O to the semicircle; the angle which this

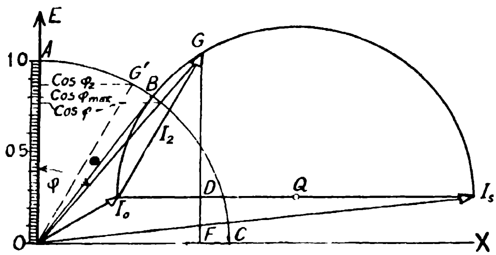


FIG. 68.—Load Diagram for a Simple Parallel Circuit [Fig. 60]

vector makes with the vertical axis is equal to the phase difference, φ, between line current and impressed E.M.F., and the cosine of this angle is equal to the power factor of the circuit. Hence if a quadrant ABC be drawn with O as centre, and the vertical radius OA be divided into 100 equal parts, then the vertical projection of the point of intersection of any line-current vector with the quadrant gives the power factor as a percentage. Obviously the power factor attains its maximum value when the line-current vector is tangential to the semicircle I_0GI_s.

The *current*, I_2, *in the branch of variable impedance*, corresponding to any particular value of resistance, is represented by a chord in the semicircle, I_0I_s, the chord being drawn from I_0 to the appropriate point, say G, on the circumference.

The *power*, P_2, *supplied* to this portion of the circuit is therefore represented by the ordinate GD in the semicircle I_0GI_s.

The *power factor*, cos φ_2, of this branch may be obtained from the quadrant ABC and scale OA, by drawing through O a parallel to I_0G, and projecting the point of intersection, G', on to the scale OA.

The *power*, P_1, *supplied to the non-variable branch* is represented by the vertical distance, DF, between the horizontal axis and the diameter, $I_0 I_s$, of the semicircle $I_0 G I_s$.

The *total power*, P, *supplied* to the circuits is represented by the ordinate, GF, drawn from the horizontal axis to the circumference of the semicircle. Since in this case the whole of the power supplied is expended against losses the ordinates representing power will also represent the losses in the circuits.

The *scale for the power ordinates* is readily obtained from the scale of the current vectors. Thus if these vectors are drawn to a scale of 1 cm. $= p$ amp., the current corresponding to a vector of length OG cm. is $OG.p$ amp. If E is the line voltage and $\cos\varphi$ the power factor, the power supplied is equal to $E.I \cos\varphi$. Now the ordinate GF, corresponding to the vector OG, has a length equal to $OG.\cos\varphi$, cm., and represents the power $E.I\cos\varphi$. Hence the scale of the power ordinates, viz. 1 cm. $= q$ watts, must be such that $q.GF = E.I\cos\varphi$, i.e. $q(OG.\cos\varphi) = E(p.OG)\cos\varphi$, whence $q = p.E$, or 1 cm. $= p.E$ watts.

SERIES CIRCUITS

Load diagram for a series circuit containing both fixed and variable impedance. Two cases of series circuits will be considered, viz. (1) a circuit in which the variable impedance consists of a non-inductive resistance, (2) a circuit in which a constant ratio exists between the resistance and reactance of the variable impedance, i.e. the power factor is constant but is less than unity. These cases have a practical application, as the conditions are representative of a simple transmission line supplying power to a single load of variable magnitude but of constant power factor, which may be either equal to, or less than, unity.

The circuit diagrams are shown in Figs. 69*a* and 69*b*. In these diagrams the fixed impedance, Z_1, represents the impedance of the transmission line, and the variable resistance (R_2, Fig. 69*a*) or the variable impedance (Z_2, Fig. 69*b*) represents the impedance of the "load," which may consist of a bank of lamps in the former case, and a motor, or other electromagnetic apparatus, in the latter case.

The construction of the no-load and short-circuit diagrams for these cases is fairly simple, as only one inversion is necessary, but the deduction of the load diagram differs in a number of respects from that for the parallel circuit.

Case 1. Construction of the diagram for a load of unity power factor. The "line" impedance, Z_1, Fig. 69*a*, is represented in

Fig. 70 by the image vector OA, the co-ordinates of the point A being R_1, X_1. The "load" impedance is represented by a straight line, $AB\infty$, drawn upwards from A parallel to the vertical axis. If this line is extended below the point A, then the lower portion represents the load impedance for the case when the resistance is negative, i.e. when the load consists of a generator operating at unity power factor. The joint impedance of the circuit, corresponding to a particular value, R_2, of the load, is therefore given by a vector drawn from O to the appropriate point on the line $AB\infty$.

To obtain the current in the circuit the impedance diagram must be inverted with respect to the pole O. Now the inverse point

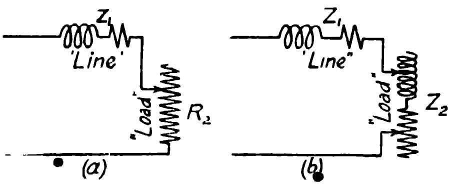


FIG. 69 Circuit Diagrams for Series Circuits

corresponding to A is at A_1, in OA, or OA produced, according to the scales adopted for impedance and admittance. The inversion of the line AB with respect to O is a portion of a semicircle which passes through the origin (O) and has its centre in the horizontal axis. The diameter of this semicircle corresponds to the inversion of OD, where D is the point of intersection of BA produced, with the horizontal axis. Therefore the scale for admittance must be selected so as to give a convenient diameter for this semicircle.

The inversion of the line DAB is therefore the semicircle OA_1H, and the inversion of the line $DC\infty$ is the semicircle OFH, the centre being at Q in both cases. The arc OG_1A_1 therefore represents the locus of the joint admittance vector when the resistance R_2 is varied from zero to ∞, and the arc $OFHA_1$ represents the locus of this vector when the resistance is negative and is varied between zero and ∞. By a suitable change of scale these arcs also represent the locus of the line current when the resistance is varied, O being the no load point (corresponding to $R_2 = \infty$ and zero current) and A_1, the short-circuit point (corresponding to $R_2 = 0$).

The performance of the circuit is obtained from the diagram as follows—

The *line current*, corresponding to a particular value, R_2, of the load resistance (which is represented by AG on the resistance scale) is given, on the current scale (which is E times the admittance

scale, E being the line voltage), by the length of the chord, OG_1, drawn from O through G, G_1 being the inverse point to G.

The *phase difference* between line current and impressed E.M.F. is given by the inclination of OG_1 to the vertical axis (which contains the vector of the impressed E.M.F.). The power factor may therefore be obtained directly by constructing a power-factor scale and quadrant, as shown in Fig. 68, and projecting the point of intersection of the current vector and the quadrant on to this scale.

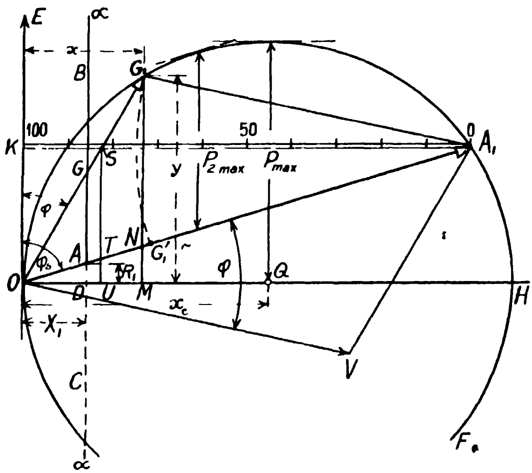


FIG. 70.—Load Diagram for a Series Circuit [Fig. 69 (a)] having a Variable Load of Unity Power Factor

The *power taken from the supply system* is given, on the power scale (which is E times the current scale), by the ordinate, G_1M of the appropriate current vector.

The *power supplied to, and utilized by, the load* is represented by G_1N, and that expended in losses in the line is represented by NM.

Proof. The diameter, OH, of the current semicircle OG_1A_1H corresponds to a current $I_M = E/X_1$, where X_1 is the reactance of the "line." Hence, if the scale for current is 1 cm. = p amp., the length of OH (in cm.) is equal to $I_m/p = E/X_1p$. Similarly the length of any radius-vector, such as OG_1, drawn from O to the semicircle, which corresponds to a current of I, is I/p cm.

If x, y, are the co-ordinates, with respect to rectangular axes passing through the origin O, of any point, G_1, on the semicircle, the equation to the circle is

$$\left(x - \frac{E}{2X_1p}\right)^2 + y^2 = \left(\frac{E}{2X_1p}\right)^2 \quad . \quad . \quad . \quad (45)$$

since the co-ordinates of the centre of the circle are $x_c = E/2X_1p$, $y_c = 0$.

Whence $\qquad x^2 + y^2 = x(E/X_1p)$,

i.e. $\qquad OG_1^2 = x(E/X_1p)$.

Now if φ is the inclination of OG_1 to the vertical axis, the ordinate at G_1 is given by $G_1M = y = OG_1.\cos\varphi = (I/p)\cos\varphi$. Multiplying each side by the line voltage, E, we have $E.G_1M = (EI\cos\varphi)/p$, whence $pE.G_1M = EI\cos\varphi$. Therefore the ordinate G_1M represents the power taken from the supply system to the scale 1 cm. $= pE$ watts.

The power, P_2, supplied to the load is equal to the difference between the power taken from the supply system and the power expended in the I^2R losses in the line. Thus

$$
\begin{aligned}
P_2 &= EI\cos\varphi - I^2R_1 \\
&= pE.G_1M - (p.OG_1)^2R_1 \\
&= pE.G_1M - p^2R_1(xE/X_1p) \\
&= pE.G_1M - x(pER_1/X_1) \\
&= pE.G_1M - x(pE\cot\varphi_s)
\end{aligned}
$$

since $OG_1^2 = x^2 + y^2 = x(E/X_1p)$, as proved above, and $R_1/X_1 = \cot\varphi_s$, where φ_s is the phase difference between the short-circuit current (I_s) and the impressed E.M.F.

Now $x\cot\varphi_s$ is equal to the ordinate, NM, of the straight line OA_1 at abscissa x. Hence

$$
\begin{aligned}
P_2 &= pE.G_1M - pE.MN \\
&= pE.G_1N
\end{aligned}
$$

i.e. for any current represented by OG_1 the power supplied to the load is represented by the intercepted length of the ordinate at this point between the semicircle and the straight line joining the origin and the short-circuit point, the scale being 1 cm. $= pE$ watts.

The line OA_1 is therefore the datum line from which the output power, P_2, is measured. Similarly the line OH, i.e. the abscissa axis, is the datum line from which the input power, P_1, and the losses are measured.

The losses are also represented, for any particular value (such as OG_1) of the line current, by the perpendicular distance of that point from the vertical axis (i.e. by the abscissa of the point G_1), but in this case the scale is 1 cm. $= (pE.\cot\varphi_s)$ watts. The vertical axis is therefore the datum line from which losses may be measured.

Hence the horizontal axis, the short-circuit line (OA_1), and the vertical axis are important lines in the load diagram.

The *maximum power taken from the supply system* is represented by the maximum ordinate, QJ, in the semicircle, and the maximum power supplied to the load is obtained by drawing a tangent parallel to the short-circuit line (OA_1) and determining the length of the intercept, between the circumference and the short-circuit line, of the ordinate drawn from the point of contact of the tangent.

The *efficiency of the circuit*—i.e. the ratio: (power supplied to the load/power taken from the supply system)—is given by the ratio G_1N/G_1M. Instead of calculating this ratio for each value of the current the efficiency may be obtained direct'y as follows—

Draw through the point A_1, a perpendicular A_1K, to the vertical axis and divide this into 100 parts, placing the zero at A_1. Then

the point, S, at which the current vector, OG_1, intersects this scale gives the percentage efficiency directly.

Proof. From S, the point of intersection of OG_1 (or OG_1 produced) with KA_1, draw the ordinate SU, and let T be the point of intersection of this ordinate with the short-circuit line, OA_1. Then, since $SU = KO$, we have the following pairs of similar triangles—

triangle G_1NO is similar to triangle STO,
triangle G_1MO is similar to triangle SUO,
triangle A_1ST is similar to triangle A_1KO.

Hence,

$$\frac{\text{Power supplied to load}}{\text{Power taken from supply}} = \frac{P_2}{P} = \frac{G_1N}{G_1M} = \frac{ST}{SU} = \frac{ST}{KO} = \frac{A_1S}{A_1K}$$

i.e. the distance A_1S expressed as a fraction of A_1K is equal to the efficiency of the transmission. Therefore, if A_1K is divided into 100 equal parts, with zero at A_1, the point of intersection of the line OG_1, or OG_1 produced, with this scale gives the efficiency directly as a percentage.

Voltage regulation. The voltage drop in the impedance of the line, and the voltage available at the load, may, on the assumption of constant supply voltage, be obtained directly from the diagram as follows, the proof being given below—

Join the points A_1 and G_1. Then the triangle OG_1A_1 is the triangle of voltages for the system; OA_1 representing the supply voltage to a scale Z_1 times the current scale (since at short circuit $I_s = E/Z_1$), OG_1 representing the voltage drop in the "line" to this scale, and G_1A_1 representing the voltage at the load to the same scale. Hence, since the percentage voltage regulation is given by: $100 \times$ (arithmetic difference between "supply" and "load" voltages/supply voltage), this quantity is represented in the diagram by the ratio $100\ (OG_1'/OA_1)$, where A_1G_1', along A_1O, is made equal to A_1G_1.

The phase difference between the "load" and "supply" voltages is given by the angle OA_1G_1.

Proof. Let E denote the supply, or impressed, E.M.F., I_s the short-circuit current, φ_s the phase difference between E and I_s, and Z_1 the impedance of the line. Also let E_1 denote the voltage drop in the line and E_2 the voltage at the load, respectively, for a line current I, which corresponds to a load resistance R_2, the phase difference between this current and the impressed E.M.F. being φ. Then, taking the vector of the impressed E.M.F., OE, Fig. 70, as the vector of reference and employing the exponential form of symbolic notation, we have

$$\dot{Z}_1 = Z_1\varepsilon^{j\varphi_s},$$
$$\dot{I}_s = \frac{E}{\dot{Z}_1} = \frac{E}{Z_1}\varepsilon^{-j\varphi_s}$$
$$\dot{I} = I\varepsilon^{-j\varphi}$$
$$\dot{E}_1 = \dot{I}\dot{Z}_1 = I\varepsilon^{-j\varphi}Z_1\varepsilon^{j\varphi_s} = IZ_1\varepsilon^{j(\varphi_s-\varphi)}$$
$$\dot{E}_2 = \dot{I}R_2 = IR_2\varepsilon^{-j\varphi}$$

Let the vector of reference be now rotated through an angle φ_s in the clockwise direction, so that it coincides with the vector representing the short-circuit current, and let the scale for the voltage vectors be Z_1 times that of the current vectors, i.e. 1 cm. $= pZ_1$ volts. Then this change of position and scale is equivalent to multiplying the original voltage vectors by the quantity $1/\dot{Z}_1 = (1/Z_1)\epsilon^{-j\varphi_s}$.

Hence the impressed E.M.F. is now represented by the vector quantity

$$E' = \frac{\dot{E}}{\dot{Z}_1} = \frac{E}{Z_1}\epsilon^{-j\varphi_s},$$

i.e. by the vector OA_1 in Fig. 70.

The voltage drop in the line impedance is represented by the vector quantity

$$E'_1 = \frac{\dot{E}_1}{\dot{Z}_1} = \frac{IZ_1\epsilon^{j(\varphi_s-\varphi)}}{Z_1\epsilon^{j\varphi_s}} = I\epsilon^{-j\varphi}$$

i.e. by the vector OG_1,
and the voltage across the load is represented by the vector quantity

$$E'_2 = \frac{\dot{E}_2}{\dot{Z}_1} = \frac{IR_2\epsilon^{-j\varphi}}{Z_1\epsilon^{j\varphi_s}} = I\left(\frac{R_2}{Z_1}\right)\epsilon^{-j(\varphi+\varphi_s)}$$

i.e. by the vector OV drawn at an angle φ below OA_1.

Since in triangle OG_1A_1 the angle $OA_1G_1 = \varphi$, the side G_1A_1 is parallel to OV. Also since G_1 is the inverse point to G, and A_1 is the inverse point to A, we have $OG.OG_1 = OA.OA_1$, so that triangle OAG is similar to triangle OA_1G_1. Therefore

$$G_1A_1 : OG_1 :: GA : OA$$

i.e. $G_1A_1 = OG_1(GA/AO)$.

Hence, since $GA/AO = R_2/Z_1$, G_1A_1 represents the quantity $I(R_2/Z_1)$.

Therefore the triangle OG_1A_1 is the voltage triangle for the system, OA_1 representing the supply voltage, OG_1 the voltage drop in the line impedance, and G_1A_1 the voltage at the load, the scale of these quantities being 1 cm $= pZ_1$ volts.

Construction of the diagram for loads having a power factor less than unity. Let the power factor of the load be $\cos\varphi_2$. Then in constructing the no-load and short-circuit diagram we set off OA, Fig. 71, to represent the image vector of the impedance of the "line," as before, and from A draw the line $AB\infty$, at an angle φ_2 to the vertical axis, to represent the image vector of the impedance of the load. Observe that if the power factor is lagging, φ_2 is set off in the clockwise direction, but if the power factor is leading, φ_2 is set off in the counter clockwise direction.

We now obtain the inversion of the lines OA and $AB\infty$ with respect to the pole O. The inversion of the line $AB\infty$ gives a portion of a semicircle which passes through the origin. The centre of this semicircle lies in a line, drawn through the origin, perpendicular to $AB\infty$, and the diameter is obtained in the same manner as in the above case.

The semicircle is shown in Fig. 71 by OG_1A_1H, the centre being at Q. The point A_1 on the circumference is the inverse point to A, and the arc OG_1A_1 is inverse to the line $AB\infty$. Thus the arc

OG_1A_1 is the locus of the joint admittance vector when the load is varied from zero to short circuit. By a suitable change of scale this arc also will represent the locus of the line-current vectors when the load is varied, O being the no-load point and A the short-circuit point.

On comparing Fig. 70 with Fig 71 it will be observed that in the former diagram the centre of the semicircle lies in the horizontal axis, but that in the latter diagram the line containing the centre is displaced from the horizontal axis by the angle φ_2, the displacement

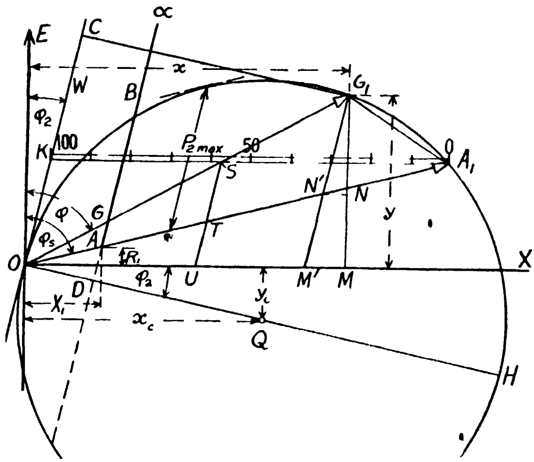


FIG 71 —Load Diagram for a Series Circuit [Fig 69 (b)] having a Variable Load of Constant Power Factor

being in the counter-clockwise (or positive) direction for a leading power factor, and in the clockwise direction for a lagging power factor. The short-circuit line, OA_1, occupies the same position in each diagram, if the constants of the "line" are the same for the two cases.

To complete the diagram we must determine the scale for efficiency and also the datum lines for the input power, output power, and the losses in the line impedance.

The datum lines for input power and output power are the horizontal axis, OX, and the short-circuit line, OA_1, respectively.

The *power input* corresponding to a current represented by the vector OG_1 is given by the ordinate G_1M at this point; the power scale being E times the current scale, as in the case above when the power factor of the load was unity. The power input is also given

by the distance, G_1M', of the point G_1 from the horizontal axis measured in a direction perpendicular to the diameter of the semicircle, i.e. in a direction parallel to the tangent at O, but the scale is now $(E \cos \varphi_2)$ times the current scale as proved below.

The phase difference between the current and the supply voltage is given by the angle which the current vector, OG_1, makes with the vertical axis.

The *power output* is given by the portion, G_1N', of the line G_1M' which is intercepted between the circumference and the short-circuit line OA_1; and the losses in the "line" are given by the portion, $N'M'$, intercepted between the short-circuit line and the horizontal axis, the scale in both cases being $(E \cos \varphi_2)$ times the current scale as proved below.

The datum line from which the I^2R losses in the "line" impedance are measured is obtained by drawing a tangent, OW, from the origin, O. This tangent is inclined at the angle φ_2 to the vertical axis, and the I^2R loss corresponding to a current OG_1 is given by the perpendicular distance of the point G_1 from the tangent as proved below.

The *voltage regulation* and the voltage at the load are obtained as before from the triangle OG_1A_1; OA_1 representing the supply voltage, OG_1 the voltage drop in the line, and A_1G_1 the voltage at the load, the scale in each case being Z_1 times the current scale.

Proof. The line current is a maximum when the reactance of the whole circuit has a value represented by OD, Fig. 71, which is the perpendicular distance of the line $AB\infty$, produced backwards, from the origin O. This value of reactance is only obtained practically by operating the load as a generator, and is equal to $Z_1 \sin(\varphi_s - \varphi_2)$, where Z_1 is the impedance of the "line." Whence the maximum current is $I_M = E/Z_1 \sin(\varphi_s - \varphi_2)$, and the diameter of the current circle, for a scale of 1 cm. = p amp. is $I_M/p = E/(pZ_1 \sin(\varphi_s - \varphi_2))$. The co-ordinates of the centre of the circle are therefore

$$x_c = (\tfrac{1}{2} I_M/p) \cos \varphi_2 = E \cos \varphi_2 / 2pZ_1 \sin(\varphi_s - \varphi_2);$$

$$y_c = -(\tfrac{1}{2} I_M/p) \sin \varphi_2 = -E \sin \varphi_2 / 2pZ_1 \sin(\varphi_s - \varphi_2).$$

Hence the equation to the current circle is

$$\left(x - \frac{E \cos \varphi_2}{2pZ_1 \sin(\varphi_s - \varphi_2)}\right)^2 + \left(y + \frac{E \sin \varphi_2}{2pZ_1 \sin(\varphi_s - \varphi_2)}\right)^2 = \left(\frac{E}{2pZ_1 \sin(\varphi_s - \varphi_2)}\right)^2$$

or
$$x^2 + y^2 = \frac{E}{pZ_1}\left\{x \frac{\cos \varphi_2}{\sin(\varphi_s - \varphi_2)} - y \frac{\sin \varphi_2}{\sin(\varphi_s - \varphi_2)}\right\}$$

$$= \frac{E}{pZ_1 \sin(\varphi_s - \varphi_2)} \sqrt{(x^2 + y^2)} \,.\, \sin(\varphi - \varphi_2) \qquad (46)$$

where φ is the inclination to the vertical axis of the line joining the point x, y, to the origin. Hence the I^2R loss in the impedance of the "line," due to a

current which is represented by OG_1 $(= \sqrt{(x^2+y^2)}$, where x, y, are the co-ordinates of the point G_1), is given by

$$R_1(p.OG_1)^2 = p^2R_1(x^2+y^2) = \left\{p^2R_1\frac{E}{pZ_1\sin(\varphi_s-\varphi_2)}\right\} OG_1.\sin(\varphi-\varphi_2)$$

$$= \left\{pE\frac{\cos\varphi_s}{\sin(\varphi_s-\varphi_2)}\right\} OG_1\,.\sin(\varphi-\varphi_2),$$

since $R_1/Z_1 = \cos\varphi_s$.

Now $OG_1\sin(\varphi-\varphi_2)$ is the perpendicular distance, G_1C, of the point G_1 from the tangent OW drawn through the origin. Hence the I^2R loss in the "line" is proportional to G_1C and the tangent OW is the datum line from which the I^2R losses in the line impedance are measured, the scale being $\{E\cos\varphi_s/\sin(\varphi_s-\varphi_2)\}$ times the current scale, or 1 cm. $= pE\cos\varphi_s/\sin(\varphi_s-\varphi_2)$ watts.

The I^2R loss due to a current represented by OG_1 is therefore given by

$$\left(pE\frac{\cos\varphi_s}{\sin(\varphi_s-\varphi_2)}\right) G_1C \text{ watts},$$

where G_1C is the perpendicular distance, measured in cm., of G_1 from the tangent OW. Now $G_1C/\sin(\varphi_s-\varphi_2)$ is equal to ON', where N' is the point of intersection of the short-circuit line, OA_1, and a line G_1M', drawn through G_1 parallel to the tangent OW. Also $ON'\cos\varphi_s$ is the ordinate at the point N'. Hence the I^2R loss is also given by

$$pE.NM,$$

where N is the projection of the point N' on the ordinate, G_1M, drawn at G_1.

The power taken from the supply system is given by $EI\cos\varphi$, or by $pE.G_1M$, since $G_1M = (p.OG_1)\cos\varphi$.

Hence the power supplied to the load is given by G_1N, the difference between G_1M and NM, the scale being E times the current scale.

Now since the angle $MG_1M' = \varphi_2$: $G_1M = G_1M'\cos\varphi_2$; $G_1N = G_1N'\cos\varphi_2$; $NM = N'M'\cos\varphi_2$. Therefore, if the power scale is changed to $(E.\cos\varphi_2)$ times the current scale, the intercepts G_1M', G_1N', $N'M'$, on a line drawn through G_1 parallel to the tangent at the origin (i.e. this line is perpendicular to the diameter of the circle) give the power input, the power output, and the line losses; the scale being 1 cm. $= pE\cos\varphi_2$ watts.

The horizontal axis is therefore the input datum line, and the short-circuit line is the output datum line.

The efficiency [i.e. the ratio: (power taken from supply/power supplied to the load)] is given by A_1S/A_1K, where S is the point of intersection of OG_1 with the horizontal line, A_1K, drawn through A_1. Thus if from the point S a line SU be drawn parallel to the tangent OW, and if T is the point of intersection of this line and the short-circuit line OA_1, then from the similar pairs of triangles OG_1M', OSU; OG_1N', OST; A_1KO, A_1ST we have

$$\frac{\text{Power supplied to load}}{\text{Power taken from supply}} = \frac{G_1N'}{G_1M'} = \frac{ST}{SU} = \frac{ST}{KO} = \frac{SA_1}{KA_1}.$$

The proof that triangle OA_1G_1 is the "voltage triangle" of the system is the same as that given on p. 135.

Construction of the load diagram from test data. The load diagrams hitherto considered have been deduced from a knowledge of the "constants" of the circuit, i.e. the resistance and reactances of "line" and the "load." But in the case of series circuits of

the type shown in Fig. 69 it is apparent that the current circle could have been drawn if the magnitude and phase of the short-circuit current and the power factor of the load had been given. For example, the centre of the current circle is obtained quite easily by setting off the short-circuit line in its correct position with reference to the rectangular axes; bisecting this line, and drawing a perpendicular to intersect a line drawn through the origin and inclined at the angle φ_2 with respect to the horizontal axis, where $\cos \varphi_2$ is the power factor of the load and the angle φ_2 is measured in the positive direction when the power factor is leading, and in the negative direction when the power factor is lagging. The diagram is completed by drawing a horizontal line through the short-circuit point and a tangent from the origin to intersect the former. The distance between the short-circuit point and the point of intersection of the horizontal line and tangent is divided into 100 parts, with the zero at the short-circuit point, thereby giving the scale for efficiency. The scale for power follows directly from the scale for current and a knowledge of the line voltage and the power factor of the load.

In the case of the parallel circuit represented in Fig. 60, the current circle, Fig. 68, can be drawn when the magnitudes and phases of the no-load and short-circuit currents are known.

Example. The following worked example is given to show the simplicity with which the performance of a simple transmission line can be obtained by means of the load diagram.

A variable load of constant power factor is supplied from a generating station 2 miles away through a transmission line consisting of copper conductors 0·372 in. in diameter spaced 18 in. apart. The pressure at the generating station is 2200 V., and the frequency is 50 cycles per second; and both are constant.

Two cases will be considered: (1) power factor of load 0·9 lagging; (2) power factor of load 0·95 leading.

Firstly, the resistance and reactance of the "line" must be calculated. Taking the specific resistance of copper as $0·69 \times 10^{-6}$ ohm per in. cube, the resistance of the line conductors is

$$R_1 = 2 \times 5280 \times 12 \times 2 \times 0·69 \times 10^{-6}/(\pi \times 0·186^2) = 1·6\ \Omega.$$

The reactance of the line conductors is obtained by multiplying their inductance by $2\pi \times$ frequency, the inductance being calculated by the method given on p. 38. Thus the inductance per cm. of double line is

$$[0·92 \log_{10} (D/r) + 0·1] \times 10^{-8} \text{ henry.}$$

In the present case $D = 18$ in., $r = 0·186$ in.; $\log_{10} (18/0·186) = 1·985$.

Hence, since 1 mile = 1609 metres, the inductance of the transmission line is

$$2 \times 1609 \times 100 \times 10^{-8}(0·92 \times 1·985 + 0·1) = 0·0062 \text{ H.},$$

and the reactance of the line is $314 \times 0·0062 = 1·95\ \Omega$.

Whence the impedance of the line is $Z_1 = \sqrt{(1·6^2 + 1·95^2)} = 2·52\ \Omega$

Instead of obtaining the current circle from the inversion of the joint impedance diagram we will determine it directly by calculating the short-circuit current. Thus

$$I_s = E/Z_1 = 2200/2{\cdot}52 = 872 \text{ A.} \quad \varphi_s = \cos^{-1}(1{\cdot}6/2{\cdot}52) = 50{\cdot}6°.$$

Selecting a current scale of 1 cm. = 50 A., we draw the line OA_1 (Fig. 72), 17·44 cm. (= 872/50) in length and inclined at an angle of 50·6° to the vertical axis, to represent the short-circuit current.

Draw through the origin lines OH_1, OH_2, inclined at angles of − 25·8° (= $\cos^{-1}0{\cdot}9$) and + 18·2° (= $\cos^{-1}0{\cdot}95$), respectively, to the horizontal axis. Bisect OA_1 at F and draw the perpendicular FQ_2Q_1 to intersect the lines

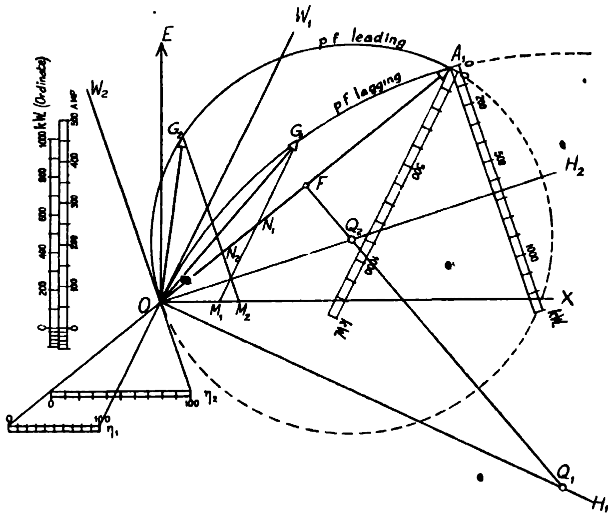


FIG. 72.—Load Diagram for Simple Transmission Line

OH_2, OH_1, at Q_2 and Q_1 respectively. Then Q_1 is the centre of the current circle when the power factor of the load is 0·9 lagging, and Q_2 is the centre of the current circle when the power factor of the load is 0·95 leading.

The diameters of the circles are

41·6 cm. (= 17·44/sin(50·6° − 25·8°) for the load of power factor 0·9 lagging).
18·7 cm. (= 17·44/sin(50·6° − 18·2°) for the load of power factor 0·95 (leading).

The circle for the load of 0·95 power factor (leading) is shown complete in Fig. 72, but only a portion of the circle for the load of 0·9 power factor (lagging) is shown on account of space restrictions. Moreover, only the portions of both circles which are above the horizontal axis and between the no-load and short-circuit points are required for the present purpose.

The diagrams are completed by drawing tangents OW_1, OW_2, at the origin and constructing the efficiency scales. To prevent confusion these scales are constructed below the horizontal axis. Thus the short-circuit line and the tangents are produced beyond the origin, and a horizontal line of *any* convenient length is drawn between the short-circuit line, produced, and each tangent. Each of these lines is divided into 100 equal parts, as shown in Fig. 72. The efficiency corresponding to a particular line current is obtained by producing the current vector backwards until it intersects the appropriate scale.

The scale for power may be obtained when the method of measuring this quantity is decided. Thus, if power is measured vertically, the scale is E times the current scale—i.e. 1 cm. = 2200 × 50 = 110,000 watts, or 1 cm. = 110 kW.—for both cases. But if power is measured in a direction parallel to the appropriate tangent (i.e. in a direction perpendicular to the diameter of the appropriate current circle, the scales for the two cases will not be the same, being 1 cm. = 2200 × 0·9 × 50 = 99,000 watts, or 1 cm. = 99 kW., for the 0·9 power-factor load; and 1 cm. = 2200 × 0·95 × 50 = 104,500 watts, or 1 cm. = 104·5 kW., for the 0·95 power-factor load.

The performance of the transmission line, as deduced from the load diagram of Fig. 72, is given in Table VII, and the results are plotted in Fig. 73.

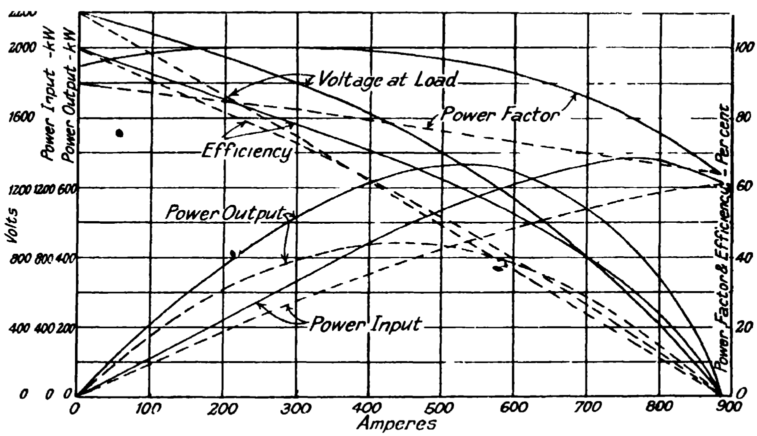


FIG. 73.—Performance Curves of Transmission Line (determined from the Load Diagram of Fig. 72)

The results show that a load of lagging power factor adversely affects the efficiency and voltage regulation of the line, and the power factor at which the generator operates.

The power and losses are obtained by measuring these quantities in directions parallel to the appropriate tangents. For example, when the power factor is lagging the input, output, and losses corresponding to the current represented by the vector OG_1, Fig. 72, are given by G_1M_1, G_1N_1, N_1M_1, respectively, the scale being 1 cm. = 99 kW. Similarly, when the power factor is leading, the input, output, and losses corresponding to the current represented by the vector OG_2, are given by G_2M_2, G_2N_2, N_2M_2, respectively, the scale being 1 cm. = 104·5 kW.

SERIES-PARALLEL CIRCUITS

In deducing the load diagram for series-parallel circuits we shall first consider a circuit in which one of the parallel branches contains a fixed impedance and the other contains a variable impedance which is variable between 0 and ∞, the ratio of resistance to reactance being constant (i.e. the power factor of this branch is constant). The more general case, where the second branch

TABLE VII

Measured and calculated quantities (from Fig. 72) for performance of transmission line.

Current (amp.)			100	200	300	400	445*	500	528†	600	7
Length OG_1 (cm.)			2	4	6	8	8·9	10	10·56	12	14
Length OG_2 (cm.)											
Input	p.f. = 0·9 (lagging)	Length G_1M_1 (cm.)	1·94	3·8	5·5	7·1	7·76	8·52		9·8	10·87
		Power (kW)	192	376	544	703	768	844		970	1076
	p.f. = 0·95 (leading)	Length G_2M_2 (cm.)	2·08	4·2	6·32	8·4		10·23	10·68	11·75	12·76
		Power (kW)	217·5	439	660	877		1068	1116	1228	1332
Output	p.f. = 0·9 (lagging)	Length G_1N_1 (cm.)	1·73	3·1	3·97	4·44	4·46	4·42		3·9	2·87
		Power (kW)	172	307	393	440	441	438		386	284
	p.f. = 0·95 (leading)	Length G_2N_2 (cm.)	1·93	3·55	4·92	5·9		6·36	6·4	6·2	5·16
		Power (kW)	201·8	371	514	616		664	669	648	540
Losses	p.f. = 0·9 (lagging)	Length N_1M_1 (cm.)	0·21	0·7	1·53	2·66	3·3	4·1		5·9	8·0
		Power (kW)	19·8	69	151	263	327	406		584	792
	p.f. = 0·95 (leading)	Length N_2M_2 (cm.)	0·15	0·65	1·4	2·5		3·87	4·28	5·55	7·6
		Power (kW)	15·7	68	146	261		404	447	580	792
Efficiency (per cent)		p.f. = 0·9 (lagging)	89·6	81·7	72·3	62·6	57·4	51·9		39·8	26·6
		p.f. = 0·95 (leading)	92·7	84·6	77·9	70·2		62·2	60	52·8	40·5
Voltage at load	p.f. = 0·9 (lagging)	Length G_1A_1 (cm.)	15·6	13·7	11·8	9·8	8·9	7·8		5·76	3·7
		Volts	1965	1726	1487	1235	1120	982		726	466
	p.f. = 0·95 (leading)	Length G_2A_1 (cm.)	16·6	15·55	14·3	12·82		11·16	10·56	8·97	6·42
		Volts	2090	1960	1802	1615		1406	1330	1130	809
Voltage drop in line	p.f. = 0·9 (lagging)	Length OG_1 (cm.)	2	4	6	8	8·9	10		12	14
		Volts	252	504	756	1080	1120	1260		1512	1764
	p.f. = 0·95 (leading)	Length OG_2 (cm.)	2	4	6	8		10	10·56	12	14
		Volts	252	504	756	1080		1260	1330	1512	1764
Power factor at generator (per cent)	load p.f. 0·9 (lagging)		87·3	85·5	82·5	79·9	78·6	76·8		73·5	70
							lag				
	load p.f. 0·95 (leading)		98·8 (lead)	99·8 (lead)	1·0	99·6		97·2	96·1	93	86·5
									lag		

Scales.—Current: 1 cm. = 50 A.
Power: 1 cm. = 99 kW., for p.f. = 0·9 (lagging); 1 cm. = 104·5 kW., for = 0·95 for p.f. (leading).
Voltage: 1 cm. = 2·52 · 50 = 126 V.

* Maximum load when power factor of load is 0·9 lagging).   † Maximum load when power factor of load is 0·95 (leading).

contains both fixed and variable impedances, will be considered later, and it will be shown that such a circuit may be reduced to an equivalent circuit of the former type.

Load diagram for a series-parallel circuit with variable impedance, of constant power factor, in one branch. The construction of the no-load and short-circuit diagram for this circuit involves the determination of (1) the diagrams for the joint admittance and joint impedance of the parallel branches, (2) the diagram for the

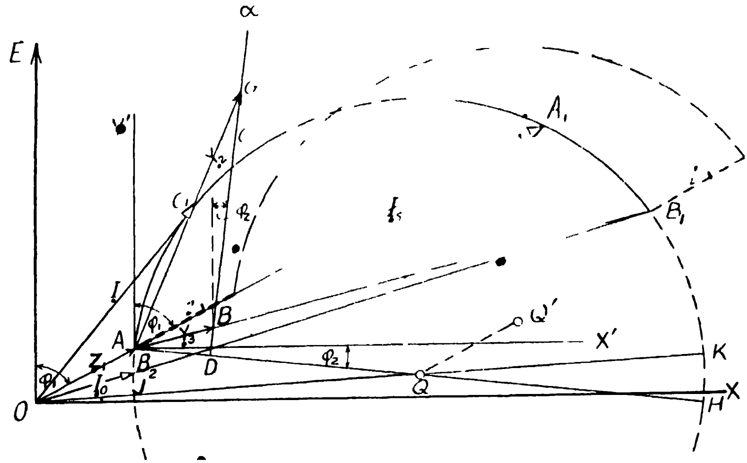


FIG. 74.—No-load and Short-circuit Diagram for Simple series-parallel Circuit

joint impedance of the complete circuit, (3) the inversion of this diagram and the change of scale so that the vectors will represent currents.

The variable impedance will be denoted by Z_2, as before, the fixed series impedance by Z_1, and the impedance of the non-variable branch by Z_3.

The no-load and short-circuit diagram is constructed as follows—Select a suitable scale for admittance and draw AB, Fig. 74, to represent the admittance, Y_3, of the branch of fixed impedance. It is preferable for reasons which will be explained later, to draw AB with respect to temporary axes, AX', AY', instead of the permanent axes of reference. From B draw the line $BC\alpha$ to represent the admittance of the variable branch, the line being inclined at the angle φ_2 with respect to the temporary vertical axis, where $\cos\varphi_2$ is the power factor of the variable branch. Vectors

drawn from A to the line $BC\infty$ therefore represent the joint admittance of the parallel branches. Hence the line $BC\infty$ is the locus of the joint admittance vector when the impedance of the variable branch is positive and is varied between zero and infinity.

The joint impedance of the parallel branches is obtained by the inversion of the locus $BC\infty$ with respect to the pole A. This inversion gives the arc AG_1B_1, the centre of which lies at Q in the line AH inclined at φ_2 with respect to the temporary horizontal axis. Vectors drawn from A to the arc AG_1B_1 therefore represent the joint impedance of the parallel branches when the impedance of the variable branch is positive and is varied from zero to infinity. Similarly, vectors drawn from A to the lower arc B_1HA represent the joint impedance of the branch circuits when the impedance of the variable branch is negative and is varied from zero to infinity (i.e. when the circuit Z_2 is acting as a generator at a constant power factor, $\cos \varphi_2$).

To obtain the joint impedance of the whole circuit the series impedance, Z_1, must be added vectorially to the joint impedance vectors for the branch circuits. If the addition is carried out in this manner, another circle, which is shown dotted in Fig. 74, will be obtained. The diameter of this new circle is the same as that of the original circle, but its centre is at Q'; the distance QQ' being equal, on the impedance scale, to Z_1, and the inclination of QQ' to the vertical axis being equal to φ_1, where $\tan \varphi_1 = X_1/R_1$. This circle is therefore the locus of the joint impedance vectors for the complete circuit when the axes AX', AY', are the axes of reference.

Instead of constructing the second circle to obtain the joint impedance of the complete circuit, we may make the original circle represent this quantity by shifting the temporary origin (A) to O, where the distance OA represents the series impedance Z_1, and the inclination of OA to the vertical is equal to φ_1. Thus the single circle AG_1B_1H may represent either (1) the locus of the vector of the joint impedance of the parallel branches of the circuit—in which case the origin is at A and the axes of reference are AX', AY'—or (2) the locus of the vector of the joint impedance of the complete circuit—in which case the origin is at O and the axes of reference are OX, OY.

The joint admittance of the complete circuit is obtained by the inversion of the circle AG_1B_1H with respect to the pole O. This inversion, as explained on p. 115, gives a circle the centre of which lies in the line joining O and Q. But by a suitable choice of the new admittance scale the original circle may be made to represent its inverse circle. For example, if J, K, are the points of intersection

of the line OQ, produced, and the circumference of the circle AG_1B_1H, then if this circle is to represent the locus of both impedance and admittance vectors, we must have $m.OJ = 1/n.OK$, or $n = 1/(m \times OJ.OK)$, where m, n, are the scales for impedance and admittance respectively.

Now in the original diagram for the admittance of the parallel branch circuits the points B and ∞ correspond to zero and infinite admittance, respectively, of the variable branch. The inversion of these points with respect to A gives the points B_1 and A respectively, on the impedance circle, and the inversion of points B_1 and A with respect to O gives the points B_2 and A_1 on the admittance circle. Hence B_2 is the " no-load " point, and A_1 is the " short-circuit " point in the final admittance diagram. By changing the scale of this diagram to E times the admittance scale, where E is the supply voltage, we obtain the current diagram for the circuit, B_2 being the " no-load " point and A_1 the " short-circuit " point. Hence the no-load current is given by the vector OB_2 and the short-circuit current is given by the vector OA_1.

The *load diagram* is obtained from the current diagram by determining the datum lines for (1) the power taken from the supply system, (2) the power expended in the series impedance, (3) the power expended in the branch of fixed impedance, and (4) the power supplied to the variable branch. These lines are shown in Fig. 75, in which only the current circle of Fig. 74 together with the no-load and short-circuit points is drawn. The additional construction involved after the no-load and short-circuit points have been determined and the circle has been drawn, is as follows –

From the origin draw a tangent OF to the circle, bisect it at D and draw a perpendicular $DUHV$ to the line OQ joining the origin and the centre of the circle. This perpendicular is called the " semi-polar " of the circle with respect to the origin.

From the short-circuit point, A_1, draw the tangent A_1V.

Join the no-load and short-circuit points, and produce the line (A_1B_2) to cut the horizontal axis at T. Join T and V (the intersection of the semi-polar and the tangent at the short-circuit point).

Join the short-circuit point and the point, U, at which the semi-polar intersects the horizontal axis.

The *datum line for obtaining the power taken from the supply system* is the horizontal, or abscissæ, axis, and is called the " input " datum line. The power taken from the supply system, when the line current is represented by the vector OG_1, is proportional to the ordinate, G_1M, drawn from the point G_1.

The *datum line for obtaining the I^2R loss in the series impedance, Z_1,*

is the "semi-polar" of the current circle with respect to the origin O. This line, UHV, is called the "primary loss" datum line. The I^2R loss in the series impedance, corresponding to a line current represented by the vector OG_1, is proportional to the perpendicular distance, G_1H, of the point G_1 from the semi-polar.

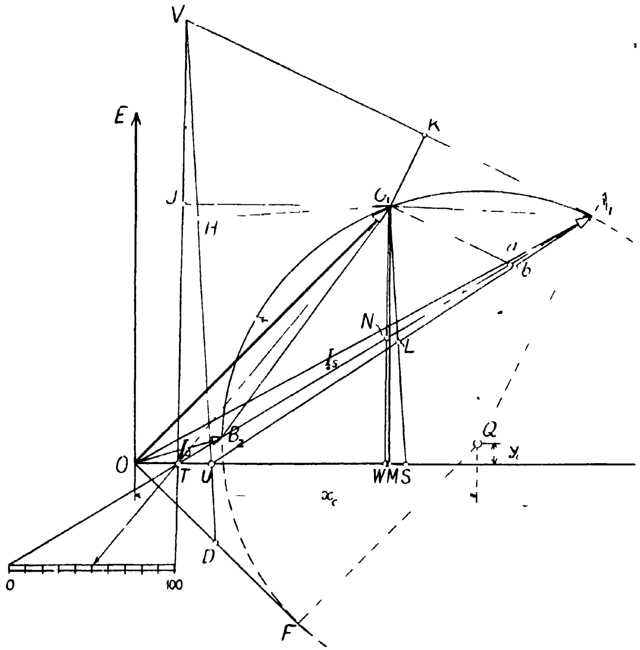


FIG. 75 Load Diagram for Simple Series-parallel Circuit

The loss is also represented by LS, which is the intercept on the line G_1S, drawn through G_1 parallel to the semi-polar, made by the horizontal axis and the line UA_1 (which joins the short-circuit point and the point at which the semi-polar intersects the horizontal axis).

Proofs are given later.

The *datum line for obtaining the power expended in the branch of fixed impedance* is the tangent drawn from the short-circuit point. The power expended in this branch, when the line current is represented by the vector OG_1, is proportional to the perpendicular distance, G_1K. of the point G_1 from this tangent.

This loss is also represented by ab, which is the intercept made by the lines A_1T and A_1U, on the line G_1b, drawn through G_1 parallel to the tangent A_1V.

The *datum line for obtaining the power supplied to the branch of variable impedance* is the line joining the no-load and short-circuit points, and is called the "output" datum line. The power supplied to this branch, when the line current is represented by the vector OG_1, is proportional to G_1N, which is the intercept, made by the circumference of the circle and the "output" line, on the line G_1W, drawn from the point G_1 parallel to the line VT. The intercept NW, made by the horizontal axis and the line, A_1T, joining the no-load and short-circuit points is proportional to the sum of the I^2R loss in the series impedance and the power expended in the branch of fixed impedance.

The scale for *efficiency* is constructed on a horizontal line drawn in any convenient position between boundary lines formed by the "output" datum line, A_1B_2T, or its extension, and a line, VT, passing through the points of intersection of (i) the "output" datum line and the horizontal axis, (ii) the "primary loss" datum line (i.e. the semi-polar) and the tangent drawn from the short-circuit point. The ratio

$$\frac{\text{power supplied to the branch of variable impedance}}{\text{power taken from the supply system}}$$

is given directly on this scale by the point at which a line joining the appropriate point on the current circle and the point, T, of intersection of the "output" datum line and the horizontal axis intersects the scale.

The triangle OG_1A_1 is the voltage triangle for the circuit; the side OA_1 representing the supply voltage, OG_1 the voltage drop in the series impedance, and G_1A_1 the voltage at the terminals of the parallel branch circuits.

The sides G_1A_1, G_1B_2, of the triangle $A_1G_1B_2$ represent, to dissimilar scales, the magnitudes of the currents in the fixed and variable branches respectively.

Proofs. Let Z_1 denote the impedance of the series portion of the circuit, Z_2, Z_3, the impedances of the two branches of the parallel portions of the circuit, the former being variable and the latter being fixed in value. Also let E denote the supply voltage, E_1 the voltage drop in the series impedance, and E_2 the voltage at the terminals of the parallel portions of the circuit.

Then, taking the supply voltage as the vector of reference, the no-load current, I_0, which corresponds to infinite impedance in the variable branch, is given by $I_0 = E/(Z_1 + Z_3)$, or $I_0 = E/\sqrt{[(R_1 + R_3)^2 + (X_1 + X_3)^2]}$. The short-circuit current, which corresponds to zero impedance in the variable branch, is given by $I_s = E/Z_1$, or $I_s = E/\sqrt{(R_1^2 + X_1^2)}$.

The line current corresponding to any particular value, Z_2, of the variable impedance is given by

$$I_1 = \frac{E}{Z_1 + [1/(Y_2 + Y_3)]}$$

where Y_2 is the value of admittance corresponding to the impedance Z_2, and Y_3 is the admittance of the non variable branch.

The corresponding currents in the branch circuits are: $I_2 = E_2Y_2$ in the variable branch, and $I_3 = E_2Y_3$ in the non-variable branch, where E_2 is the voltage at the terminals of these branches and is given by $E_2 = E - I_1Z_1$.

Since the line current I_1 is equal to the vector sum of the currents in the parallel branches of the circuit, we have, when the current in the variable branch is I_2,

$$I_1 = I_2 + I_3 = I_2 + Y_3(E - I_1Z_1),$$

and when the current in this (variable) branch is zero, the line current is equal to the no-load current and is given by

$$I_0 = Y_3(E - I_0Z_1).$$

Whence, by subtraction,

$$\begin{aligned} I_1 - I_0 &= I_2 + Y_3(E - I_1Z_1) - Y_3(E - I_0Z_1) \\ &= I_2 - Y_3Z_1(I_1 - I_0) \end{aligned}$$

or

$$I_2 = (I_1 - I_0)\,(1 + Z_1/Z_3).$$

Thus the current in the variable branch is equal to the vectorial difference of the line and no-load currents multiplied by the complex number $(1 + Z_1/Z_3)$, which is a constant quantity for a given circuit. The magnitude of this current is therefore proportional to the vector difference of the line and no-load currents. Now B_2G_1, Fig. 75, is the difference of the line current and no-load current vectors. Hence, B_2G_1 represents the magnitude of the current in the variable branch to a scale $(1 + Z_1/Z_3)$ times the scale of the line current. For example, if the scale for the line current is 1 cm. $= p$ amp., the scale for the current in the variable branch is 1 cm. $= p(1 + Z_1/Z_3)$ amp. It is necessary to observe, however, that the vector B_2G_1 does not give the phase of the current I_2.*

The current I_3 in the branch of fixed impedance is proportional to the voltage, E_2, at the terminals of the parallel branches. Now the triangle OG_1A_1 is the voltage triangle for the circuit (see p. 135), and the voltage at the terminals of the parallel branches is represented by G_1A_1, the scale being Z_1 times the scale for the line current. Hence the magnitude of the current in the branch of fixed impedance is given by $I_3 = E_2/Z_3 = (pZ_1/Z_3)G_1A_1$: i.e. the magnitude of this current is represented by G_1A_1, the scale being 1 cm. $= (pZ_1/Z_3)$ amp. But the vector G_1A_1 only gives the phase of this current in the special case when the reactance of this branch is zero.

Since at no load the current in the branch of fixed impedance has a value equal to I_0, and the voltage across this portion of the circuit is represented by B_2A_1, we have $I_3 = I_0(G_1A_1/B_2A_1) = p(OB_2/B_2A_1)G_1A_1$.

Thus the scale for the current I_3 may be obtained from the scale for line current either by calculation or by measurement.

Power taken from the supply system. At no load the power taken from the supply system is given by: $P_0 = EI_0 \cos\varphi_0$, where φ_0 is the phase difference between the no-load current and the supply voltage. The power taken from the supply system at short circuit is given by $P_s = EI_s \cos\varphi_s$.

Now if the diagram of Fig. 75 is drawn to a scale of 1 cm. $= p$ amp., the power taken at no load is represented by $E.p.OB_2.\cos\varphi_0$, or by pEy_0, where

* The current I_2 has a constant phase difference, φ_2, with respect to the voltage, E_2, across the parallel branches of the circuit.

y_0 is the ordinate at B_2. Similarly the power taken at short circuit is represented by $E.p.OA_1 \cos \varphi_s$, or by pEy_s, where y_s is the ordinate at A_1, and generally, if any particular line current, I, is represented by the vector OG_1, the power taken from the supply system is represented by pEy, where y is the ordinate at G_1. Therefore the power taken from the supply system is represented by the ordinate of the extremity of the current vector, the scale being 1 cm. = pE watts.

Power expended in series impedance. The equation representing the I^2R loss in the series impedance, Z_1, is deduced as follows:-

Let the line current I be represented by the vector OG_1, the co-ordinates of the point G_1 being x, y.

Then, $\quad I^2R_1 = (p.OG_1)^2 R_1 = p^2(x^2 + y^2)R_1$.

If r is the radius of the current circle and x_c, y_c, are the co-ordinates of its centre, the equation to the circle is

$$(x - x_c)^2 + (y - y_c)^2 = r^2,$$

from which we obtain

$$x^2 + y^2 = 2xx_c + 2yy_c - (x_c^2 + y_c^2 - r^2) \qquad . \qquad . \qquad (47)$$

Hence the I^2R loss in the series impedance is represented by the equation

$$I^2R_1 = p^2(x^2 + y^2)R_1$$
$$= p^2R_1[2xx_c + 2yy_c - (x_c^2 + y_c^2 - r^2)]$$
$$= 2p^2R_1[xx_c + yy_c - \tfrac{1}{2}(x_c^2 + y_c^2 - r^2)]$$
$$= [2p^2R_1\sqrt{(x_c^2 + y_c^2)}]\frac{xx_c + yy_c - \frac{1}{2}(x_c^2 + y_c^2 - r^2)}{\sqrt{(x_c^2 + y_c^2)}}$$
$$= [2p^2R_1\sqrt{(x_c^2 + y_c^2)}].G_1H,$$

since the expression $[xx_c + yy_c - \frac{1}{2}(x_c^2 + y_c^2 - r^2)]/\sqrt{(x_c^2 + y_c^2)}$ represents the perpendicular distance G_1H of the point x, y (i.e. the point G_1), from the straight line which is represented by the equation

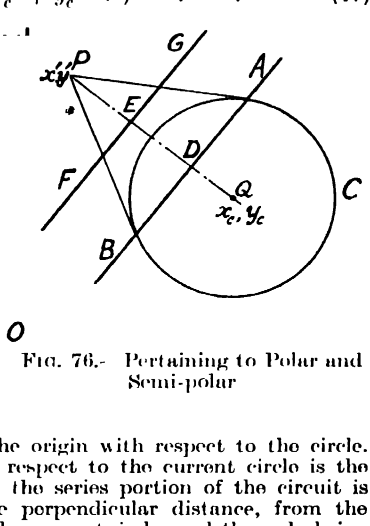


FIG. 76.- Pertaining to Polar and Semi-polar

$$xx_c + yy_c - \tfrac{1}{2}(x_c^2 + y_c^2 - r^2) = 0,$$

this line being the "semi-polar"* of the origin with respect to the circle. Thus the semi-polar of the origin with respect to the current circle is the datum line from which the I^2R loss in the series portion of the circuit is measured, the loss being given by the perpendicular distance, from the semi-polar of the appropriate point on the current circle, and the scale being 1 cm. = $2p^2R_1\sqrt{(x_c^2 + y_c^2)}$ watts.

Construction for the polar and semi-polar of a given point with respect to a circle. Let Q be the centre of the circle, C. Fig. 76, and P the point from which the polar with respect to the circle is required. From P draw tangents to the circle, and join the points of contact by the line ADB. This line is the *polar* of the point P with respect to the circle, and is perpendicular to the line PQ, joining the point P and the centre Q of the circle.

To obtain the semi-polar, bisect DP at E and draw through E a line FEG, parallel to the polar. Then the line FEG is called the *semi-polar* of the point P with respect to the circle.

* The semi-polar is the line which is parallel to the polar and is mid-way between the polar and the origin. Both the semi-polar and polar, in the present case, are perpendicular to the line joining the origin and the centre of the circle, the equation to this line being $xy_c - yx_c = 0$.

In cases where the determination of the points of contact of the tangents is difficult, or may lead to an inaccurate construction for the polar, and semi-polar, the following construction may be adopted: Divide the line PQ, joining the centre of the circle and the point P, at D and E such that $PQ.QD = r^2$, or $PD = PQ[1 - (r/PQ)^2]$; and $PE = \frac{1}{2}PQ[1 - (r/PQ)^2]$. Draw through the points D, E, lines perpendicular to the line PQ. Then the line through D is the polar, and that through E is the semi-polar.

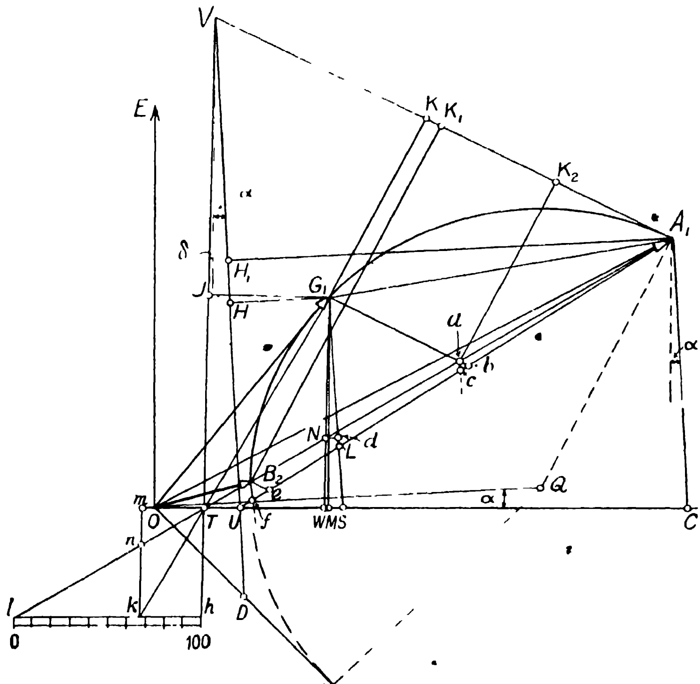


FIG. 77. —Load Diagram for Simple Series-parallel

The proof for this construction is as follows: Let the equation of the circle be $(x - x_c)^2 + (y - y_c)^2 = r^2$, and let x', y', be the co-ordinates of the point, P, for which the polar is required. Then the equation of the polar is $(x - x_c)(x' - x_c) + (y - y_c)(y' - y_c) = r^2$, or

$$x(x' - x_c) + y(y' - y_c) - x_c(x' - x_c) - y_c(y' - y_c) - r^2 = 0,$$

which is the equation to a straight line. Moreover, since the equation to the line joining the point P and the centre, Q, of the circle is

$$\frac{x - x'}{x_c - x'} - \frac{y - y'}{y_c - y'} = 0,$$

or

$$x(y_c - y') - y(x_c - x') - x'y_c + x_c y' = 0,$$

this line is perpendicular to the polar.

Hence the distance between the centre, Q, of the circle and the point of intersection, D, of the line PQ and the polar (i.e. the perpendicular distance of the centre of the circle from the polar) is

$$DQ = \frac{x_c(x_c - x') + y_c(y_c - y') - x_c(x_c - x') - y_c(y_c - y') + r^2}{\sqrt{[(x_c - x')^2 + (y_c - y')^2]}}$$

$$= \frac{r^2}{\sqrt{[(x_c - x')^2 + (y_c - y')^2]}}$$

Also the distance between the points P and Q is

$$PQ = \sqrt{[(x_c - x')^2 + (y_c - y')^2]}$$

Therefore $\qquad PQ.DQ = r^2.$

The proof for the alternative method of obtaining the I^2R loss in the series impedance (i.e. from the intercept, made by the line A_1U and the horizontal axis, on a line drawn from G_1 parallel to the semi-polar) is as follows :-

If from the points G_1 and A_1 the lines G_1S, A_1C, and G_1H, A_1H_1, Fig. 77, are drawn parallel to, and perpendicular to, respectively, the semi-polar VU, then from the similar triangles USL, UCA_1, we have

$$\frac{LS}{A_1C} = \frac{US}{UC} = \frac{G_1H}{A_1H_1}$$

i.e. $$\frac{LS}{G_1H} = \frac{A_1C}{A_1H_1}$$

Now G_1H, as already shown, represents the I^2R loss in the series impedance when the current is represented by OG_1, the scale being 1 cm. $= 2p^2R_1\sqrt{(x_c^2 + y_c^2)}$ watts. Also A_1H_1 and A_1C both represent the I^2R loss at short circuit, the scale for the former being 1 cm. $= 2p^2R_1\sqrt{(x_c^2 + y_c^2)}$, and that for the latter being 1 cm. $= pE/\cos\alpha$ watts, where α is the angle which the semi-polar makes with the vertical axis.

[*Note.* α is also the inclination of OQ with respect to the horizontal axis.]

Hence,

$$A_1C(pE/\cos\alpha) = A_1H_1[2p^2R_1\sqrt{(x_c^2 + y_c^2)}]$$

i.e. $$\frac{A_1C}{A_1H_1} = \frac{2p^2R_1\sqrt{(x_c^2 + y_c^2)}}{pE/\cos\alpha}$$

Substituting this value for A_1C/A_1H_1 in the above equation, we have

$$\frac{LS}{G_1H} = \frac{2p^2R_1\sqrt{(x_c^2 + y_c^2)}}{pE/\cos\alpha}$$

Whence, $\qquad LS(pE/\cos\alpha) = G_1H[2p^2R_1\sqrt{(x_c^2 + y_c^2)}].$

Therefore, LS represents the I^2R loss in the series impedance, to the scale 1 cm. $= pE/\cos\alpha$ watts, when the current is represented by OG_1.

Power expended in the branch circuit of constant impedance. The power expended in this portion of the circuit is given by $P_3 = E_2^2/R_3$.

Now the triangle OA_1G_1, Fig. 75, is the voltage triangle for the system, the side OA_1 representing the supply voltage, OG_1 the voltage drop in the series impedance, and G_1A_1 the voltage at the terminals of the parallel branches.

Since the vector OA_1 represents the short-circuit current to a scale of 1 cm. $= p$ amp., and this current is given by E/Z_1, the same vector, OA_1, will also represent the supply voltage to a scale Z_1 times the current scale, i.e. the scale for voltage is 1 cm. $= pZ_1$ volts. Therefore

$$P_3 = E_2^2/R_3 = (pZ_1.G_1A_1)^2/R_3 = (p^2Z_1^2/R_3)[(x_s - x)^2 + (y_s - y)^2],$$

since $G_1A_1^2 = (x_s - x)^2 + (y_s - y)^2$, where x_s, y_s, are the co-ordinates of the short-circuit point A_1.

Expanding this expression and substituting for $x^2 + y^2$ from equation (47), we have

$$P_3 = (p^2Z_1^2/R_3)\{x(x_c - x_s) + y(y_c - y_s) - [(x_c^2 - x_s^2) + (y_c^2 - y_s^2) - r^2]\}.$$

Whence,

$$P_3 = \frac{p^2Z_1^2}{R_3}\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}$$
$$\frac{x(x_c - x_s) + y(y_c - y_s) - [(x_c^2 - x_s^2) + (y_c^2 - y_s^2) - r^2]}{\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}}$$
$$= \frac{p^2Z_1^2}{R_3}\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}\,.G_1K,$$

since the expression

$$\frac{x(x_c - x_s) + y(y_c - y_s) - [(x_c^2 - x_s^2) + (y_c^2 - y_s^2) - r^2]}{\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}}$$

represents the perpendicular distance, G_1K of the point x, y (i.e. the point G_1) from the tangent at A_1. Therefore the tangent at the short-circuit point is the datum line from which the power expended in the branch of constant impedance is measured, the scale being 1 cm. $= (p^2Z_1^2/R_3)\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}$ watts.

The proof for the alternative method of obtaining this power is as follows: If from the no-load point, B_2, and the point G_1, Fig. 77, perpendiculars, B_2K_1, G_1K, and parallels, B_2e, G_1b, be drawn to the tangent, A_1V, then from the similar quadrilaterals $A_1eB_2K_1$, A_1baK_2,

$$\frac{B_2e}{ab} = \frac{B_2K_1}{aK_2} = \frac{B_2K_1}{G_1K}$$

Also, if from the points B_2 and a parallels are drawn to the semi polar VU, we have, from the similar triangles B_2fe, acb, VUA_1,

$$\frac{B_2e}{B_2f} = \frac{ab}{ac} = \frac{VA_1}{VU}$$

or

$$\frac{B_2e}{ab} = \frac{B_2f}{ac}$$

Whence,

$$\frac{B_2f}{ac} = \frac{B_2K_1}{G_1K}$$

$$\frac{B_2f}{B_2K_1} = \frac{ac}{G_1K}$$

Now B_2f and B_2K_1 both represent the same quantity, viz. the power expended in the fixed branch at no load, the scale for the former being 1 cm. $= pE/\cos a$ watts, and that for the latter being

$$1 \text{ cm.} = (p^2Z_1^2/R_3)\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]} \text{ watts.}$$

Therefore

$$B_2f(pE/\cos a) = B_2K_1\{(p^2Z_1^2/R_3)\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}\}$$

or

$$\frac{B_2f}{B_2K_1} = \frac{(p^2Z_1^2/R_3)\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}}{pE/\cos a}$$
$$= \frac{ac}{G_1K}$$

i.e. $$ac(pE/\cos a) = G_1K\{(p^2Z_1^2/R_3)\sqrt{[(x_c - x_s)^2 + (y_c - y_s)^2]}\}$$

Therefore ac represents the power expended in the branch of fixed impedance when the current is represented by OG_1, the scale being 1 cm. = $pE/\cos a$ watts.

Power expended in the branch circuit of variable impedance. The power expended in this portion of the circuit is given by $P_2 = E_2 I_2 \cos \varphi_2$. Since φ_2 is constant, and E_2 and I_2 are represented by $pZ_1.G_1A_1$ and $G_1B_2[p(1+Z_1/Z_3)]$ respectively, the power P_2 is proportional to the scalar product of G_1A_1 and G_1B_2, which is proportional to the length of the perpendicular drawn from G_1 to the line, A_1B_2, joining the no-load and short-circuit points.

This power, however, may be represented more conveniently by the intercept, G_1N, made by the circumference and the line A_1B_2T, on the line, G_1W, drawn from G_1 parallel to the line, VT, which joins the point V (at which the tangent, A_1V, intersects the semi-polar, VU) and the point T (at which the line, A_1B_2T, joining the no-load and short-circuit points intersects the horizontal axis), as the scale is then 1 cm. = $pE/\cos \delta$, where δ is the inclination of the line VT with respect to the vertical axis. The proof is as follows—

If from a a parallel to A_1U is drawn to intersect G_1S at d, and from d a horizontal line, dN, is drawn to intersect the line G_1S (which is drawn parallel to VT), the point of intersection, N, is coincident with that of the lines A_1B_1T and G_1S, as is apparent from the geometry of Fig. 77, the triangle aNd being similar to the triangle A_1TU.

[NOTE. The line joining the points a and d is not shown in Fig. 77.]

Now the portion dS, of G_1S, is made up of two parts, dL and LS; the former representing the power expended in the branch of fixed impedance, and the latter representing the I^2R loss in the series impedance, the scale in each case being 1 cm. = $pE/\cos a$ watts. Therefore the ordinate at d, or, alternatively, the ordinate at N, represents the power expended in the fixed portion of the circuit, the scale being 1 cm. = pE watts. Hence, NW represents this power to the scale 1 cm. = $pE/\cos \delta$ watts, where δ is the inclination of VT to the vertical axis.*

The remaining portion, G_1N, of G_1W must therefore represent the power supplied to the variable branch to the scale 1 cm. = $pE/\cos \delta$ watts. Since δ is usually a very small angle, the power scale when measuring parallel to VT is practically equal to the power scale when measuring along the ordinate.

Therefore in the load diagram the power expended in, or supplied to, all parts of the circuit is obtained by drawing from G_1 (i) the ordinate G_1M, (ii) a parallel, G_1W, to VT, (iii) a parallel, G_1S, to the semi-polar VU. The power taken from the supply system is then given by the ordinate, G_1M, the scale being 1 cm. = pE watts. The power expended in the series impedance is given by the intercept LS, or, alternatively, by the ordinate at L, the scales being 1 cm. = $pE/\cos a$ watts in the former case, and 1 cm. = pE watts in the latter case. The power expended in the fixed portions of the circuit (i.e. the series impedance and the fixed branch of the parallel portion of the circuit) is given by NW, or, alternatively, by the ordinate at N, the scales being 1 cm. = $pE/\cos \delta$ watts in the former case, and 1 cm. = pE watts in the latter case. The power supplied to the variable branch of the circuit is given by G_1N, or, alternatively, by the difference in the ordinates at G_1 and N, the scales being 1 cm. = $pE/\cos \delta$ watts in the former case, and 1 cm. = pE watts in the latter case.

The *efficiency*, i.e. the ratio (power supplied to branch of variable impedance/power taken from supply system) is given by G_1N/G_1W.

Now, if the lines A_1B_2T and VT are produced beyond T, and any horizontal line be drawn to intersect their extensions at l and h, respectively; and if G_1 be joined to T and produced so as to cut this horizontal line at k, the efficiency is given by lk/lh. Thus, if from k a line, km, is drawn parallel to VT to

* The line VT may be called the "total loss" datum line, since the length of the perpendicular, G_1J, drawn from G_1, is proportional to the intercept NW.

intersect Tl and the horizontal axis at n and m respectively, we have, from the similar triangles Thl, nkl

$$\frac{nk}{Th} = \frac{lk}{lh}$$

But

$$\frac{nk}{mk} = \frac{nk}{Th} = \frac{G_1N}{G_1H}$$

Therefore

$$\frac{lk}{lh} = \frac{G_1N}{G_1H}$$

Hence if lh be divided into 100 equal parts, with the zero point at l and the 100 point at h, the percentage efficiency is given directly by the scale reading at k (i.e. the point at which the line joining G_1 and T intersects the scale).

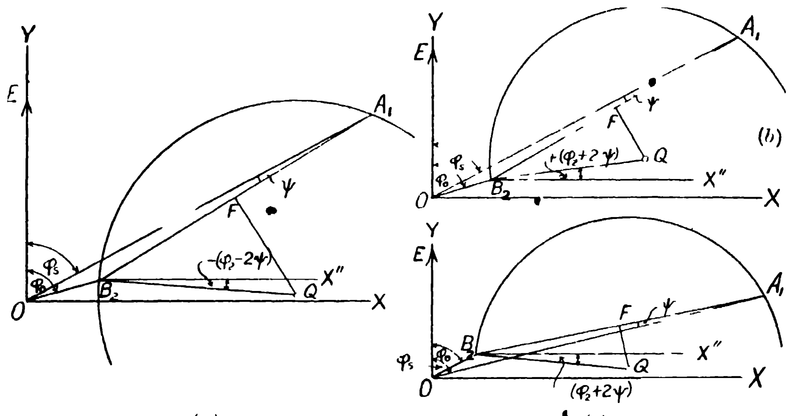


FIG. 78. Construction of No-load and Short-circuit Diagram from Data of No-load and Short-circuit Tests.

Construction of the load diagram from test data. For the case, discussed above, where the power factor of the variable branch circuit is constant and the impedance of this portion is variable between zero and infinity, the following construction for the no-load and short-circuit diagram may be adopted when the magnitude and phases of the no-load and short-circuit currents are known.

Draw rectangular axes OX, OY, and from the origin O draw the vectors OB_2, OA_1, Fig. 78, to represent the no-load and short-circuit currents, respectively, to a convenient scale, the inclination of these vectors to the vertical axis being φ_o and φ_s respectively. Join the points B_2 and A_1; bisect this line at F, and draw the perpendicular at FQ. From B_2 draw the line B_2Q to intersect this perpendicular at Q, which is the centre of the current circle. The angle which B_2Q makes with the horizontal axis, B_2X'', at B_2 is

equal to $\pm\varphi_2 + 2\psi$,*where $\cos\varphi_2$ is the power factor of the variable branch (φ_2 being positive for a leading power factor and negative for a lagging power factor), and ψ is the angle OA_1B_2. Hence when the power factor, $\cos\varphi_2$, of the variable branch is *lagging*, $B_2X''Q$ is drawn *below* the horizontal axis, B_2X'', the angle $X''B_2Q$ being equal, numerically, to $(\varphi_2 - 2\psi)$* ; but if the power factor is *leading*, B_2Q is drawn *above* the horizontal axis, B_2X'', and the angle $X''B_2Q$ is equal, numerically, to $(\varphi_2 + 2\psi)$.*

Proof. A reference to Fig. 74, which refers to the case when the power factor of the variable branch is lagging, and the no-load point lies below the short-circuit line, will show that the line AQ containing the centre of the current circle is inclined at the angle φ_2 with respect to the horizontal axis AX', this angle being below the horizontal axis because the power factor is lagging. Now since the angles AQB_2, AA_1B_2, both subtend the same arc, AB_2, of the circle B_2AA_1H, but the former angle is at the centre, and the latter is at the circumference, of the circle, the angle AQB_2 is double the angle AA_1B_2. Hence the angle which the line B_2Q makes with a horizontal axis drawn through B_2 is equal, numerically, to $\varphi_2 - 2\psi$. Therefore the line B_2Q, Fig. 78(*a*), which is inclined at an angle equal to $-\varphi_2 + 2\psi = -(\varphi_2 - 2\psi)$, with respect to the horizontal axis B_2X', passes through the centre of the circle.

In a similar manner it may be shown that when the power factor of the variable branch is leading, the line B_2Q, Fig. 78(*b*), which is inclined at an angle equal to $+\varphi_2 + 2\psi$ with respect to the horizontal axis B_2X'', passes through the centre of the circle.

The diagram is completed by constructing the efficiency scale and drawing the datum lines for input, output, etc., in the manner already described.

Application of the load diagram to practical circuits. We have shown how the no-load and short-circuit diagram may be constructed for simple circuits consisting of impedances arranged in series or series-parallel, and how the performance of these circuits may be obtained from the extended form of this diagram. We shall now show how the diagram may be applied to the more general case of the series-parallel circuit in which one branch contains both fixed and variable impedances. This type of circuit is shown in Fig. 79, and is representative of the equivalent circuit of a commercial transmission line, as well as the equivalent circuits of a static transformer, a polyphase induction motor, etc. It is apparent, therefore, that the load diagram of Fig. 75 has an extensive application in practice.

Although the no-load and short-circuit diagram for a series-parallel circuit of the type shown in Fig. 79 may be constructed without difficulty, the performance of the circuit cannot be

* If the no-load point lies above the short-circuit line, i.e. if $\varphi_s > \varphi_0$, as in Fig. 78 (*c*), the negative sign must be given to 2ψ.

determined completely from the diagram without a knowledge of constants of the variable branch. Such a circuit, however, may be replaced by an equivalent circuit of the simple series-parallel type, and the performance of the former may be deduced from that of the latter in the manner shown later.

Equivalent series-parallel circuit which replaces a general circuit. The general series-parallel circuit shown in Fig. 79 is the equivalent of a large number of practical electric circuits, and the performance of the latter may be deduced from that of the former, provided that the inductance and capacity are constant with varying currents, and that the supply voltage and frequency are constant.

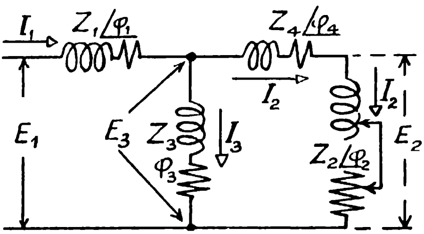


FIG. 79.—General Series-parallel Circuit

In the general circuit one of the parallel branches consists of a fixed impedance, Z_4/φ_4, connected in series with a variable impedance, Z_2/φ_2 (called the "load"); the other branch consists of a fixed impedance, Z_3/φ_3. A fixed impedance, Z_1/φ_1 (called the "line" impedance), is connected between the parallel branches and the supply system.

Consider now the voltages and currents at the supply and load ends of the circuit when (*a*) the load is open circuited, (*b*) the load is short circuited. In the former case let the line current be denoted by I_o, and the voltages at the supply and load by E_{1o}, E_2, respectively. In the latter case let E_{1s} denote the value of the supply E.M.F. necessary to obtain a current I_{2s} in the short-circuited load, and let I_{1s} denote the corresponding value of the supply current. Circuit and vector diagrams representing these conditions are given in Fig. 80.

Then,

$$\dot{I}_o = \dot{E}_{1o}/(\dot{Z}_1 + \dot{Z}_3) = \dot{E}_{1o}\dot{Y}_o \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where $\dot{Y}_o = 1/(\dot{Z}_1 + \dot{Z}_3)$ is the "no-load admittance" for the circuit.

Also,

$$\dot{E}_2 = \dot{E}_{1o} - \dot{I}_o\dot{Z}_1 = \dot{E}_{1o} - \dot{E}_{1o}\dot{Z}_1\dot{Y}_o = \dot{E}_{1o}(1 - \dot{Z}_1\dot{Y}_o) \quad . \quad (49)$$

Hence,

$$\dot{E}_{1o}/\dot{E}_2 = 1/(1 - \dot{Z}_1\dot{Y}_o) = 1 + \dot{Z}_1/\dot{Z}_3 = \dot{C}_1 \quad . \quad . \quad (50)$$

where $\dot{C}_1$ denotes the complex number $1/(1 - \dot{Z}_1\dot{Y}_o) = 1 + \dot{Z}_1/\dot{Z}_3$.

If ψ_1 is the phase difference between E_{1o} and E_2, then $C_1 = C'_1\varepsilon^{j\psi_1}$

[NOTE.—ψ_1 is positive or negative according to whether E_2 is leading or lagging with respect to E_{1o}.]

Again, when the load is short-circuited,

$$I_{1s} = I_{2s} + I_{3s},$$

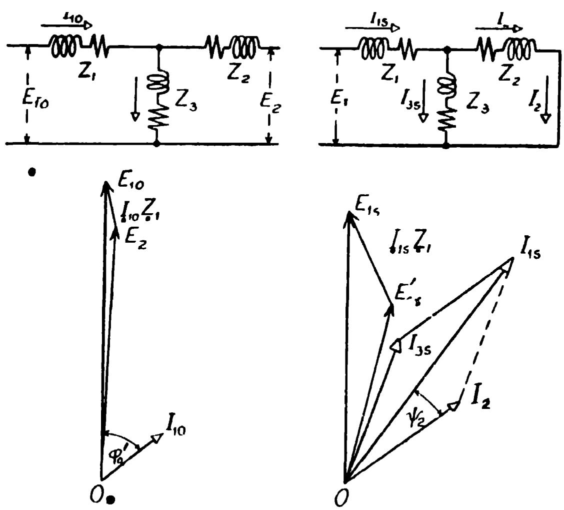


FIG. 80 Circuit and Vector Diagrams representing No load and Short circuit Conditions for a General Series parallel Circuit

where I_{3s} is the current in the parallel branch of fixed impedance Z_3 But $I_{3s} = I_{2s}(Z_4/Z_3)$. Hence,

$$I_{1s} = I_{2s}(1 + Z_4/Z_3)$$

i.e. $$I_{1s}/I_{2s} = 1 + Z_4/Z_3 = C'_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where C'_2 denotes the complex number $(1 + Z_4/Z_3)$. If ψ_2 is the phase difference between I_{1s} and I_{2s}, then $C'_2 = C_2\varepsilon^{j\psi_2}$

[NOTE.—ψ_2 is positive or negative according to whether I_{2s} is leading or lagging with respect to I_{1s}.]

Also,

$$E_{1s} = I_{1s}[(Z_1 + Z_3Z_4/(Z_3 + Z_4)] = I_{1s}Z_s \quad . \quad . \quad (52)$$

where $Z_s\ [= Z_1 + Z_3Z_4/(Z_3 + Z_4)]$ is called the "short-circuit impedance" of the circuit.

Whence,

$$Y_o Z_s = 1 - 1/C'_1C'_2 \text{*} \qquad \qquad (53)$$

Let the load impedance now have a value Z_2, and let the supply voltage be adjusted so as to give a voltage of E_2 at the terminals of the load. Moreover, let the load current under these conditions be I_2, the supply voltage E_1, and the line current I_1. Then the line current may be considered to be made up of two components, one being the no-load current corresponding to the supply voltage E_1, and the other being the line short-circuit current corresponding to the load short-circuit current I_2. Similarly the supply voltage may be considered to be made up of two components, one being the supply voltage necessary to obtain a voltage E_2 at the terminals of the load at no-load, and the other being the supply voltage necessary to obtain a load current equal to I_2 when the load is short-circuited.

Hence,

$$\begin{aligned} I_1 &= I_{1s} + I_{1o} \\ &= C'_2 I_2 + E_{1o} Y_o \\ &= C'_2 I_2 + C'_1 E_2 Y_o \qquad \qquad (54) \\ &= C'_2 E_2/Z_2 + C'_1 E_2 Y_o \\ &= E_2(C'_2/Z_2 + C'_1 Y_o) \qquad \qquad (54a) \end{aligned}$$

since $I_2 = E_2/Z_2$.

Also,

$$E_1 = E_{1o} + E_{1s} = C'_1 E_2 + C'_2 I_2 Z_s \qquad \qquad (55)$$

* This expression is obtained by substituting for $Y_o Z_s$, reducing, and then substituting in terms of $C'_1C'_2$. Thus

$$\begin{aligned} Y_o Z_s &= \left(\frac{1}{Z_1 + Z_3}\right)\left(Z_1 + \frac{Z_3 Z_4}{Z_3 + Z_4}\right) \\ &= \frac{Z_1}{Z_1 + Z_3} + \frac{Z_3 Z_4}{(Z_1 + Z_3)(Z_3 + Z_4)} \end{aligned}$$

Now since $C'_1 = 1 + Z_1/Z_3$, and $C'_2 = 1 + Z_4/Z_3$, we have

$$Z_1 + Z_3 = C'_1 Z_3, \qquad Z_1/Z_3 = C'_1 - 1;$$
$$Z_3 + Z_4 = C'_2 Z_3, \qquad Z_4/Z_3 = C'_2 - 1.$$

Hence,

$$\begin{aligned} Y_o Z_s &= \frac{Z_1}{C'_1 Z_3} + \frac{Z_3 Z_4}{C'_1 C'_2 Z_3^2} \\ &= \frac{C'_1 - 1}{C'_1} + \frac{C'_2 - 1}{C'_1 C'_2} \\ &= 1 - \frac{1}{C'_1 C'_2} \end{aligned}$$

If equation (54) be multiplied throughout by Z_s, and is then subtracted from equation (55), we obtain

$$E_1 - I_1Z_s = C_1E_2 + C_2I_2Z - (C_2I_2Z_s + C_1E_2Y_oZ_s)$$
$$= C_1E_2(1 - Y_oZ_s)$$
$$= E_2/C_2,$$

since $(1 - Y_oZ_s) = 1/C_1C_2$

Whence,

$$E_2 = C_2(E_1 - I_1Z_s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

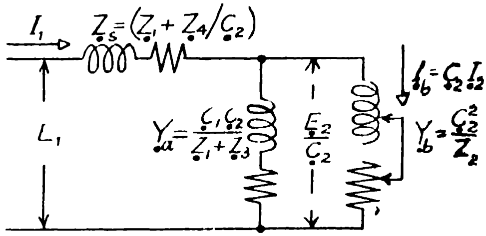


FIG. 81 Simple Series parallel Circuit equivalent to the General Circuit of Fig. 79

Substituting this value in equation (54*a*), we obtain

$$I_1 = C_2(E_1 - I_1Z_s)(C_2/Z_2 + C_1Y_o)$$
$$= (E_1 - I_1Z_s)(C_2^2/Z_2 + C_1C_2Y_o) \quad . \quad . \quad . \quad (54b)$$

I.e. in the general circuit the line current, corresponding to a given supply voltage, E_1, and a particular value, Z_2, of the load impedance, is given by the product of two quantities; of which one—$(C_2^2/Z_2 + C_1C_2Y_o)$—represents the joint admittance of the two parallel branches of a simple series-parallel circuit, and the other—$(E_1 - I_1Z_s)$ represents the voltage across these branches when the supply voltage is E_1, the line current is I_1 and the series impedance is Z_s.

Therefore the general series-parallel circuit of Fig 79 may be replaced by the equivalent simple series-parallel circuit shown in Fig. 81, in which the series impedance has a value equal to the short-circuit impedance of the general circuit, i.e. $Z_s = Z_1 + Z_4/C_2$; the fixed branch of the parallel portion of this circuit, Fig. 81, has an admittance equal to C_1C_2 times the "no-load admittance" of the general circuit, i.e. $Y_a = C_1C_2Y_o = C_1C_2/(Z_1 + Z_3) = C_1C_2Y_o\varepsilon^{j(\psi_1 + \psi_2 + \varphi_o)}$; and the variable branch has an admittance equal to C_2^2 times the admittance of the "load" in the general circuit, i.e. $Y_b = C_2^2/Z_2 = C_2^2Y_2\varepsilon^{j(\varphi_2 + 2\psi_2)}$.

In comparing the general circuit of Fig. 79 with its equivalent circuit, Fig. 81, it is found that, for any particular value of the load impedance in the general circuit, the same value is obtained for the joint impedance* of both circuits, and therefore if the same voltage is impressed on both circuits the line current and the power taken from the supply system will be the same for each circuit.

The voltage at the terminals of the parallel branches of the equivalent circuit is given by $(E_1 - I_1 Z_s)$, which, as is shown by equation (56), is equal to E_2/C_2, where E_2 is the voltage at the terminals of the load in the general circuit.

The current, I_b, in the variable branch of the equivalent circuit is given by

$$I_b = Y_b E_2/C_2 = E_2(C_2^2 Y_2)/C_2 = C_2 E_2 Y_2 = C_2 I_2 \quad . \qquad (57)$$

Hence the power, P_b, supplied to the variable branch of the equivalent circuit is given by

$$\begin{aligned} P_b &= \frac{E_2}{C_2} C_2 I_2 \cos(\varphi_2 + 2\psi_2) \\ &= E_2 I_2 \cos(\varphi_2 + 2\psi_2) \\ &= P_2 \frac{\cos(\varphi_2 + 2\psi_2)}{\cos\varphi_2} \quad . \quad . \quad . \quad . \quad . \qquad (58) \end{aligned}$$

where P_2 is the power supplied to the load in the general circuit. Thus, according to the signs and magnitudes of φ_2 and ψ_2 the

* The joint impedance of the general circuit of Fig. 79 is given by

$$Z = Z_1 + \frac{1}{(1/Z_3) + [1/(Z_2 + Z_4)]} = Z_1 + \frac{Z_3(Z_2 + Z_4)}{Z_2 + Z_3 + Z_4}$$

and that of the equivalent circuit is given by

$$Z_e = Z_s + [1/(C_2^2 Y_2 + C_1 C_2 Y_0)].$$

Substituting for Z_s, $C_2^2 Y_2$, $C_1 C_2 Y_0$ the values

$$Z_s = Z_1 + \frac{Z_3 Z_4}{Z_3 + Z_4}; \qquad C_2^2 Y_2 = \frac{1}{Z_2}\left(1 + \frac{Z_4}{Z_3}\right)^2;$$

$$C_1 C_2 Y_0 = \frac{Y_0}{(1 - Y_0 Z_s)} = Z_3 - \frac{Z_3 Z_4}{Z_3 + Z_4};$$

we have

$$\begin{aligned} Z_e = Z_1 + \frac{Z_3 Z_4}{Z_3 + Z_4} + \frac{Z_3^2 Z_2}{(Z_3 + Z_4)^2 + Z_2(Z_3 + Z_4)} \\ = Z_1 + \frac{Z_3 Z_4}{Z_3 + Z_4} + \frac{Z_2 Z_3^2}{(Z_3 + Z_4)(Z_2 + Z_3 + Z_4)} \\ = Z_1 + \frac{Z_3(Z_2 + Z_4)}{Z_2 + Z_3 + Z_4} \\ = Z \end{aligned}$$

power supplied to the variable branch of the equivalent circuit may be equal to, greater than, or less than the power supplied to the load in the general circuit, but the ratio of the quantities is constant for a given circuit.

The power, P_s, expended in the series impedance, Z_s, of the equivalent circuit is given by $P_s = I_1{}^2R_s$, where R_s is the "short-circuit resistance" of the general circuit, and includes the resistance of the line impedance, Z_1, as well as the joint resistance of the branch impedances Z_3, Z_4, but not the resistance of the load. Hence the power expended in the series impedance of the equivalent circuit for a given value of the line current represents the I^2R, or copper, losses in all parts, except the load, of the general circuit for the same value of the line current.

The power, P_a, supplied to the fixed branch of the parallel portion of the equivalent circuit is best expressed in symbolic notation and is given by

$$P_a = (E_2/C_2)^2/Z_a - E_2{}^2Y_oC_1/C_2 \quad . \quad . \quad . \quad (59)$$

If the power supplied to the general circuit at no-load is denoted by P_o, then $P_o = E_1Y_o$, whence $Y_o = P_o/E_1$. Therefore

$$P_a = P_o(E_2/E_1)^2C_1/C_2,$$

or $$P_a(C_2/C_1) = P_o(E_2/E_1)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (59a)$$

Now the power supplied to the general circuit at no-load is expended in (1) the magnetic (hysteresis and eddy-current) losses in the iron portions of the circuit, (2) the losses in the dielectrics surrounding the conductors, and in the dielectrics of any condensers included in the circuit, (3) the I^2R losses due to the no-load current. Hence if the no-load I^2R losses are ignored, or P_o is corrected to allow for these losses, then the quantity $P_a(C_2/C_1)$ represents the magnetic and dielectric losses in the general circuit, since the dielectric losses vary as the square of the E.M.F. impressed when the frequency is constant, and the magnetic iron losses vary approximately as the square of the induced E.M.F.

[NOTE.—Magnetic losses are considered in detail in Chapter XI.]

Thus the performance of any circuit which can be reduced to a general series-parallel circuit may be obtained from the performance of the equivalent series-parallel circuit; the constants of which may be determined, as already shown, either directly from those of the general circuit, or indirectly from no-load and short-circuit tests as described later.

Deduction of performance of general circuit from the load diagram for the equivalent circuit. The no-load and short-circuit diagram

for the equivalent circuit is constructed in the same manner as that for the simple series-parallel circuit, but care must be exercised in obtaining the correct phase angles for the equivalent admittances and impedances. For example, the admittance vector, AB, Fig. 82, for the branch, Y_a, of constant admittance is set off at an angle equal to $(\pm \varphi_o \pm \psi_1 \pm \psi_2)$ with respect to the vertical axis, φ_o being the phase angle for the no-load admittance (i.e. $\cos \varphi_o$ is the no-load power factor of the general circuit), and ψ_1, ψ_2, are determined as shown later (p. 167). The length of this vector on the

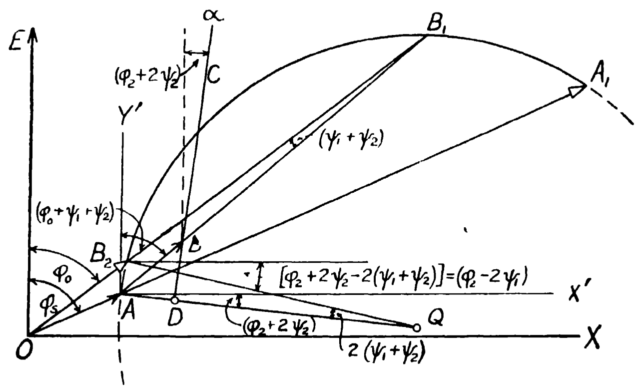


FIG. 82 No-load and Short circuit Diagram for Circuit of Fig 81

admittance scale is equal to $C_1C_2Y_o$. The admittance vector, $AB\infty$, Fig. 82, for the variable branch is then set off at an angle equal to $(\pm \varphi_2 \pm 2\psi_2)$ with respect to the vertical axis, $\cos \varphi_2$ being the power factor of the load in the general circuit. The inversion of this line with respect to A gives, therefore, the locus of the joint impedance vector for the parallel branches of the equivalent circuit. Observe that the line AQ, Fig. 82, is now inclined at an angle equal to $(\pm \varphi_2 \pm 2\psi_2)$ with respect to the horizontal axis.

The vector representing the series, or line, impedance in the equivalent circuit is then drawn in position, its inclination to the vertical axis being equal to $\pm \varphi_s$, where $\cos \varphi_s$ is the power factor for the general circuit when the load is short circuited. Finally, the inversion of the impedance circle with respect to the origin, O, is obtained, and the scale is changed so as to obtain the no-load and short-circuit points, B_2, A_1, respectively.

The load diagram, Fig. 83, is obtained by drawing the semi-polar, VU; the tangent, A_1V, at the short-circuit point; joining

the no-load and short-circuit points and producing this line so as to cut the horizontal axis at T; and drawing a line through the short-circuit point, A_1, and the point of intersection, U, of the semi-polar and the horizontal axis, exactly as in the case for the simple series-parallel circuit.

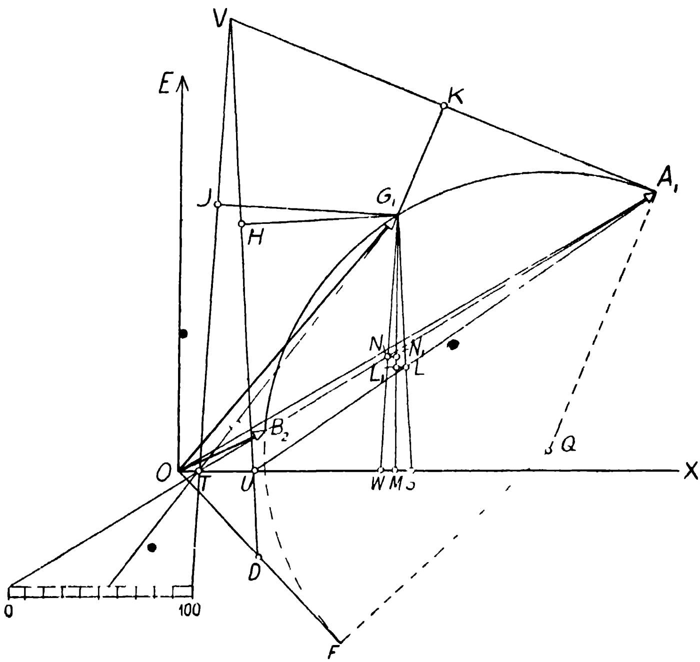


FIG. 83.—Load Diagram for Circuit of Fig. 81

The performance of the general circuit, when the line current is represented by the vector OG_1, is deduced from the load diagram as follows—

The *power input* from the supply system is given by the ordinate G_1M.

The *power supplied to the load* is given by $G_1N_1[\cos\varphi_2/\cos(\varphi_2+2\psi_2)]$ where G_1N_1 is the difference of the ordinates at G_1 and N; N being the point of intersection of the lines G_1W and A_1B_2, the former being drawn from G_1 parallel to VT and the latter being drawn through the no-load and short-circuit points.

The I^2R, or *copper losses*, in the circuit are given by L_1M, which is the ordinate at L; L being the point of intersection of the lines

G_1S and A_1U, the former being drawn from G_1 parallel to the semi-polar, and the latter being drawn through the short-circuit point and the point at which the semi-polar intersects the horizontal axis. This loss may be expressed as a percentage of the power input by constructing a "percentage I^2R loss" scale as follows: Draw any convenient line, parallel to the horizontal axis, to intersect the semi-polar, VU, and the line A_1U; divide the intercepted portion into 100 equal parts, the zero being at the point of intersection with the semi-polar. The percentage I^2R loss is then given by the scale reading at the point where the line joining the points G_1 and U intersects the scale.

The *magnetic, dielectric, and losses other than I^2R losses* are most conveniently obtained by calculating the difference of the power input and the power supplied to the load together with the I^2R losses. If the losses are to be obtained from the diagram, it is necessary to subtract from N_1L_1 (which is the difference of the ordinates at N and L) the quantity

$$G_1N_1\left(\frac{\cos\varphi_2}{\cos(\varphi_2+2\psi_2)}-1\right)$$

The *scale* for the above quantities is the "ordinate power scale" which is E times the current scale of the diagram.

The *efficiency scale* is obtained by drawing any convenient line, parallel to the horizontal axis, to intersect the lines A_1T, VT, or their extensions. The portion thus intercepted is then divided into $100[\cos\varphi_2/\cos(\varphi_2+2\psi_2)]$ equal parts. The efficiency of the general circuit, i.e. the ratio

$$\frac{\text{power supplied to load}}{\text{power taken from supply system}}$$

is then given directly by the scale reading at the point where the line joining the points G_1 and T, produced if necessary, intersects the scale.

The *current in the load* is given by G_1B_2/C_2, where C_2 is the absolute value of the complex number $\dot{C}_2$ The scale is the same as that for the line current.

The voltage at the load is given by $G_1A_1 \times C_2$.

Determination of the no-load and short-circuit constants of a general circuit. The construction of the no-load and short-circuit diagram for an equivalent circuit involves a knowledge of the four quantities $\dot{C}_1, \dot{C}_2, \dot{Y}_o, \dot{Z}_s$. These quantities are called the "no-load" and "short-circuit" constants of the general circuit. As already

shown, they are not independent of one another, but are connected by equation (53), p. 158, which may be expressed in the form

$$C_1C_2 = \frac{1}{1 - Y_oZ_s} = \frac{1/Z_s}{(1/Z_s) - Y_o}$$

If numerator and denominator are multiplied by E_1, we obtain

$$C_1C_2 = \frac{E_1/Z_s}{(E_1/Z_s) - E_1Y_o}$$
$$= I_s/(I_s - I_o) \quad . \quad . \quad . \quad . \quad . \quad (60)$$

where $I_s(=E_1/Z_s)$ is the short-circuit current, and $I_o(=E_1Y_o)$ is the no-load current, taken by the general circuit when the supply voltage is E_1. Thus the product C_1C_2 is equal to the quotient of the short-circuit current at normal supply voltage and the vector difference of the short-circuit current and the no-load current: it is therefore readily obtained from no-load and short-circuit tests. For example, in the no-load and short-circuit diagram, Fig. 82, the absolute value of the product C_1C_2 is given by $C_1C_2 = OA_1/A_1B_2$, and the argument, $\psi_1 + \psi_2$, is given by the angle OA_1B_2, the value of which depends upon the relative phase differences of the no-load and short-circuit currents, e.g.

$$\tan(\psi_1 + \psi_2) = [I_o \sin(\varphi_o - \varphi_s)]/[I_s - I_o \cos(\varphi_o - \varphi_s)] \quad . \quad (61)$$

In cases where the phase differences of the no-load and short-circuit currents, with respect to the supply voltage, are approximately equal, the angle OA_1B_2 will be small, and the value of the product C_1C_2 may be obtained by a simple calculation instead of by geometrical construction or complex algebra. For example, when φ_o is approximately equal to φ_s, the product C_1C_2 is given with sufficient accuracy for practical purposes by

$$C_1C_2 = I_s/[I_s - I_o \cos(\varphi_o - \varphi_s)] \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Moreover, with symmetrical circuits, i.e. those for which $Z_1 = Z_4$, $C_1 = C_2 = C$,* and $\psi_1 = \psi_2 = \psi$: whence, from equations (53), (60), we have

$$C = \sqrt{[1/(1 - Y_oZ_s)]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53a)$$
$$= \sqrt{[I_s - I_o)]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60a)$$
$$C = \sqrt{[I_s/\{I_s - I_o \cos(\varphi_o - \varphi_s)\}]} \quad . \quad . \quad . \quad (62a)$$
$$\tan 2\psi = [I_o \sin(\varphi_o - \varphi_s)]/[I_s - I_o \cos(\varphi_o - \varphi_s)] \quad . \quad . \quad (61a)$$

* This follows from equation (51) by substituting $Z_4 = Z_1$, thus

$$C_2 = 1 + (Z_1/Z_3) = C_1$$

With unsymmetrical circuits, i.e. those in which Z_1 and Z_4 are unequal, the angles ψ_1, ψ_2, may, under suitable conditions, be determined by measuring directly the phase difference between the E.M.Fs. at the supply and load terminals at no-load, and the phase difference between the currents in the line and load at short circuit.* But if ψ_1 and ψ_2 are small it will be difficult to obtain these angles accurately by measurement. Hence in this case, and in other cases where it is impossible to measure these phase differences directly, the angles ψ_1, ψ_2, must be obtained indirectly by carrying out either a no-load or a short-circuit test from the load end of the circuit in addition to the ordinary no-load and short-circuit tests from the supply end.

For example, if in the circuit of Fig. 79 the supply and load are interchanged, i.e. the impedance Z_4 is placed in the supply circuit and the impedance Z_1 is placed in the load circuit, and if the no-load admittance and the short-circuit impedance under these conditions are denoted by Y'_o, Z'_s, respectively, we have

$$Y'_o = 1/(Z_3 + Z_4), \quad Z'_s = Z_4 + Z_1 Z_3/(Z_1 + Z_3).$$

Now for the original circuit the no-load admittance and the short-circuit impedance are given by $Y_o = 1/(Z_1 + Z_3)$, $Z_s = Z_1 + Z_3 Z_4/(Z_3 + Z_4)$, respectively.

Whence

$$\frac{Y'_o}{Y_o} = \frac{Z_1 + Z_3}{Z_3 + Z_4} = \frac{1 + Z_1/Z_3}{1 + Z_4/Z_3} = \frac{C_1}{C_2}$$

and

$$\frac{Z_s}{Z'_s} = \frac{Z_1 + [Z_3 Z_4/(Z_3 + Z_4)]}{Z_4 + [Z_1 Z_3/(Z_1 + Z_3)]} = \frac{[Z_1(Z_3 + Z_4) + Z_3 Z_4](Z_1 + Z_3)}{[Z_4(Z_1 + Z_3) + Z_1 Z_3](Z_3 + Z_4)}$$

$$= \frac{Z_1 + Z_3}{Z_3 + Z_4} = \frac{C_1}{C_2}$$

Therefore,

$$\frac{C_1}{C_2} = \frac{Y_o'}{Y_o} = \frac{Z_s}{Z_s'}$$

or

$$\frac{C_1}{C_2} = \frac{Y_o'}{Y_o} \varepsilon^{j(\varphi_o' - \varphi_o)} \frac{Z_s'}{Z_s} \varepsilon^{j(\varphi_s - \varphi_s)}$$

Whence

$$C_1/C_2 = Y_o'/Y_o = Z_s/Z_s' \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

and

$$\psi_1 - \psi_2 = \varphi_o' - \varphi_o = \varphi_s - \varphi_s' \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

* Methods of determining these quantities are described in Chapter XV.

or, since

$$2(\psi_1 - \psi_2) = \varphi_o' - \varphi_o + \varphi_s - \varphi_s',$$

$$\psi_1 - \psi_2 = \tfrac{1}{2}(\varphi_o' - \varphi_o + \varphi_s - \varphi_s') \quad . \quad . \quad . \quad . \quad (64a)$$

These equations and those (60), (61), given on page 165 enable the four quantities C_1, C_2, ψ_1, ψ_2, to be readily calculated. Thus

$$C_1 = \sqrt{[Y_o'Y_o'(1 - Y_oZ_s)]} = \sqrt{[I_oI_s/I_o'(I_s - I_o)]} \quad . \quad . \quad (65)$$

$$C_2 = \sqrt{[Y_o'/Y_o(1 - Y_oZ_s)]} = \sqrt{[I_o'I_s/I_o(I_s - I_o)]} \quad . \quad . \quad (66)$$

where I_o' is the no-load current, at normal voltage, when the supply and load are in erchanged.*

$$\psi_1 = \tfrac{1}{2}[(\psi_1 + \psi_2) + (\psi_1 - \psi_2)]$$

$$= \tfrac{1}{2}\left[\tan^{-1}\frac{I_o\sin(\varphi_o - \varphi_s)}{I_s - I_o\cos(\varphi_o - \varphi_s)} + \tfrac{1}{2}(\varphi_s - \varphi_s' + \varphi_o' - \varphi_o)\right] . \quad (67)$$

$$\psi_2 = \tfrac{1}{2}[(\psi_1 + \psi_2) - (\psi_1 - \psi_2)]$$

$$= \tfrac{1}{2}\left[\tan^{-1}\frac{I_o\sin(\varphi_o - \varphi_s)}{I_s - I_o\cos(\varphi_o - \varphi_s)} - \tfrac{1}{2}\{\varphi_s - \varphi_s' + \varphi_o - \varphi_o'\}\right] . \quad (68)$$

The phase differences φ_o, φ_s, φ'_o, φ_s', are calculated from the power and volt-ampere inputs at no-load and short-circuit respectively; the power input being measured by a wattmeter and the volt-ampere input being measured by a voltmeter and ammeter. Thus—

$\cos\varphi_o = P_o/E_{1o}I_o$, where P_o is the power input, E_{1o} the supply voltage, and I_o the line current at no-load

Similarly,

$\cos\varphi_s = P_s/E_{1s}I_{1s}$, where P_s is the power input, E_{1s} the supply voltage, and I_{1s} the line current at short-circuit.

Construction of the load diagram for a general circuit from test data. If the impedances of the several branches of a general circuit are unknown the no-load and short-circuit diagram must be constructed from data obtained from no-load and short-circuit tests. The data required are: (1) the no-load and short-circuit currents at normal supply voltage, (2) the phase differences of these currents

* Alternatively, if the voltages at the supply and load terminals can be measured accurately at no-load, and these voltages are denoted by E_{1o}, E_{2o}, respectively, then $C_1 = E_{1o}/E_{2o}$.

Again, if the currents in the line and load portions of the circuit can be measured under short-circuit conditions, and these currents are denoted by I_{1s}, I_{2s}, respectively, then $C_2 = I_{1s}/I_{2s}$.

with respect to the supply voltage, (3) either the phase difference, φ'_o, between the no-load current and the supply voltage when the supply and load terminals are interchanged, or the phase difference, φ'_s, between the short-circuit current and the supply voltage when the supply and load terminals are interchanged.

The vectors OA_1, OB_2, Fig. 84 (a), representing the normal, no-load, and short-circuit currents are drawn in their correct positions with respect to the axis of reference, and the centre of the current circle is determined as follows: Join the no-load and short-circuit

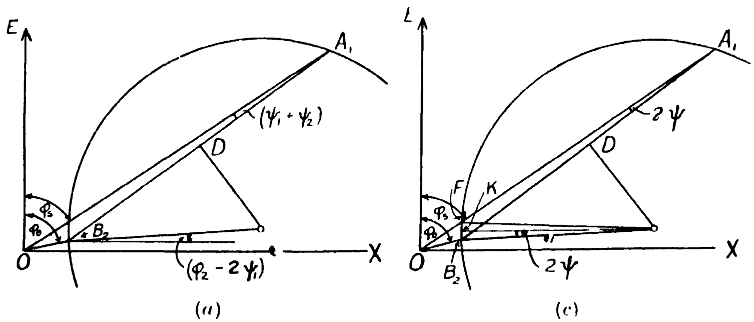


FIG. 84.—Construction for Determining Centre of Current Circle

points, bisect this line at D, and draw the perpendicular DQ. From B_2 draw the line B_2Q inclined at the angle $(\pm\,\varphi_2 \mp 2\psi_1)$* with the horizontal axis B_2X'. Then the point, Q, where this line intersects the perpendicular DQ is the centre of the circle. Observe that the angle $(\varphi_2 - 2\psi_1)$, Fig. 84 (a), may be expressed as

$$[\varphi_2 - (\psi_1 + \psi_2) - (\psi_1 - \psi_2)] = \varphi_2 - \angle OA_1B_2 - (\psi_1 - \psi_2),$$

and if $(\psi_1 - \psi_2)$ is neglected the angle $X'B_2Q$ becomes equal to $(\varphi_2 - \angle OA_1B_2)$; the sign of the angle OA_1B_2 being determined with reference to the short-circuit line OA_1, i.e. the angle OA_1B_2 is positive when the no-load point lies above the short-circuit line, and *vice versâ*. Moreover, if $\varphi_2 = 0$, and $\psi_1 = \psi_2$, the angle $X'B_2Q$ is equal to the angle OA_1B_2, and the construction shown in Fig. 84 (b) may be adopted for obtaining the centre of the circle. In this case a vertical, B_2F, is erected from the no-load point to intersect the short-circuit line at F. The portion B_2F is bisected at K, and a perpendicular KQ is drawn to intersect the line DQ at Q, which is the centre of the circle.

* See remarks on p. 155 for meaning of signs.

Proof. In Fig. 82 the line AQ, which contains the centre of the circle, is inclined at the angle $-(\varphi_2 + 2\psi_2)$ with respect to the horizontal axis AX'. Now the angle AQB_2 is equal to twice the angle AB_1B_2, and since the latter is equal to $(\psi_1 + \psi_2)$, the angle AQB_2 is equal to $2(\psi_1 + \psi_2)$. Hence, since the angle $X''B_2Q$, Fig. 82, is equal to the difference of the angles $X'AQ$ and AQB_2, i.e. $\angle X''B_2Q = (\varphi_2 + 2\psi_2) - 2(\psi_1 + \psi_2) = \varphi_2 - 2\psi_1$, a line drawn from B_2 at this angle with respect to the horizontal axis contains the centre of the circle.

With the construction shown in Fig. 84(b), which may be adopted when $\varphi_2 = 0$, and $\psi_1 = \psi_2$, we have

$$\angle B_2QK = \angle KQF = \tfrac{1}{2}\angle B_2QF = \angle B_2A_1O = 2\psi.$$

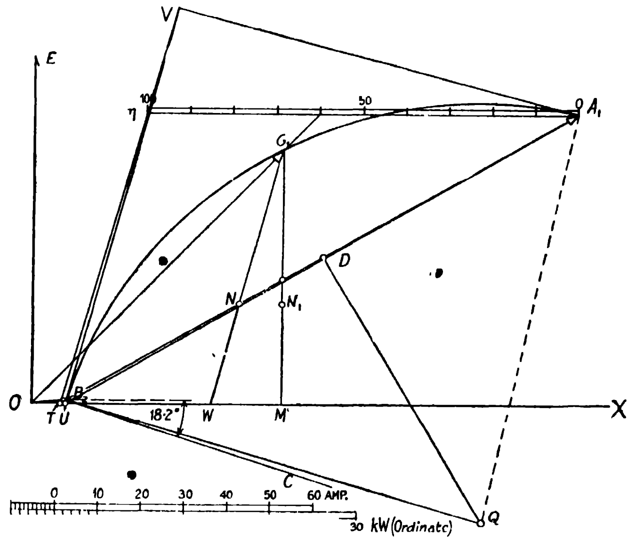


FIG. 85.- Load Diagram for Series-parallel General Circuit (Data in Example and Table VIII)

Example. To illustrate the application of the above principles we will now calculate the performance of a symmetrical series-parallel general circuit for a variable load of 0·95 power factor (lagging), the no-load and short-test readings for the circuit being as follows -

No-load	Volts 500.	Amperes 8·25.	Watts 430.
Short-circuit	Volts 500.	Amperes 145.	Watts 34,800.

The performance will be calculated for a constant supply pressure of 500 volts.

From the no-load and short-circuit readings we obtain

$$\cos\varphi_o = \frac{430}{500 \times 8{\cdot}25} = 0{\cdot}1044$$

$$\varphi_o = 84^\circ$$

$$\cos\varphi_s = \frac{34800}{500 \times 145} = 0{\cdot}485$$

$$\varphi_s = 61^\circ$$

Selecting a current scale of 1 cm. = 5 amperes, we commence the construction of the no-load and short-circuit diagram by drawing OB_2, OA_1, Fig. 85, to represent the no-load and short-circuit currents respectively; the length of OB_2 being 1·65 (= 8·25/5) cm. and its inclination to the vertical being 84°, the length of OA_1 being 29 (= 145/5) cm. and its inclination to the vertical being 61°.

Next join B_2A_1, bisect at D, and draw the perpendicular DQ.

Since the power factor of the load is 0·95, lagging, $\varphi_2 = \cos^{-1}0{\cdot}95 = 18{\cdot}2°$, and is negative. Also since the circuit is symmetrical and $\varphi_0 > \varphi_s$, the angle OA_1B_2 is equal to 2ψ and is positive. Hence from B_2 a line B_2Q is drawn inclined at the angle $-(\varphi_2 - 2\psi) = -(18{\cdot}2 - \angle OA_1B_2)°$ with respect

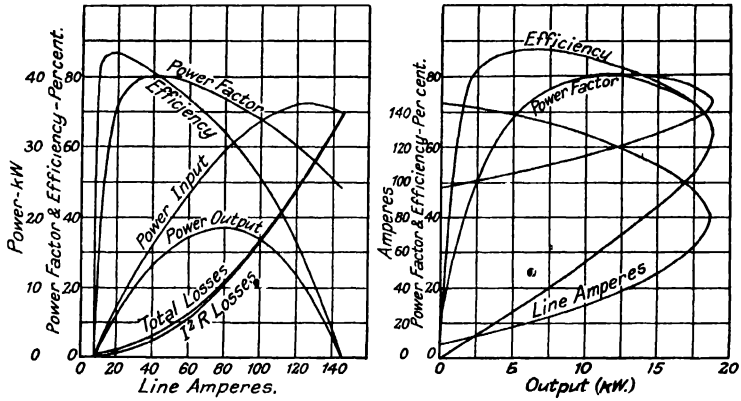


FIG. 86.—Performance Curves of General Circuit (Determined from the Load Diagram of Fig 85)

to the horizontal axis.* The point of intersection of this line with the perpendicular drawn from D is the centre of the current circle.

The diagram is completed by drawing the semi polar, VU, the tangent, A_1V, at the short-circuit point, determining their point of intersection, V, and joining this point to the point, T, where the line joining the no-load and short-circuit points intersects the horizontal axis. The short-circuit point is also joined to the point, U, at which the semi-polar intersects the horizontal axis.

The power scales may be determined when the angles of inclination of VT and VU with respect to the vertical are known. By measurements on the diagram, Fig. 85, these angles are found to be 15·5° and 14·9° respectively. Hence the power scales are

1 cm. = 5 × 500 = 2500 watts, for measurements along the ordinate;

1 cm. = 2500 × cos 15·5° = 2408 watts, for measurements parallel to the "total loss line," VT;

1 cm. = 2500 × cos 14·9° = 2415 watts, for measurements parallel to the semi-polar, VU.

The value of 2ψ, obtained from equation (61a), is

$$\tan 2\psi = \frac{I_0 \sin(\varphi_0 - \varphi_s)}{I_s - I_0 \cos(\varphi_0 - \varphi_s)} = \frac{8{\cdot}25 \sin(84° - 61°)}{145 - 8{\cdot}25 \cos(84° - 61°)} = 0{\cdot}02345,$$

whence $2\psi = 1{\cdot}36$.

* This construction is best effected by drawing a line, B_2C, at an angle of 18·2° below the horizontal axis and then constructing the angle CB_2Q equal to the angle OA_1B_2.

The correction factor $[\cos\varphi_2/\cos(\varphi_2 + 2\psi)]$ for the efficiency scale and for obtaining the power supplied to the load from the intercept, G_1N, Fig. 85, is equal to

$$\cos 18{\cdot}2/\cos(18{\cdot}2 + 1{\cdot}36) = 0{\cdot}95/0{\cdot}9422 = 1{\cdot}008,$$

and is so close to unity that it may be neglected for practical purposes.

Finally, the value of C is calculated from equation (62*a*), thus

$$C = \sqrt{\frac{I_s}{I_s - I_0\cos(\varphi_0 - \varphi_s)}} = \sqrt{\frac{145}{145 - 8{\cdot}25\cos(84^\circ - 61^\circ)}} = \sqrt{1{\cdot}055} = 1{\cdot}027$$

The performance of the circuit as deduced from measurements on the load diagram, Fig. 85, is given in Table VIII, and in the curves of Fig. 86.

TABLE VIII

Measured and Calculated Quantities from Fig. 85 for Performance of Series-parallel General Circuit

Line current (amp.)		20	40	60	80	100	120
Length OG_1 (cm.)		4	8	12	16	20	24
Input	Length G_1H * (cm.)	3	6·64	9·73	12·28	14·07	15
	Power (kW.)	7·24	16·03	23·5	29·65	33·97	36·22
Output	Length G_1N (cm.)	2·6	5·42	7·12	7·76	7·05	5
	Power (kW.)	6·28	13·09	17·2	18·73	17·02	12·07
Total losses	Length NH' (cm.)	0·4	1·22	2·61	4·52	7·02	10
	Power (kW.)	0·96	2·94	6·3	10·92	16·95	24·15
I^2R losses	Length LH'† (cm.)	0·2	1·03	2·44	4·38	6·91	9·91
	Power (kW.)	0·48	2·49	5·89	10·57	16·68	24
Efficiency (per cent.)		86·8	81·7	73·2	65·4	50·1	33·3
Power factor (per cent.)		72·4	80	78·4	74·1	68	60·4

* Parallel to VU.

† L is point of intersection (not marked in Fig. 85) of GH' and A_1U.

CHAPTER VIII

POLYPHASE CURRENTS AND SYSTEMS

The simplest form of polyphase alternator. In Chapter I, when considering the simple alternator, it was shown that if the conductors forming the coil-sides of the rotating coil were distributed over the surface of the supporting cylinder, the E.M.Fs. generated in the several turns were not of the same phase (see Figs. 6, 7). Hence, if instead of a single coil, a number of coils, displaced from one another, are arranged on the cylinder and each coil is provided with slip rings and brush gear, as shown in Figs. 87 (a) and 88 (a), —which show the arrangement for two and three coils respectively—the E.M.Fs. between the several pairs of slip rings will differ in phase. We have then the simplest type of polyphase alternator.

The mutual phase differences between the several E.M.Fs. are equal to the mutual angular displacements, in electrical degrees, between the respective coils. For example, with two coils fixed 90 electrical degrees apart [Fig. 87 (a)] the phase differences between the E.M.Fs. will be 90 degrees, and with three coils fixed 120 electrical degrees apart [Fig. 88 (a)] the phase differences will be 120 degrees.

Assuming the coils to be rotating with constant angular velocity in a magnetic field of uniform density, the E.M.Fs. will vary as a sine function of the time, and may be represented by the equations

$$e_1 = E_{1m} \sin \omega t, \qquad e_2 = E_{2m} \sin(\omega t - \tfrac{1}{2}\pi),$$

in the former case [Fig. 87 (a)]; and

$$e_1 = E_{1m} \sin \omega t,\ e_2 = E_{2m} \sin(\omega t - \tfrac{2}{3}\pi),\ e_3 = E_{3m} \sin(\omega t - \tfrac{4}{3}\pi)$$

in the latter case.

In each case the E.M.Fs. have the same frequency because all the coils rotate with the same angular velocity.

Graphical representations of the E.M.F. equations are given, in rectangular co-ordinates, in Figs. 87 (b), 88 (b); and vector diagrams showing the R.M.S. values of the several E.M.Fs. are given in Figs. 87 (c), 88 (c).

Conventional methods of representing the alternators in circuit diagrams are shown in Figs. 87 (d), 88 (d).

Polyphase systems. If each coil of the alternators of Figs. 87, 88, is connected to a separate circuit as represented by the conventional diagrams in Figs. 89, 90, the currents, assuming these

circuits to be similar, will have phase differences of 90 degrees in the former case and 120 degrees in the latter case. Each of these

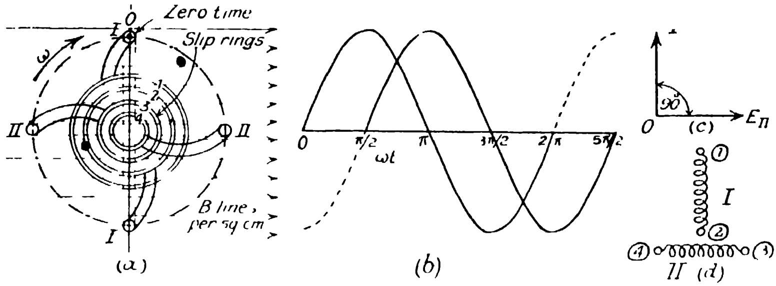


FIG. 87. Simplest Form of Two phase Alternator

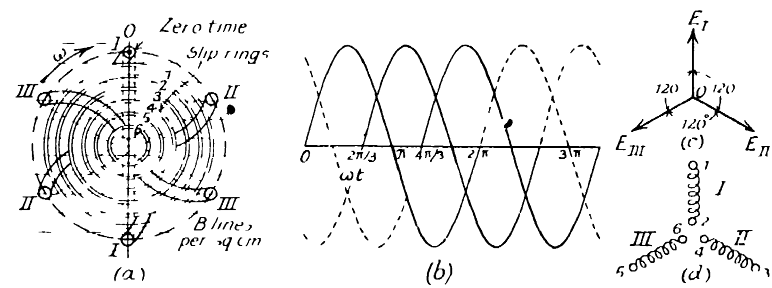
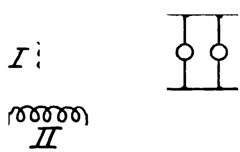


FIG. 88.—Simplest Form of Three-phase Alternator



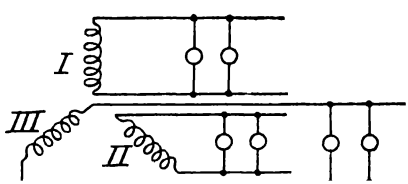


FIG. 89 FIG. 90
Circuit Connections of Alternators of Figs. 87 and 88

combinations constitutes a polyphase system, the former, Fig. 89, being called a *two-phase system,* and the latter, Fig. 90, a *three-phase system.*

[Observe that the term " phase " is here employed in a different

sense to that used in Chapter I. In the present case the term "phase" refers to two or more circuits—forming part of, or being supplied by, a single alternator (or other source of electric power)—in which a phase difference exists between the generated, or impressed E.M.Fs. associated with the respective circuits. Thus the circuits of a polyphase system are called the "phases" of the system.]

In addition to the classification according to the number of phases or circuits, polyphase systems may be classified according to (1) the symmetry and balance of the currents and E.M.Fs., and (2) the manner in which the phases are interconnected.

Symmetrical systems. A polyphase system is symmetrical when the several E.M.Fs., of the same frequency, have equal maximum values and are displaced from one another by equal time angles. Thus in a symmetrical n-phase system the n E.M.Fs. are displaced from one another by $1/n$th of a period. The E.M.Fs., if sinusoidal, are represented by the equations

$$e_1 = E_m \sin \omega t,\ e_2 = E_m \sin\left(\omega t - \frac{2\pi}{n}\right),\ e_3 = E_m \sin\left(\omega t - 2\left[\frac{2\pi}{n}\right]\right),$$

$$\ldots\ e_n = E_m \sin\left[\omega t - (n-1)\frac{2\pi}{n}\right].$$

The E.M.F. vectors, therefore, form a regular closed polygon, and their vector sum is zero. Moreover, the algebraic sum of the instantaneous E.M.Fs. is zero at every instant. For example, with a three-phase system ($n = 3$) we have

$$e_1 = E_m \sin \omega t, \quad e_2 = E_m \sin(\omega t - \tfrac{2}{3}\pi), \quad e_3 = E_m \sin(\omega t - \tfrac{4}{3}\pi);$$

whence

$$\begin{aligned} e_1 + e_2 + e_3 &= E_m[\sin \omega t + \sin(\omega t - \tfrac{2}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= E_m[\sin \omega t + 2 \sin(\omega t - \pi) \,.\, \cos \tfrac{1}{3}\pi] \\ &= 0. \end{aligned}$$

Similarly, with a four-phase system ($n = 4$) we have

$$e_1 = E_m \sin \omega t, \quad e_2 = E_m \sin(\omega t - \tfrac{1}{2}\pi),$$

$$e_3 = E_m \sin(\omega t - \pi), \quad e_4 = E_m \sin(\omega t - \tfrac{3}{2}\pi),$$

whence

$$\begin{aligned} e_1 + e_2 + e_3 + e_4 = E_m[\sin \omega t + \sin(\omega t - \tfrac{1}{2}\pi) &+ \sin(\omega t - \pi) \\ &+ \sin(\omega t - \tfrac{3}{2}\pi)] \\ = 0. \end{aligned}$$

The two-phase system, in which there are two E.M.Fs. having a phase difference of 90 degrees and which are represented by the equations $e_1 = E_m \sin \omega t$, $e_2 = E_m \sin(\omega t - \frac{1}{2}\pi)$, is accordingly unsymmetrical, but, as will be shown later, this system is a special case of the four-phase system.

The principal symmetrical systems in practical use are the three-phase and six-phase systems, of which the latter is used almost entirely in connection with converting machinery (rotary converters) and is obtained from a three-phase system. Other symmetrical systems in use are the four-phase, nine-phase, and twelve-phase systems. These systems are used in connection with converting machinery; the four-phase system being used with rotary converters supplied from a two-phase system, and the nine and twelve-phase systems being used with rotary and motor converters supplied from three-phase systems. The six-phase and twelve-phase systems are also used in connection with mercury-arc power rectifiers.

Balanced systems. A polyphase system is balanced when the loads on the several circuits, or phases, are equal and have the same power factor. Under these conditions the instantaneous power in the system as a whole is constant, notwithstanding that the power in each phase is pulsating. This feature gives polyphase systems a great advantage over a single-phase system for the supply of power to motors and converting machinery.

Power in a polyphase balanced system. The power in a balanced polyphase system is, at any instant, equal to the sum of the instantaneous power in the separate phases. Thus if in an n-phase system the several E.M.Fs. are given by the equations

$$e_1 = E_m \sin \omega t,\quad e_2 = E_m \sin(\omega t - 2\pi/n),\quad e_3 = E_m \sin(\omega t - 2(2\pi/n)),$$
$$\ldots\ e_n = E_m \sin(\omega t - 2\pi(n-1)/n),$$

and the currents are given by the equations

$$i_1 = I_m \sin(\omega t - \varphi),\quad i_2 = I_m \sin(\omega t - \varphi - 2\pi/n),$$
$$i_3 = I_m \sin(\omega t - \varphi - 2(2\pi/n)),\ \ldots$$

the instantaneous power is given by

$$\begin{aligned} p &= e_1 i_1 + e_2 i_2 + e_3 i_3 + \ldots + e_n i_n \\ &= E_m I_m [\sin \omega t \,.\, \sin(\omega t - \varphi) + \sin(\omega t - 2\pi/n) \,.\, \sin(\omega t - \varphi - 2\pi/n) \\ &\qquad + \sin(\omega t - 4\pi/n) \,.\, \sin(\omega t - \varphi - 4\pi/n) + \ldots] \\ &= E_m I_m \tfrac{1}{2} [\cos\varphi - \cos(2\omega t - \varphi) + \cos\varphi - \cos(2\omega t - \varphi - 4\pi/n) \\ &\qquad + \cos\varphi - \cos(2\omega t - \varphi - 8\pi/n) + \ldots] \end{aligned}$$

$$p = E_m I_m \tfrac{1}{2}[n\cos\varphi - \cos(2\omega t-\varphi) - \{\cos(2\omega t-\varphi)\,.\,\cos 4\pi/n + \sin(2\omega t-\varphi)\,.\,\sin 4\pi/n\} - \{\cos(2\omega t-\varphi)\,.\,\cos 8\pi/n + \sin(2\omega t-\varphi)\,.\,\sin 8\pi/n\} - \ldots]$$

$$= \tfrac{1}{2}E_m I_m[n\cos\varphi - \cos(2\omega t-\varphi)\{1+\cos 4\pi/n + \cos 8\pi/n + .\} - \sin(2\omega t-\varphi)\{\sin 4\pi/n + \sin 8\pi/n + \ldots\}]$$

$$= \tfrac{1}{2}nE_m I_m \cos\varphi$$

$$= n\,E\,I\cos\varphi \, . \qquad (69)$$

[NOTE.— $1 + \cos 4\pi/n + \cos 8\pi/n + \cos 12\pi/n + \ldots = 0$,

$\sin 4\pi/n + \sin 8\pi/n + \sin 12\pi/n + \ldots = 0$,

n being an integer.]

Hence the instantaneous power is expressed in terms of the constant quantities E, I, n, $\cos\varphi$. Therefore the total power in any polyphase system with balanced loads is entirely free from pulsation.

Production of rotating magnetic fields by polyphase currents. In addition to the above advantage a symmetrical polyphase system possesses the further advantage that a uniformly rotating magnetic field, of constant magnitude, may be produced by means of *stationary* coils.

The general conditions to be satisfied for the production of a bi-polar magnetic field from an n-phase system are (1) the number of magnetizing coils must equal the number of phases, (2) these coils must be similar and must be arranged symmetrically with respect to a central axis, and (3) the angle between their magnetic axes, taken in order, must equal the mutual phase difference between the currents supplied to the coils (i.e. $360/n$ degrees).

The speed of rotation of the magnetic field is equal to the angular velocity of the supply currents, i.e. the speed in revolutions per second is equal to the frequency of the supply currents.

When a multipolar magnetic field is required there must be n similar coils per pair of poles, and the angle, measured in degrees, between the magnetic axes of the successive coils must equal $(360/\tfrac{1}{2}np)$, where p is the number of poles. In this case the speed of the field, in revolutions per second, is equal to $f/\tfrac{1}{2}p$, where f is the frequency of the supply currents.

In all cases the magnitude of the rotating field is given by

$$H = \tfrac{1}{2}nH_m,$$

where H_m is the maximum magneto-motive-force due to one coil.

Analytical proof. Consider three coils arranged, as in Fig. 91, with their magnetic axes 120° apart. Let these coils be supplied from a symmetrical three-phase system, and let the instantaneous values of their M.M.Fs. be represented by the equations

$$h_A = H_m \sin \omega t, \qquad h_B = H_m \sin(\omega t - \tfrac{2}{3}\pi), \qquad h_C = H_m \sin(\omega t - \tfrac{4}{3}\pi).$$

These M.M.Fs. have directions in space along the magnetic axes of the respective coils and they are each alternating at the frequency of the supply currents.

The value of the resultant M.M.F., in space, due to the joint action of the coils may, at any particular instant, be obtained from a knowledge of their components, at this instant, along two arbitrary axes perpendicular to each other.

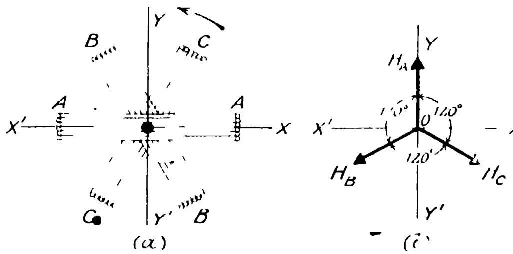


FIG. 91.—Space and Vector Diagrams for Analytical Proof of Theorem of Rotating Magnetic Field

Thus, taking one axis (Y) along the magnetic axis of coil A, and measuring space angles in the counter-clockwise direction from the positive or right-hand horizontal axis, we have for the sum of the components of M.M.Fs. along the vertical axis at the instant t

$$\begin{aligned}
h_Y &= h_A \sin 90^\circ + h_B \sin 210^\circ + h_C \sin 330^\circ \\
&= H_m[\sin \omega t - \tfrac{1}{2}\sin(\omega t - \tfrac{2}{3}\pi) - \tfrac{1}{2}\sin(\omega t - \tfrac{4}{3}\pi)] \\
&= H_m[\sin \omega t - \tfrac{1}{2}(\sin \omega t . \cos \tfrac{2}{3}\pi - \cos \omega t . \sin \tfrac{2}{3}\pi) \\
&\qquad - \tfrac{1}{2}(\sin \omega t . \cos \tfrac{4}{3}\pi - \cos \omega t . \sin \tfrac{4}{3}\pi)] \\
&= H_m\left[\sin \omega t - \frac{1}{2}\left(-\frac{1}{2}\sin \omega t - \frac{\sqrt{3}}{2}\cos \omega t\right) - \frac{1}{2}\left(-\frac{1}{2}\sin \omega t + \frac{\sqrt{3}}{2}\cos \omega t\right)\right] \\
&= \tfrac{3}{2} H_m \sin \omega t.
\end{aligned}$$

Similarly, the sum of the components of M.M.Fs. along the perpendicular axis (X) at the instant t are

$$\begin{aligned}
h_X &= h_A \cos 90^\circ + h_B \cos 210^\circ + h_C \cos 330^\circ \\
&= H_m\left[-\frac{\sqrt{3}}{2}\sin\left(\omega t - \frac{2}{3}\pi\right) + \frac{\sqrt{3}}{2}\sin\left(\omega t - \frac{4}{3}\pi\right)\right] \\
&= \frac{\sqrt{3}}{2} H_m\left[-\left(\sin \omega t . \cos \frac{2}{3}\pi - \cos \omega t . \sin \frac{2}{3}\pi\right) + \left(\sin \omega t . \cos \frac{4}{3}\pi - \cos \omega t . \sin \frac{4}{3}\pi\right)\right] \\
&= \frac{\sqrt{3}}{2} H_m\left(+\frac{1}{2}\sin \omega t + \frac{\sqrt{3}}{2}\cos \omega t - \frac{1}{2}\sin \omega t + \frac{\sqrt{3}}{2}\cos \omega t\right) \\
&= +\tfrac{3}{2} H_m \cos \omega t
\end{aligned}$$

Therefore the magnitude of the resultant M.M.F. at the instant t is given by

$$h = \sqrt{(h_X^2 + h_Y^2)} = \sqrt{\{(\tfrac{3}{2} H_m \sin \omega t)^2 + (\tfrac{3}{2} H_m \cos \omega t)^2\}}$$
$$= \tfrac{3}{2} H_m,$$

i.e. the resultant M.M.F. is constant in magnitude and is equal to 3/2 × maximum M.M.F. of one coil.

Let θ represent the space angle which the resultant M.M.F. makes with the X axis at the time t.

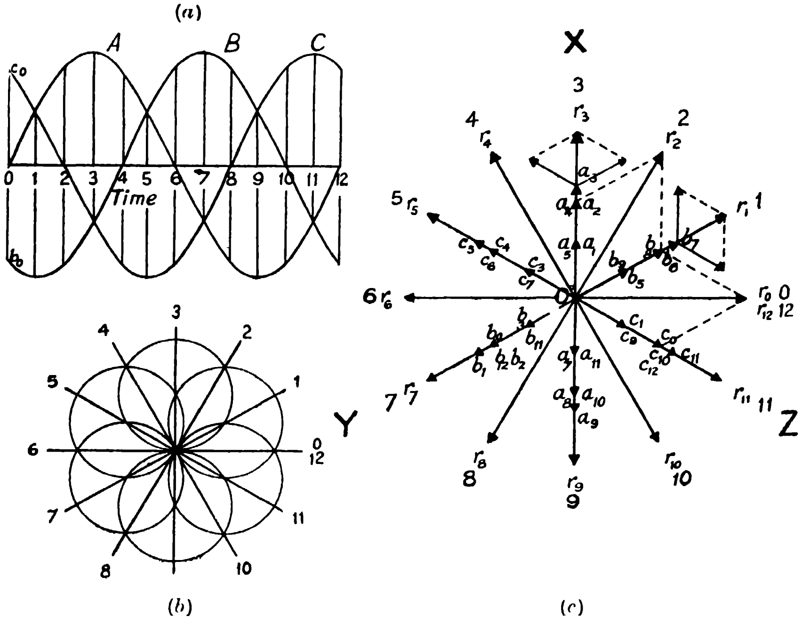


FIG. 92. Time and Space Diagrams for Graphical Proof of Theorem of Rotating Magnetic Field

Then
$$\tan \theta = \frac{h_Y}{h_X} = \frac{\tfrac{3}{2} H_m \sin \omega t}{\tfrac{3}{2} H_m \cos \omega t} = \tan \omega t,$$
$$\theta = \omega t.$$

Therefore the magnetic field rotates in the counter-clockwise direction with an angular velocity of ω, and the number of revolutions per second is equal to the frequency of the supply currents.

Graphical proof. The construction for the graphical proof of the theorem is extremely simple for the case of two coils supplied with current from a two-phase system, the coils being arranged about a common centre with their magnetic axes perpendicular to each other. We shall, however, give the construction for the three-phase case.

First, a time diagram is required showing the instantaneous values of the M.M.Fs. of the coils at successive instants of the period. This diagram may be drawn either in rectangular co-ordinates, as shown in Fig. 92(a), or in polar co-ordinates, as shown in Fig. 92(b).

Second, a space vector diagram is required showing the directions of these M.M.Fs., as well as their magnitudes, in space.

The space vector diagram for the three phase case under consideration is shown in Fig. 92(c), and contains three axes of reference, OX, OY, OZ, having a mutual displacement of 120°.

To obtain the direction and magnitude of the resultant M.M.F. at successive instants, the instantaneous values of the M.M.Fs. of the coils are obtained from the time diagram and are transferred to the space vector diagram from which the resultant is obtained by constructing the vector parallelogram.

For example, let the period be divided into twelve equal parts and the time intervals be numbered 0 to 12, as in Fig. 92(a). At zero time let the M.M.F. of coil A be zero and the M.M.Fs. of coils B and C be represented by the ordinates b_0, c_0, in Fig. 92(a). Setting these quantities off as Ob_0, Oc_0, in the space vector diagram of Fig. 92(c), we obtain, by constructing the vector parallelogram, the direction and magnitude of the resultant Or_0. Since $b_0 = -0.866\,H_m$, and $c_0 = 0.866\,H_m$, the resultant Or_0 is equal to

$$2 \times 0.866\,H_m \cos 30° = \tfrac{3}{2}H_m$$

When one twelfth of a period has elapsed the M.M.Fs. of the coils are represented in the time diagram by the ordinates a_1, b_1, c_1, and when these quantities are set off in their correct positions in the space vector diagram we obtain the resultant Or_1. Since $a_1 = \tfrac{1}{2}H_m$, $b_1 = -H_m$, $c_1 = \tfrac{1}{2}H_m$, the resultant Or_1 is equal to $H_m + 2 \times \tfrac{1}{2}H_m \cos 60° = \tfrac{3}{2}H_m$. Moreover, the angle between Or_0 and Or_1 is 30°, which is equal to the time angle (i.e. $\tfrac{1}{12} \times 360°$) between the points 0 and 1 in the time diagram. Similar results will be obtained for other points of the period.

Hence the magnitude of the resultant M.M.F. is constant and is equal to $\tfrac{3}{2}H_m$.

The locus of the resultant M.M.F. vector is a circle, and the radius vector makes one revolution during each period of the supply current.

Interconnection of the phases of a polyphase system. If the phases of a polyphase system supply separate circuits, as in Figs. 89, 90, then a pair of line wires is required for each circuit. But by suitably interconnecting the phases the number of line wires can be reduced without affecting the operation of the system. Thus the three-phase, six-wire system of Fig. 90 can be reduced to one having only three line wires. Similarly, four and six-phase systems can be operated with four and six line wires respectively, and, in general, any symmetrical n-phase system can be operated with n line wires.

The interconnection must be carried out in such a manner that if closed circuits are formed the sum of the instantaneous E.M.Fs. in them must be zero in order that there shall be no circulating currents in these circuits.

The principal methods of interconnection are (1) the star connection, (2) the mesh, or ring, connection. These are shown diagrammatically in Figs. 93, 94.

Star connection of polyphase systems. The star connection, Fig. 93, is formed by joining one end of each phase to a common point—which is called the "neutral point"—and connecting the

other ends to the line wires. In making this connection for a generator, transformer, motor or apparatus in which the electric circuits are interlinked with magnetic circuits, the line wires must be connected to the terminals, or the ends of the phases, which have

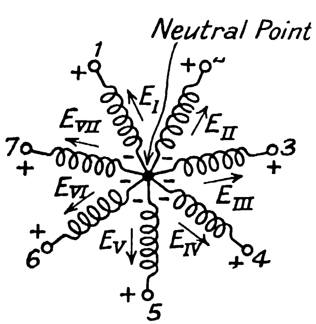


Fig. 93

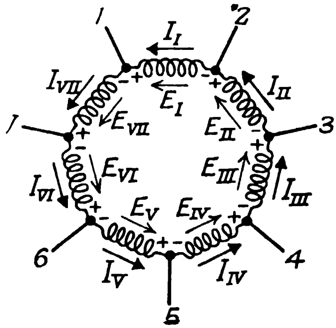


Fig. 94

Star- and Mesh connections of Polyphase System

like polarity at successive instants, as if this condition is not fulfilled the interconnected system will be unsymmetrical. [Examples of dissymmetry due to incorrect connections are, for the three-phase system, given on p. 195.]

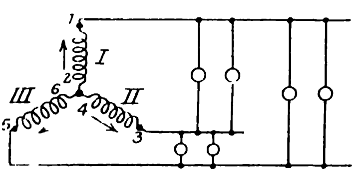


Fig. 95

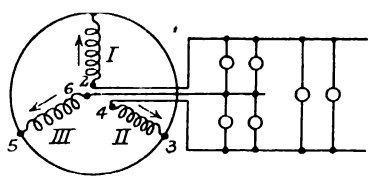


Fig. 96

Methods of Interconnecting the Three phase Alternator of Figs. 88, 90, to obtain Three Line Wires

Hence in these cases *similar ends* (i.e. either "starting" or "finishing" ends) of the phases—assuming the coils to be wound in the same direction and connected in the same manner—must be connected together to form the neutral point. For example, with the simple three-phase alternator, shown in Fig. 88, the neutral point may be formed either, as shown in Fig. 95, by connecting together the coil-ends which were originally connected to slip rings 2, 4, 6, in which case the line wires are connected to slip rings 1, 3, 5; or, as shown in the alternative method of Fig. 96,

by connecting together the coil-ends which were originally connected to slip rings 1, 3, 5, in which case the line wires are connected to slip rings 2, 4, 6.

In the case of load circuits containing resistance or reactance, the particular ends of the circuits which must be connected together to form the neutral point are immaterial, provided that, with inductively-reactive circuits the several magnetic circuits are not interlinked magnetically. If the magnetic circuits are interlinked, however, as in the case of the three-phase reactance, or choking coil, shown in Fig. 97, then only similar ends of the coils may be connected together.

Line E.M.F. and current relations for star connection. The magnitude of the E.M F. between any pair of line wires, or terminals, of a star-connected polyphase alternator is given simply by the vector difference between the E.M.Fs. of the phases to which these lines are connected, since in making the star connection of the windings the ends of like instantaneous polarity were connected together. In the case of an unloaded n-phase symmetrical system with sinusoidal E.M.Fs. each of the line voltages is equal to

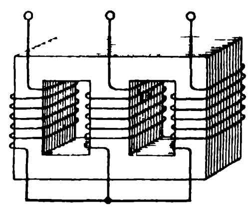


FIG. 97. Connections of Three phase, Star connected, Choking Coil

$$2\sin(\pi/n) \times \text{phase voltage},$$

and leads the corresponding phase voltage by $[\frac{1}{2}(\pi - 2\pi/n)]$ radians, or $(90 - 180/n)$ degrees, i e. the voltage between the lines connected to, say, phases 1 and 2 is $(90 - 180/n)$ degrees in advance of the voltage of phase 1.

Proof. Taking the positive direction of the E.M F generated in each phase to be outwards, or away from the neutral point, as indicated in Fig. 93, the R.M.S. values E_1, E_2, of the E M Fs. in two adjacent phases are represented in the vector diagram of Fig. 98, by the vectors OA, OB. The E.M.F. between the terminals of these phases is given by the difference of the vectors OA, OB, i.e. by the vector OC.

Now in a symmetrical system $E_1 = E_2 = E$, say. Hence, in Fig. 98, $OA = OB$, and $OC = AB$.

If AB is bisected at F and this point is joined to O, the line OF is perpendicular to AB and bisects the angle between OA and OB. Therefore, $AB = 2AF = 2OA \sin \frac{1}{2}(2\pi/n)$. Whence $OC = E_1 - E_2 = 2E \sin(\pi/n) = 2 \sin(\pi/n) \times$ phase E.M.F. The angle between the vectors OC and OA is $[\frac{1}{2}\pi - \frac{1}{2}(2\pi/n)]$ radians, or $(90 - 180/n)°$.

The current in any line is equal to the current in the phase to which the line is connected.

Mesh, or ring, connected polyphase system. The mesh, or ring, connection (Fig. 94) is formed by joining the several phases in

series to form a closed circuit and connecting the line wires to the junctions of the phases. In making this connection for a generator, transformer, motor, or apparatus in which the electric circuits are interlinked with magnetic circuits *dissimilar ends* of adjacent phases must be connected together. Hence the resultant E.M.F. acting in the closed circuit is equal to the vector sum of the several

E.M.Fs., the resultant E.M.F.

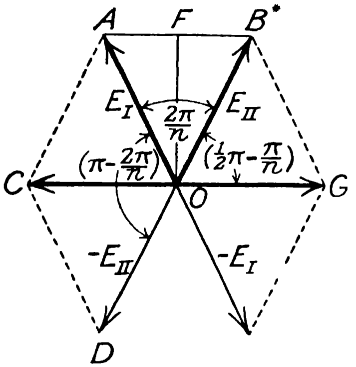


FIG. 98

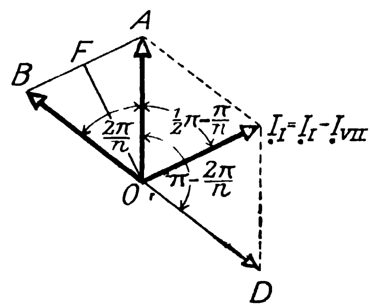


FIG. 99

Vector Diagrams for Star- and Mesh connected Polyphase Systems

currents will flow in the closed circuit. With non-sinusoidal E.M.Fs. the resultant E M F. may not be zero, and in this case circulating currents will flow in the closed circuit.

Line E.M.F. and current relations for mesh connection. The E.M.F. between two adjacent line wires is equal to the E.M.F. in the phase to which these line wires are connected.

The current in any line wire is equal to the vector difference of the currents in the phases connected to that line wire. In the case of a balanced n-phase system in which the currents are sinusoidal the line current is equal to

$$2\sin(\pi/n) \times \text{phase current},$$

and leads the currents in the phases connected to that line wire by angles of $(\frac{1}{2}\pi + \pi/n)$ and $(\frac{1}{2}\pi - \pi/n)$ radians, as shown below.

Proof. Let the positive direction of the currents in the phases and line wires be that marked in the circuit diagram of Fig. 94. Then, assuming balanced loads, the phase currents are represented, in the vector diagram of Fig. 99, by the vectors OA, OB, for the two phases under consideration. The current in the line wire connected to these phases is equal to $I_1 - I_2$,

and is represented by the difference of the vectors OA and OB, i.e. by the vector OC.

Now for a symmetrical and balanced system $I_1 = I_2 = I$, say. Hence, in Fig. 99, $OA = OB$, and $OC = BA = 2\,OA \sin \frac{1}{2}(2\pi/n)$. Whence the line current ($= OC$) $= I_1 - I_2 = 2\,I \sin(\pi/n) = 2 \sin(\pi/n) \times$ phase current.

The angle between OC and OA—i.e. the phase difference between the current in line wire A, Fig. 99, and the current in phase I_1—is equal to $\frac{1}{2}(\pi - 2\pi/n) = (\frac{1}{2}\pi - \pi/n)$ radians, or $(90 - 180/n)°$. Similarly, the angle between OC and OB is equal to $\frac{1}{2}(\pi + \pi/n)$ radians.

Weight of line conductors for single-phase and inter-connected polyphase systems. In comparing the weight of copper required for the line conductors of single-phase and polyphase systems it is essential to base the comparison upon similar conditions in each case. Thus the total power, the power factor, and the distance over which the power is transmitted must obviously be the same for the several cases under consideration. Other conditions which must be co-related are: the efficiency of the transmission, the percentage drop in voltage, the current density in the line wires, the voltage between the line wires, and the voltage between each line wire and earth. Some of these conditions are independent of one another, while others are inter-dependent. For example, in systems in which the neutral point is earthed the voltage between line wires depends upon the number of phases and the phase voltage. Hence, if the voltage between any line wire and earth (i.e. the phase voltage of the system) is fixed, the voltage between the line wires of a given system will depend upon the number of phases in that system.

Again, if the reactance of the line is neglected the percentage drop in voltage in the transmission system is equal to the I^2R loss in the line wires expressed as a percentage of the total power transmitted,* and since the efficiency of transmission is given by

* Thus consider for simplicity a single phase system at unity power factor. Let V = supply voltage, v = voltage drop in line wires, and I = line current. Then the voltage at the load, or receiver end of the system, is given by $(V - v)$, and the power utilized is given by $(V - v)I$. The power, P, supplied from the generator is given by VI. Hence the efficiency of transmission is given by

$$\eta = \frac{I(V - v)}{IV} = 1 - \frac{v}{V}$$

Also, if R is the total resistance of the line wires the voltage drop (v) in the lines is given by IR, and the power expended in the line wires is given by $vI = I^2R$. Wh nce

$$\frac{vI}{VI} = \frac{I^2R}{VI}$$

or
$$\frac{v}{V} \times 100 = \frac{I^2R}{P} \times 100$$

[1 – (power expended in line wires/total power transmitted)], the percentage drop in voltage and the efficiency of transmission are inter-dependent.

Case I. Star-connected systems with earthed neutral point (Fig. 100). With symmetrical and balanced star-connected systems, in which the neutral point is earthed and the phase voltage is the same in each case, the cross section of the line wires of the different systems, for the transmission of a given amount of power over a given distance at a given power factor and efficiency, is determined in the following manner.

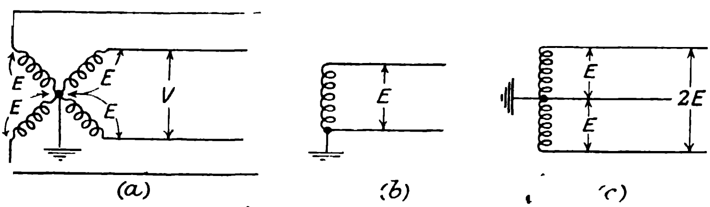


FIG. 100.—Circuit Diagrams of Polyphase and Single-phase Transmission Systems with Earthed Neutral Point

Let n = number of phases, P = total power supplied to the transmission lines, E = phase voltage, $\cos\varphi$ = power factor. Then, assuming sinusoidal E.M.Fs., the voltage between adjacent line wires, taken in order, is equal to $2E\sin\pi/n$. Also with balanced loads the power transmitted by each phase is equal to P/n, and the current in each wire is equal to $P/(nE\cos\varphi)$. Hence the I^2R loss in each line is given by $R_p(P/nE\cos\varphi)^2$, where R_p is the resistance of each line. If the efficiency is to be the same for each case the loss in the line wires must bear a fixed ratio to the power transmitted, and therefore

$$\frac{R_p(P/nE\cos\varphi)^2}{P/n} = 1 - \eta = \zeta, \text{ a constant,}$$

where η is the efficiency of transmission.

Now $R_p = \rho l/a$, where ρ is the specific resistance of the material of the conductors and l, a, are the length and cross section, respectively, of each. Hence

$$\zeta = \frac{\rho Pl}{E^2\cos^2\varphi}\cdot\frac{1}{na}$$

i.e. the product na is a constant. But na is proportional to the total weight of the line conductors.

Thus the amount of copper required for the line wires for the transmission of a given power under similar conditions is the same for all star-connected polyphase systems with earthed neutral point.

Comparison with single-phase system. To obtain a comparison with the single-phase system under similar conditions we shall consider the cases in which (α) one terminal of the generator is earthed, Fig. 100 (b), (β) the mid-point of the generator winding is earthed, Fig. 100 (c). In the first case the voltage between line wires is equal to the phase voltage, E, of the polyphase systems, but in the second case the voltage between line wires is equal to twice the phase voltage of the polyphase systems.

For case (α) we have

$$\text{line current} = P/E \cos\varphi$$

I^2R loss in line wires $= 2R_s\,(P/E\cos\varphi)^2$, where R_s is the resistance of each line. Hence for the same efficiency of transmission as in the polyphase systems we must have

$$\frac{2R_s(P/E\cos\varphi)^2}{P} = \frac{R_p(P/nE\cos\varphi)^2}{P/n}$$

Whence $$2R_s = R_p/n.$$

or, since resistance is inversely proportional to the cross section when the length is constant,

$$a_s = 2na_p,$$

where a_s, a_p, are the cross sections of the line conductors for the single and polyphase systems respectively.

But the total weight of line conductors for the single-phase system under consideration is proportional to $2a_s$, and that for the polyphase systems is proportional to na_p.

Hence

$$\frac{\text{Weight of line conductors for single-phase system}}{\text{Weight of line conductors for polyphase system}}$$

$$= \frac{2a_s}{na_p} = \frac{2 \times 2na_p}{na_p} = 4,$$

i.e. the weight of the line conductors for a single-phase system with one terminal earthed is four times that for a star-connected polyphase system with earthed neutral point, the amount of power transmitted, the efficiency, the voltage between line wires and earth, etc., being the same in each case.

For case (β), in which the mid-point of the generator winding is earthed, we have

Voltage between line wires $= 2E$.

Line current $= P/2E \cos\varphi$.

I_2R loss in line conductors $= 2R'_s(P/2E \cos\varphi)^2$

where R'_s is the resistance of each line in the present case.

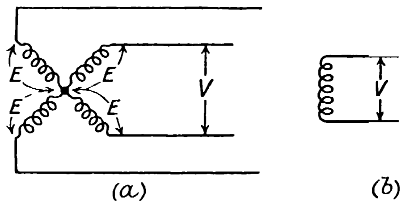


Fig. 101 — Circuit Diagrams of Polyphase and Single phase Transmission Systems with Insulated Neutral Point.

Hence for the same efficiency of transmission as in the polyphase systems we must have

$$\frac{2R'_s(P/2E \cos\varphi)^2}{P} = \frac{R_p(P/nE \cos\varphi)^2}{P/n}$$

Whence $$R'_s = \frac{2}{n} R_p,$$

and $$a'_s = \tfrac{1}{2} na_p,$$

i.e. $$2a'_s = na_p$$

But $2a'_s$ is proportional to the total weight of the line conductors for the single-phase system under consideration, and na_p is proportional to the total weight of the line conductors for the polyphase systems. Hence in this case, which is a special one, the weight of the line conductors is the same as that for the polyphase systems.

It is of interest to note that the current density in the line conductors for this case is the same as that in the polyphase cases, but in the previous single-phase case the current density is one-half of that for the polyphase cases.

Case II. Non-earthed systems (Fig. 101). In this case the polyphase system may be either star or mesh connected, but with the star connection the neutral point is insulated. The comparison will be based upon equal line voltages in all cases, the other governing conditions being the same as in the cases considered above.

Considering star-connected systems for convenience, let V = line voltage. Then the phase voltage, E, of an n-phase system is equal to $V/2 \sin \pi/n$, and the line current is equal to

$$P/nE \cos \varphi = (2P \sin \pi/n)/(nV \cos \varphi).$$

If R_p' is the resistance of one line, the I^2R loss in each line is given by $R_p'[(P \sin \pi/n)/(nV \cos \varphi)]^2$. Hence for the same efficiency of transmission as before we must have

$$\frac{R_p'[(2P \sin \pi/n)/(nV \cos \varphi)]^2}{P/n} = \zeta, \text{ a constant,}$$

i.e. $$4R_p' \frac{\sin^2 \pi/n}{n} \cdot \frac{P}{V^2 \cos^2 \varphi} = \zeta.$$

If ρ is the specific resistance of the material of the line conductors, and l, a_p', are the length and cross section, respectively, we have

$$\frac{4\sin^2 \pi/n}{na_p'} \cdot \frac{P\rho l}{V^2 \cos^2 \varphi} = \zeta,$$

whence $$na_p' = 4\sin^2 \pi/n \times \text{a constant.}$$

Now na'_p is proportional to the total weight of the line conductors. Hence in this case the weight of the line conductors varies with the number of phases, becoming smaller as the number of phases is increased. For example, for the three, four, and six-phase systems the relative weights of the line conductors are in the ratio $\sin^2 \pi/3 : \sin^2 \pi/4 : \sin^2 \pi/6$, or in the ratio $[(\sqrt{3}/2)^2 =]0{\cdot}75 : [(1/\sqrt{2})^2 =]0{\cdot}5 : [(1/2)^2 =]0{\cdot}25$.

The decrease in weight of the line conductors due to the increase in the number of phases results in an increase in the current density and heating, the current densities for the above cases being in the ratio $\operatorname{cosec} \pi/3 : \operatorname{cosec} \pi/4 : \operatorname{cosec} \pi/6$ or $2/\sqrt{3} : \sqrt{2} : 2$, or $1{\cdot}15 : 1{\cdot}414 : 2$.

Comparison with single-phase system. In the single-phase system the line current is given by $P/V \cos \varphi$, and if the resistance of each line is R_s'' the total line I^2R loss is given by $2R_s'' (P/V \cos \varphi)^2$. Hence for the same efficiency of transmission as in the polyphase systems we must have

$$\frac{2R''_s(P/V \cos \varphi)^2}{P} = 4R_p' \frac{\sin^2 \pi/n}{n} \cdot \frac{P}{V^2 \cos^2 \varphi}$$

Whence $R_s'' = (2R_p' \sin^2 \pi/n)/n$

and $2a_s'' = na_p'/\sin^2 \pi/n$

where a_s'', a_p', are the cross sections of the line conductors for the single-phase and polyphase systems now under consideration.

But the weight of the line conductors for the single-phase system is proportional to $2a_s''$, and that for the polyphase system is proportional to na_p'. Therefore

$$\frac{2a_s''}{na_p'} = \frac{1}{\sin^2 \pi/n}$$

i.e.

$$\frac{\text{Weight of line conductors for single-phase system}}{\text{Weight of line conductors for polyphase system}}$$

$$= \frac{1}{\sin^2 \pi/n}$$

The values of the quantity $[4/(n \sin^2 \pi/n)]$ for the three, four, and six-phase systems are

$\dfrac{1}{(\sqrt{3}/2)^2} = 1{\cdot}33$ for the three-phase system ;

$\dfrac{1}{(1/\sqrt{2})^2} = 2$ for the four-phase system ;

$\dfrac{1}{(1/2)^2} = 4$ for the six-phase system.

Therefore for non-earthed systems and equal voltages of transmission the weight of the line conductors for the single-phase system is

1·33 times that of the line conductors for the three-phase system,
twice that of the line conductors for the four-phase system, and
4 times that of the line conductors for the six-phase system.

It will be of interest to determine the relative current densities for these cases. Thus if a''_s a'_p, are the current densities in the single-phase and polyphase systems now under consideration we have

$$a_s'' = (P/V \cos \varphi)/a_s''.$$

$$a_p' = [2(P \sin \pi/n)/nV \cos \varphi)]/a_p'$$

Whence $$\frac{a_s''}{a_p'} = \sin \frac{\pi}{n},$$

or $$a_p' = a_s''\left(\frac{1}{\sin \pi/n}\right)$$

Therefore for the three-phase system and the specified conditions

the current density is 1·155 times that for the corresponding single-phase system, i.e.

$$a_3' = \left(\frac{1}{\sqrt{3}/2}\right) a_s'' = 1\cdot155\, a_s''.$$

For the four-phase system the current density is 1·414 times that for the corresponding single-phase system, i.e.

$$a_4' = \left(\frac{1}{1/\sqrt{2}}\right) a_s'' = 1\cdot414\, a_s'',$$

and for the six-phase system the current density is twice that for the corresponding single-phase system, i.e.

$$a_6' = \left(\frac{1}{1/2}\right) a_s'' = 2a_s''$$

Thus both earthed and non-earthed polyphase transmission systems require less weight of copper in the line conductors than single-phase transmission.

Summary of advantages of polyphase systems. The principal advantages of polyphase systems compared with single-phase systems are—

(1) A polyphase alternator is smaller and less costly than a single-phase alternator of the same output, voltage, and frequency, because in the former case the armature periphery may be utilized more effectively than is possible in the latter case. Moreover, in the polyphase alternator the armature reaction, with balanced loads, is constant, whereas in the single-phase alternator the armature reaction is pulsating, and this feature requires a more costly construction in the single-phase alternator than in the polyphase machine.

(2) Polyphase transmission requires less weight of copper in the line conductors than single-phase transmission.

(3) With polyphase currents rotating magnetic fields may be produced by means of stationary coils.

(4) The power in a polyphase balanced system is constant and non-pulsating.

(5) Polyphase motors and converting machinery have a higher efficiency and better performance than single-phase machines, these features being due to items (3) and (4).

Application of polyphase systems. Of the various polyphase systems available the three-phase system is the one most extensively employed at the present day, firstly, because this system requires the minimum number (three) of line conductors of any polyphase

system; and, secondly, because it is possible, by means of ***stationary*** transformers, to obtain other polyphase systems—such as the two, six, nine, and twelve-phase systems—from a three-phase system.

COMMERCIAL POLYPHASE SYSTEMS

We will now consider the three-phase, six-phase, four-phase, and two-phase systems more in detail.

THREE-PHASE SYSTEM

In a symmetrical system the phase E.M.Fs. are represented by the equations

$$e_{\text{I}} = E_m \sin \omega t, \quad e_{\text{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi), \quad e_{\text{III}} = E_m \sin(\omega t - \tfrac{4}{3}\pi)$$

Star-connected system. The circuit diagram for a star-connected system is given in Fig. 102, in which the assumed positive directions for E.M.Fs. and currents are indic[illegible] values of the terminal, or line, [illegible] equations

$$\begin{aligned} v_{1-2} &= e_{\text{I}} - e_{\text{II}} \\ &= E_m[\sin \omega t - \sin(\omega t - \tfrac{2}{3}\pi)] \\ &= 2E_m \sin \tfrac{1}{3}\pi \,.\, \cos(\omega t - \tfrac{1}{3}\pi) \\ &= \sqrt{3}\,.\,E_m \cos(\omega t - \tfrac{1}{3}\pi) \\ &= \sqrt{3}\,.\,E_m \sin(\omega t + \tfrac{1}{6}\pi) \end{aligned}$$

$$\begin{aligned} &= E_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= 2E_m \sin \tfrac{1}{3}\pi \,.\, \cos(\omega t - \pi) \\ &= \sqrt{3}\,.\,E_m \cos(\omega t - \pi) \\ &= \sqrt{3}\,.\,E_m \sin(\omega t - \tfrac{1}{2}\pi) \end{aligned}$$

$$\begin{aligned} v_{3-1} &= e_{\text{III}} - e_{\text{I}} \\ &= E_m[\sin(\omega t - \tfrac{4}{3}\pi) - \sin \omega t] \\ &= 2E_m \sin \tfrac{1}{3}\pi \cos(\omega t - \tfrac{5}{3}\pi) \\ &= \sqrt{3}\,.\,E_m \cos(\omega t - \tfrac{5}{3}\pi) \\ &= \sqrt{3}\,.\,E_m \sin(\omega t - \tfrac{7}{6}\pi) \end{aligned}$$

Thus the line E.M.Fs. are equal to one another and have a mutual phase difference of 120 degrees, or $\frac{2}{3}\pi$ radians Also, the line E.M.Fs. have a phase difference (leading) of 30 degrees, or $\frac{1}{6}\pi$ radians, with respect to the phase E.M.Fs. For example, the E.M.F. between lines 1 and 2 is 30 degrees in advance of the E.M.F. of phase I; that between lines 2 and 3 is 30 degrees in advance of the E.M.F. of phase II; and that between lines 3 and 1 is 30 degrees in advance of the E.M.F. of phase III.

The R.M.S. values of the line E.M.Fs. are

$$V_{1-2} = \sqrt{3}\,.\,E; \quad V_{2-3} = \sqrt{3}\,.\,E; \quad V_{3-1} = \sqrt{3}\,.\,E.$$

The vector diagram for a star-connected symmetrical system with balanced loads is shown in Fig. 103, in which the vectors OE_{I}, OE_{II}, OE_{III}, represent the phase E.M.Fs., and the vectors OV_{1-2}, OV_{2-3}, OV_{3-1}, represent the line E.M.Fs.; OV_{1-2} being the difference between the vectors OE_{I} and OE_{II}; OV_{2-3}, the difference between the vectors OE_{II} and OE_{III}; OV_{3-1}, the difference between the vectors OE_{III} and OE_{I}. These vector differences

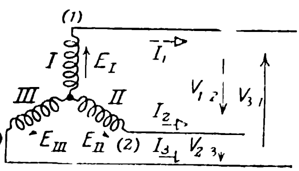


FIG. 102

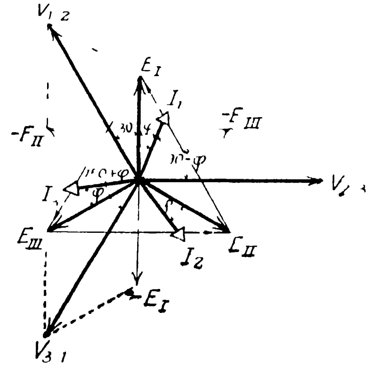


FIG. 103

Circuit and Vector Diagrams of Three-phase, Star-connected (Three-wire) System

may also be represented by the sides of the triangle formed by joining the extremities of the vectors representing the phase E.M.Fs. For example, the side $E_{\text{II}}E_{\text{I}}$, taken in the direction $E_{\text{II}} - E_{\text{I}}$, is equal and parallel to OV_{1-2}. Similarly, the sides $E_{\text{III}}\,E_{\text{II}}$ and $E_{\text{I}}\,E_{\text{III}}$ are equal and parallel to the vectors OV_{2-3} and OV_{3-1} respectively. Hence in a star-connected system the line E.M.Fs. may be represented by the triangle (or polygon, when the number of phases exceeds three) formed by joining the extremities of the vectors representing the phase E.M.Fs.

The line and phase currents are shown, in Fig. 103, by the vectors OI_1, OI_2, OI_3, each of which has a phase difference of φ with respect to the phase E.M.Fs., φ being the power factor of each of the balanced loads. Hence the phase difference between the current in any line and the voltage across adjacent pairs of line wires may be $(30 + \varphi)^\circ$, $(150 + \varphi)^\circ$, or $(90 - \varphi)^\circ$, according to the particular phase and line wires considered. For example, the phase difference between the current in line 1 and the voltage across lines 1 and 2 is $(30 + \varphi)^\circ$ lagging; that between the current in line 1 and the voltage across lines 3 and 1 is $(150 + \varphi)^\circ$ lagging; but that between

the current in line 1 and the voltage across lines 2 and 3 is $(90 - \varphi)^\circ$ leading, the appropriate line E.M.F. vector being considered as the vector of reference in each case. These phase differences are shown clearly in the vector diagram, which has been drawn for φ lagging. If φ is leading the phase differences become $(30 + \varphi)^\circ$, $(150 - \varphi)$, $(90 + \varphi)^\circ$.

Star-connected system with neutral wire. This system, which is shown diagrammatically in Fig. 104 and is called the *three-phase four-wire system,* is used in cases where unbalanced loads have to be supplied from a three-phase system. The principal application of the system in practice is for supplying single-phase lighting networks from a three-phase system which also supplies a power load; and its advantage over the ordinary three-phase star-connected system without neutral wire (which is usually called the *three-phase three-wire system*) is that the power load may be supplied at a higher voltage than the lighting load, since the latter is supplied at the "phase" voltage, while the former may be supplied at the "line" voltage of the system. For example, if the lighting load is supplied at a pressure of 230 volts, the power load may be supplied at a pressure of $230 \times \sqrt{3} = 400$ volts.

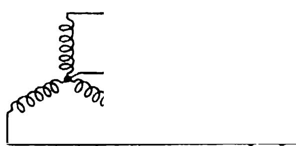


FIG. 104.—Circuit Diagram of Three-phase, Four-wire System

The current in the neutral wire is equal to the vector sum of the currents in the line wires, or "outers." Thus when the single-phase loads are balanced there is no current in the neutral wire, since the vector sum of three equal quantities having a mutual phase difference 120° is zero. When, however, the single-phase loads are unbalanced the magnitude of the current in the neutral wire and its phase relations with respect to the line currents depend upon the manner in which the unbalanced loads are distributed on the system. Vector diagrams for a number of cases are shown in Fig. 105, and a worked example is given in Chapter IX. This example shows also the manner in which the voltage drop in the neutral wire affects the symmetry and balance of the voltages across the loads.

Mesh-connected system. The circuit diagram for a mesh-connected* three-phase system is shown in Fig. 106 (*a*), in which the phases are drawn in the same relative positions as in the

* The mesh connection of three-phase circuits is frequently called the "delta" connection, and is represented by the symbol Δ.

diagram, Fig. 102, for the star-connected system. A comparison of the diagrams will show what changes in connections are necessary to convert one system to the other.

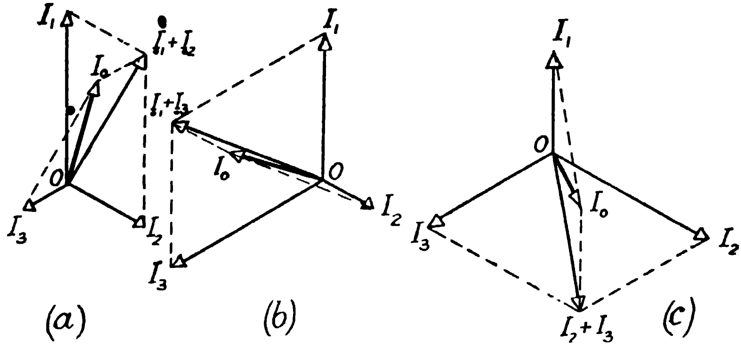


FIG. 105. Vector Diagrams showing Method of Determining Current (I_0) in Neutral Wire of Three-phase Four-wire System

The conventional circuit diagram for the mesh-connected system is shown in Fig. 106(b), this form of the circuit diagram being used in practice in preference to that of Fig. 106(a). In both diagrams the assumed positive directions for E.M.Fs. and currents are indicated by arrows.

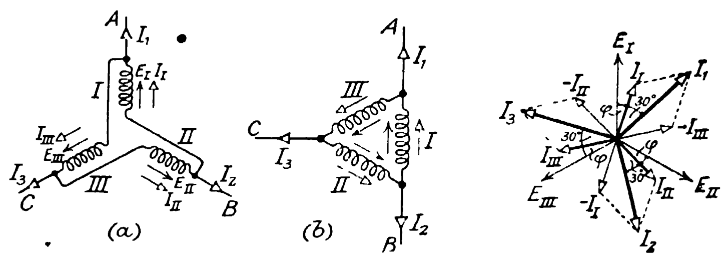


FIG. 106 FIG. 107
Circuit and Vector Diagrams of Three-phase Delta-connected System

In the mesh-connected three-phase system the line E.M.Fs. are equal to the phase E.M.Fs., and their instantaneous values are therefore represented by the equations

$$v_{1-2} = e_{\mathrm{I}} = E_m \sin \omega t,$$
$$v_{2-3} = e_{\mathrm{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi),$$
$$v_{3-1} = e_{\mathrm{III}} = E_m \sin(\omega t - \tfrac{4}{3}\pi).$$

Hence the R.M.S. values of the line E.M.Fs. are

$$V_{1-2} = E\ ; \quad V_{2-3} = E\ ; \quad V_{3-1} = E.$$

If the system is balanced the instantaneous values of the phase currents may be represented by the equations

$$i_{\mathrm{I}} = I_m \sin(\omega t - \varphi), \quad i_{\mathrm{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi - \varphi), \quad i_{\mathrm{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi)$$

and the instantaneous values of the line currents will then be represented by the equations

$$\begin{aligned}
i_1 = i_{\mathrm{I}} - i_{\mathrm{III}} &= I_m \sin(\omega t - \varphi) - I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi) \\
&= \sqrt{3}.I_m \sin(\omega t - \tfrac{1}{6}\pi - \varphi) \\
i_2 = i_{\mathrm{II}} - i_{\mathrm{I}} &= I_m \sin(\omega t - \tfrac{2}{3}\pi - \varphi) - I_m \sin(\omega t - \varphi) \\
&= \sqrt{3}.I_m \sin(\omega t - \tfrac{5}{6}\pi - \varphi) \\
i_3 = i_{\mathrm{III}} - i_{\mathrm{II}} &= I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi) - I_m \sin(\omega t - \tfrac{2}{3}\pi - \varphi) \\
&= \sqrt{3}.I_m \sin(\omega t - \tfrac{9}{6}\pi - \varphi)
\end{aligned}$$

Thus the line currents are equal to one another and have a mutual phase difference of 120°, or $\frac{2}{3}\pi$ radians. Also the current in any line has a phase difference, lagging, of 30°, or $\frac{1}{6}\pi$ radians, with respect to the current in the lagging phase connected to that line. For example, the current in line 1 (which is connected to the junction of phases I and III) has a phase difference of 30°, lagging, with respect to the current in phase 1. Similarly, the current in line 2 has a phase difference of 30°, lagging, with respect to the current in phase II, and the current in line 3 has a phase difference of 30°, lagging, with respect to the current in phase III.

The R.M.S. values of the line currents are

$$I_1 = I\sqrt{3}\ ; \quad I_2 = I\sqrt{3}\ ; \quad I_3 = I\sqrt{3}.$$

The *vector diagram* for a mesh-connected system with balanced loads is shown in Fig. 107, in which the vectors OE_{I}, OE_{II}, OE_{III} represent the phase E.M.Fs —and also the E.M.Fs. between the line wires—OI_{I}, OI_{II}, OI_{III} represent the phase currents, each lagging φ with respect to the appropriate phase E.M.F., and OI_1, OI_2, OI_3 represent the line currents. Observe that the phase difference between the current in any line wire, e.g. 1, and the voltage across the adjacent line wires, e.g. 1 and 2, is $(30 + \varphi)°$, and is lagging with respect to the line voltage.

Therefore, the phase relations between line currents and line voltages are the same for star and mesh-connected systems; and in every symmetrical and balanced system the line-voltage vectors are displaced from the line-current vectors by an angle of $(30 \pm \varphi)°$, ***φ being the***

phase difference between the E.M.F. and current in each phase, the plus sign to be employed when φ is lagging and the minus sign when φ is leading.

Effect of incorrect connections. Two cases of incorrect *star* connections are shown in Fig. 108, and the vector diagrams are shown in Fig. 109. In one case, Fig. 108(a), one phase has been incorrectly connected, and in the other case, Fig. 108(b), two

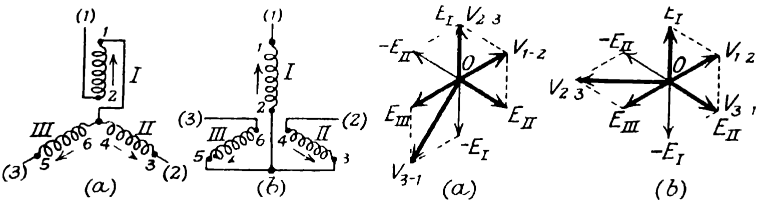


Fig. 108 Fig. 109

Circuit and Vector Diagrams of Incorrect Star Connections

phases have been incorrectly connected. In both cases the vector diagrams show that the line voltages are unbalanced and are not 120° apart.

The line voltages for the case of Fig. 108(a) are given by the equations

$$v_{1-2} = e_{\mathrm{I}} + e_{\mathrm{II}} = E_m[\sin \omega t + \sin(\omega t - \tfrac{2}{3}\pi)]$$
$$= 2E_m \cos \tfrac{1}{3}\pi \,.\, \sin(\omega t - \tfrac{1}{3}\pi) = E_m \sin(\omega t - \tfrac{1}{3}\pi),$$
$$v_{2-3} = -e_{\mathrm{II}} - e_{\mathrm{III}} = -E_m[\sin(\omega t - \tfrac{2}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi)]$$
$$= 2E_m \cos \tfrac{1}{3}\pi \,.\, \sin(\omega t - \pi) = E_m \sin \omega t,$$
$$v_{3-1} = e_{\mathrm{III}} - e_{\mathrm{I}} = E_m[\sin(\omega t - \tfrac{4}{3}\pi) - \sin \omega t]$$
$$= 2E_m \sin \tfrac{1}{3}\pi \,.\, \cos(\omega t - \tfrac{2}{3}\pi) = \sqrt{3} \,.\, E_m \sin(\omega t - \tfrac{7}{6}\pi)$$

For the case of Fig. 108(b) the line voltages are given by the equations

$$v_{1-2} = e_{\mathrm{I}} + e_{\mathrm{II}} = E_m[\sin \omega t + \sin(\omega t - \tfrac{2}{3}\pi)]$$
$$= 2E_m \cos \tfrac{1}{3}\pi \,.\, \sin(\omega t - \tfrac{1}{3}\pi) = E_m \sin(\omega t - \tfrac{1}{3}\pi),$$
$$v_{2-3} = -e_{\mathrm{II}} + e_{\mathrm{III}} = -E_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)]$$
$$= -2E_m \sin \tfrac{1}{3}\pi \,.\, \cos(\omega t - \pi) = \sqrt{3} \,.\, E_m \sin(\omega t - \tfrac{1}{2}\pi),$$
$$v_{3-1} = -e_{\mathrm{III}} - e_{\mathrm{I}} = -E_m[\sin(\omega t - \tfrac{4}{3}\pi) + \sin \omega t]$$
$$= -2E_m \cos(-\tfrac{2}{3}\pi) \,.\, \sin(\omega t - \tfrac{2}{3}\pi) = E_m \sin(\omega t - \tfrac{2}{3}\pi).$$

Thus, with a symmetrical system the equality of the line voltages is an absolute check on the correctness of the connections.

The two cases of incorrect mesh, or delta, connections are shown in Fig. 110 and the vector diagrams are shown in Fig. 111. In both cases the resultant E.M.F. acting in the closed circuit is equal to *twice* the phase E.M.F. For example, if one phase, III, is connected incorrectly, i.e. reversed, the resultant of the phase E.M.Fs. is given by

$$\begin{aligned} e = e_{I} + e_{II} - e_{III} &= E_m[\sin \omega t + \sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= E_m[\sin(\omega t - \tfrac{1}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= E_m[\sin(\omega t - \tfrac{1}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi + \pi)] \\ &= 2E_m \sin(\omega t - \tfrac{1}{3}\pi). \end{aligned}$$

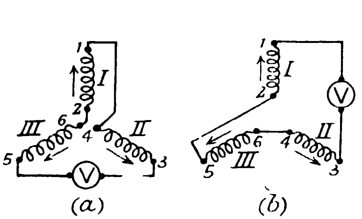


FIG. 110

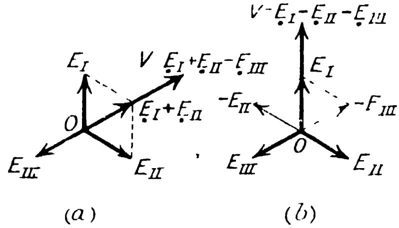


FIG. 111

Circuit and Vector Diagrams of Incorrect Delta Connections

If two phases, II and III, are reversed the resultant of the phase E.M.Fs. is given by

$$\begin{aligned} e = e_{I} - e_{II} - e_{III} &= E_m[\sin \omega t - \sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] \\ &= 2E_m \sin \omega t. \end{aligned}$$

Thus the correctness of the connections may be checked by leaving one of the junctions of the phases open and connecting a voltmeter between these phase ends. If the reading on the voltmeter is zero the phase ends to which the instrument is connected may be permanently connected together, but if the reading is equal to twice the phase E.M.F. then the connections are incorrect. With non-sinusoidal wave-forms, however, a reading will be obtained on the voltmeter, even if the connections are correct, but this reading will usually be less than twice the phase E.M.F. In such cases the non-sinusoidal E.M.F. results in circulating currents in the closed circuit, and the value of this current will depend upon the magnitudes and frequencies of the harmonic components of the E.M.F. wave, as well as the impedances of the closed circuit at these frequencies. A worked example is given on page 316. On account of this feature the Δ-connection is not usually employed in connection with alternators.

Conversion of a balanced star-connected load into an equivalent mesh-connected load and vice versa. In consequence of the above relationship (p. 194) between the line and phase currents and voltages, any symmetrical and balanced star-connected system may be replaced by an equivalent mesh-connected system, and *vice versâ.* For example, a three-phase star-connected system in which the line voltage is V and the line current is I may be replaced by a delta-connected system in which the phase voltage is V and the phase current is $I/\sqrt{3}$.

Similarly a balanced star-connected load for which the impedance of each branch is equal to $Z\underline{/\varphi}°$ may be replaced by an equivalent

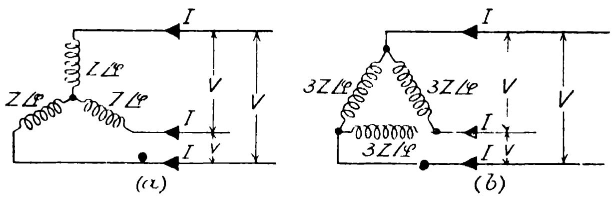


FIG. 112.—Equivalent Star and Delta Circuits

delta-connected load for which the impedance per phase is equal to $3Z\underline{/\varphi}°$. These equivalent conditions are shown in the circuit diagrams of Fig. 112.

Proof. Let V = line voltage, I = line current, and Z_Y = impedance per phase for the star connected system. Then the phase voltage is equal to $V/\sqrt{3}$, and the phase current is equal to I. Whence $Z_Y = V/(\sqrt{3}.I)$.

Now in the equivalent delta-connected system the line voltage and current must have the same values as in the star-connected system, and therefore we must have

$$\text{phase voltage} = V, \qquad \text{phase current} = I/\sqrt{3}.$$

$$\text{impedance per phase} \quad Z_\Delta = V/(I/\sqrt{3}) = \sqrt{3}.V/I = 3Z_Y,$$

since $V/I = \sqrt{3}.Z_Y$.

Moreover, as the phase difference between the line voltage and line current must be the same in both systems, we must have, therefore, the same phase difference between the phase voltages and currents. Hence $Z_\Delta\underline{/\varphi}° = 3Z_Y\underline{/\varphi}°$.

The cases of unbalanced loads are considered in Chapter IX.

Zigzag star and open-delta connections of three-phase systems. These connections are modifications of the star and delta connections and are occasionally employed in connection with generators and transformers.*

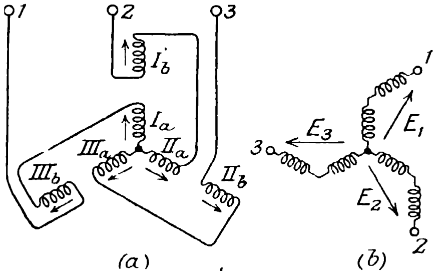
The zigzag-star connection (called also the "inter connected-star" connection) is shown diagrammatically in Fig. 113, and requires each phase of the generator, or transformer, to be wound in two equal sections, which are connected in the manner shown in Fig. 113.

The vector diagram for this connection is shown in Fig. 114, from which it follows that if the E.M.F. of each half-section of a phase-winding is equal to $\frac{1}{2}E$, the phase voltage of the system is equal to $\sqrt{3}(\frac{1}{2}E) = 0{\cdot}866E$, and the

* For other connections which are suitable for instrument transformers, see the Author's *Power Wiring Diagrams* (Pitman), p. 110.

line voltage is equal to $\sqrt{3}(\frac{1}{2}E.\sqrt{3}) = 1{\cdot}5E$. Therefore, with this connection, the line E.M.F. is only 86·6 per cent of that which would be obtained from the same machine with the normal star connection.

The vector diagram also shows that the vectors representing the phase voltages of the system have a phase difference of 30°, lagging, with respect to the vectors representing the induced E.M.Fs.; and that the vectors representing the line voltages are in phase with the latter. Hence if φ is the phase difference between the phase E.M.Fs. and currents, the phase difference between the line E.M.Fs. and currents will be $(30 + \varphi°)$, as in an ordinary three-phase system.



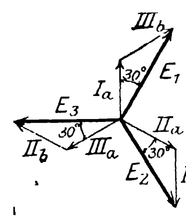


FIG. 113 FIG. 114

Circuit and Vector-Diagrams of Zigzag Connection of Three-phase System

The zigzag star connection therefore results in a reduction of the output of the alternator, or transformer, in which it is employed, and its use is restricted to cases where the ordinary star connection is inadmissible. For example, if the E.M.F. wave-form of an alternator is known to contain a component of triple frequency, this component may be eliminated from the phase E.M.F. by employing the zigzag-star connection, as the triple-frequency components of the E.M.Fs. in the sections of the phases which are connected in series neutralize each other, due to these sections being 120° apart and reversed relatively to each other.

The open-delta connection (called also the "V"-connection) is occasionally employed in connection with transformers, but it is never employed with alternators. It is obtained from the delta connection by removing one phase, as shown in the circuit diagram of Fig. 115. When equal E.M.Fs., having a phase difference of 120°, are induced in the remaining two phases, the E.M.Fs. between the line wires are equal and have a phase difference of 120°, as in a symmetrical three-phase system.

The vector diagram for this connection is shown in Fig. 116. The E.M.Fs. induced in phases I and II are represented by the vectors $OV_{1\text{-}2}$, and OV_2 [Fig. 116(*a*)], which also represent the E.M.Fs. between lines 1 and 2, and 2 and 3, respectively. The E.M.F. between lines 3 and 1 is equal to the reversed vector sum of the E.M.Fs. induced in phases I and II, and is represented by the vector $OV_{3\text{-}1}$. Thus, the line E.M.Fs. are equal and have a mutual phase difference of 120°.

If a balanced load is connected to the line wires the line currents will be equal to one another and will have a mutual phase difference of 120°, as in a symmetrical three-phase system. The currents in the phase windings will be equal to the currents in the line wires. Thus if I_A, I_B, I_C, are the load currents [see Fig. 116(*b*)], the current, I_1, in line 1 is given by $I_1 = I_A - I_C$, that, I_2, in line 2 is given by $I_2 = I_B - I_A$; and that, I_3, in line 3 is given by $I_3 = I_C - I_B$. These currents are represented, in the vector diagram of

Fig. 116(*b*), by the vectors OI_1, OI_2, OI_3; and the currents in each phase of the load are represented by the vectors OI_A, OI_B, OI_C.

The current in phase I of the transformer is equal to that in line 1, and is represented by the vector OI_1; but the current in phase II is equal to the vector sum of the currents in phase I and line 2, i.e. $I_{II} = I_1 + I_2 = -I_3$.

Hence the currents in the phases of the transformer have a phase difference of 60°, instead of 120° as in the normal delta connection. Moreover, the current in phase I has a phase difference of $(30 + \varphi)°$, lagging, with respect to the voltage between lines 1 and 2; but the current in phase II has a phase difference of $(30 - \varphi)°$, leading, with respect to the voltage between lines 2 and 3.

The total power supplied by the two phases is, therefore, equal to

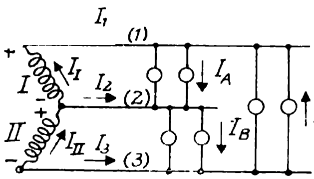


FIG. 115

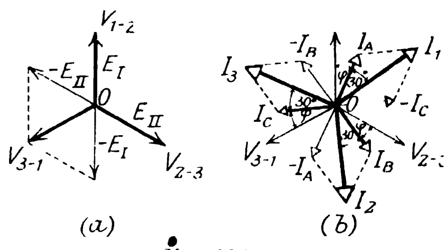


FIG. 116

Circuit and Vector Diagrams of Open delta or V connection of Three-phase System

$VI\cos(30 + \varphi)° + VI\cos(30 - \varphi)° = \sqrt{3}.VI\cos\varphi$. Thus the power supplied by each phase is equal to $\frac{1}{2}(\sqrt{3}.VI\cos\varphi)$; whereas, with the normal delta connection (i.e. with three phases) and the same phase currents, the power supplied by each phase is equal to $VI\cos\varphi$. Hence the output per phase with the open-delta connection is only $\frac{1}{2}\sqrt{3} = 0{\cdot}866$ of that obtainable with the delta connection and same phase current. In consequence of this reduced output the open-delta connection is employed only in cases where the saving of the cost of the third transformer is important, as in transformer-starting apparatus for polyphase motors. (See *Power Wiring Diagrams*, pp. 77, 78, 92.) This connection, however, is useful in practice for obtaining a temporary (three-phase) supply from a group of three delta-connected transformers in the event of one transformer becoming defective.

Power in three-phase circuits. The power in a balanced polyphase system has already (p. 175) been shown to be equal to $nEI\cos\varphi$, where E, I, are the phase E.M.Fs. and currents, and φ is the phase difference between them. Hence in a balanced three-phase system the power is given by $P = 3EI\cos\varphi$.

In the case of a star-connected system the line voltage, V, is equal to $\sqrt{3}E$, and the power is given by $P = \sqrt{3}\,.VI\cos\varphi$. Similarly, with a delta-connected system the line current (I') is equal to $\sqrt{3}\,.I'$, and the power is given by $P = \sqrt{3}\,.EI'\cos\varphi$. Therefore, in general, the power in a balanced three-phase system is given by

$$\sqrt{3} \times \text{line voltage} \times \text{line current} \times \cos\varphi.$$

In the case of an *unbalanced system* the total power must be obtained by taking the sum of the powers in the separate phases. Thus if the instantaneous values of the phase E.M.Fs. are given by the equations

$$e_{\mathrm{I}} = E_{\mathrm{I}m} \sin \omega t, \quad e_{\mathrm{II}} = E_{\mathrm{II}m} \sin(\omega t - \tfrac{2}{3}\pi), \quad e_{\mathrm{III}} = E_{\mathrm{III}m} \sin(\omega t - \tfrac{4}{3}\pi);$$

and the instantaneous values of the phase currents are given by

$$i_{\mathrm{I}} = I_{\mathrm{I}m} \sin(\omega t - \varphi_1), \quad i_{\mathrm{II}} = I_{\mathrm{II}m} \sin(\omega t - \tfrac{2}{3}\pi - \varphi_2), \quad i_{\mathrm{III}} = I_{\mathrm{III}m} \sin(\omega t - \tfrac{4}{3}\pi - \varphi_3);$$

the instantaneous power will be given by

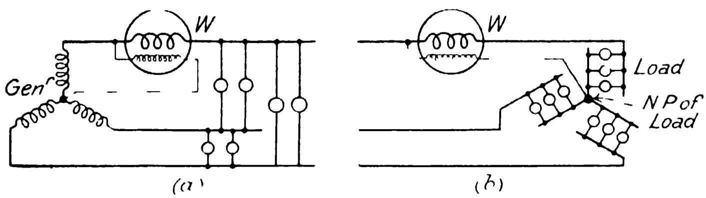


FIG. 117.—Methods of Measuring Power in Three-phase Star-connected System and Balanced Loads

$$\begin{aligned}
p &= e_{\mathrm{I}} i_{\mathrm{I}} + e_{\mathrm{II}} i_{\mathrm{II}} + e_{\mathrm{III}} i_{\mathrm{III}} \\
&= E_{\mathrm{I}m} I_{\mathrm{I}m} \sin \omega t \,.\, \sin(\omega t - \varphi_1) + E_{\mathrm{II}m} I_{\mathrm{II}m} \\
&\qquad \sin(\omega t - \tfrac{2}{3}\pi) \,.\, \sin(\omega t - \tfrac{2}{3}\pi - \varphi_2) + E_{\mathrm{III}m} I_{\mathrm{III}m} \\
&\qquad \sin(\omega t - \tfrac{4}{3}\pi) \,.\, \sin(\omega t - \tfrac{4}{3}\pi - \varphi_3) \\
&= \tfrac{1}{2} E_{\mathrm{I}m} I_{\mathrm{I}m} [\cos \varphi_1 - \cos(2\omega t - \varphi_1)] + \tfrac{1}{2} E_{\mathrm{II}m} I_{\mathrm{II}m} \\
&\qquad [\cos \varphi_2 - \cos(2\omega t - \tfrac{4}{3}\pi - \varphi_2)] + \tfrac{1}{2} E_{\mathrm{III}m} I_{\mathrm{III}m} \\
&\qquad [\cos \varphi_3 - \cos(2\omega t - \tfrac{8}{3}\pi - \varphi_3)] \\
&= E_{\mathrm{I}} I_{\mathrm{I}} [\cos \varphi_1 - \cos(2\omega t - \varphi_1)] + E_{\mathrm{II}} I_{\mathrm{II}} \\
&\qquad [\cos \varphi_2 \{1 - \cos(2\omega t - \tfrac{4}{3}\pi)\} + \sin \varphi_2 \,.\, \sin(2\omega t - \tfrac{4}{3}\pi)] \\
&\quad + E_{\mathrm{III}} I_{\mathrm{III}} [\cos \varphi_3 \{1 - \cos(2\omega t - \tfrac{8}{3}\pi)\} + \sin \varphi_3 \,.\, \sin(2\omega t - \tfrac{8}{3}\pi)]
\end{aligned}$$

Hence the mean power is given by the mean value of the above expression taken over a period. Thus

$$P = E_{\mathrm{I}} I_{\mathrm{I}} \cos \varphi_1 + E_{\mathrm{II}} I_{\mathrm{II}} \cos \varphi_2 + E_{\mathrm{III}} I_{\mathrm{III}} \cos \varphi_3 \quad . \quad . \quad (70)$$

since all terms of double frequency become zero when averaged over a period.

Therefore the mean power in an unbalanced three-phase, or any polyphase, system is equal to the sum of the mean power in each phase.

[NOTE.—The case for which the E.M.F. and current are non-sinusoidal is discussed in Chapter X.]

Measurement of power in three-phase circuits. With *balanced*

circuits the total power may be measured by a single wattmeter in a number of ways.* For example, if the neutral point of the system is available the wattmeter may be connected as shown in Fig. 117, in which the current coil of the wattmeter is inserted in one of the line wires and the potential coil is connected between that line wire and the neutral point. The total power is equal to three times the power indicated by the wattmeter. When, however, this method is employed in practice it is customary to mark the scale of the instrument in terms of the total power instead of the actual power indicated by the instrument.

If the neutral point of the system is not available, the total power may still be measured by means of a single wattmeter, but in this case it is necessary to impress on the potential coil of the wattmeter a voltage which is equivalent to the phase voltage of the system. With low-voltage circuits and the dynamometer type of wattmeter this equivalent phase voltage may be obtained by the use of a star-connected potential circuit, as shown in Fig. 118, in which one branch consists of the potential coil of the wattmeter and the other branches consist of resistances equal in value to that of the potential coil. For extreme accuracy the inductance of each of these branches must equal that of the potential coil. Thus the wattmeter indicates the true power in one phase of the system, as in the previous case.

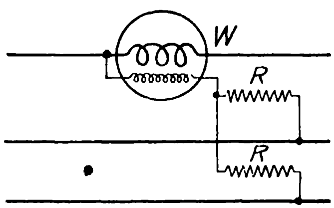


FIG. 118.—Method of Measuring Power in Balanced Three-phase System with Isolated Neutral Point.

With *unbalanced* circuits the total power may be determined by measuring the power in each phase, by separate wattmeters, and adding the results. Such a method requires three wattmeters and is not used in practice because the same result may be obtained by means of two wattmeters, or a single wattmeter of the polyphase type (p. 406), connected in the manner shown in Fig. 119. The total power is then given by the *algebraic* sum of the readings of the wattmeters, when two instruments are employed. When a polyphase wattmeter is employed, the total power is given by a single reading of the instrument, and this feature renders this type of instrument very serviceable for commercial circuits. It should be noted that in each case the results are true, whether the system is balanced or unbalanced.

* See *Power Wiring Diagrams*, p. 112.

The theory of the *two-wattmeter method* of power measurement is best developed by considering the case when two separate wattmeters are employed, as we are then able to show how the power factor of a balanced system may be obtained from the readings of the instruments. Moreover, since a polyphase wattmeter consists essentially of two single-phase instruments with a common moving system, the proof developed for the case of two separate wattmeters may readily be interpreted for the case of the polyphase wattmeter.

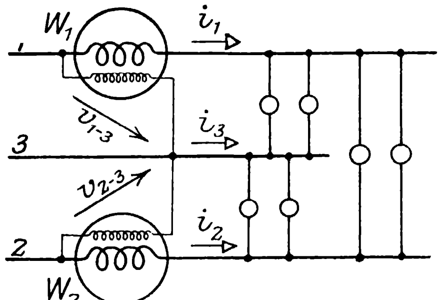


FIG. 119.—Circuit Diagram for Two-wattmeter Method of Measuring Power in Three-phase System

Theory of the two-wattmeter method of measuring power in three-phase circuits. Let the positive direction of E.M.Fs. and currents in the circuit be that marked by the arrows in Fig. 119. Then if the instantaneous values of the currents in lines 1 and 2 (in which the current coils of the wattmeters W_1, W_2 are inserted) are denoted by i_1, i_2 respectively, the instantaneous power measured by wattmeter W_1 is equal to $i_1 v_{1-3}$, and that measured by wattmeter W_2 is equal to $i_2 v_{2-3}$. Observe that the positive directions for the E.M.Fs. impressed on the potential coils of the wattmeters are: v_{1-3} for wattmeter W_1, and v_{2-3} for wattmeter W_2.

The sum of these quantities is

$$i_2 v_{2-3} + i_1 v_{1-3}.$$

But if the phase E.M.Fs. of the system are e_{I}, e_{II}, e_{III}, we have $v_{2-3} = e_{\text{II}} - e_{\text{III}}$, $v_{1-3} = e_{\text{I}} - e_{\text{III}}$, and therefore

$$i_2 v_{2-3} + i_1 v_{1-3} = i_2(e_{\text{II}} - e_{\text{III}}) + i_1(e_{\text{I}} - e_{\text{III}}) = i_1 e_{\text{I}} + i_2 e_{\text{II}} - e_{\text{III}}(i_1 + i_2)$$

Now in any three-phase three-wire system, whether the loads are balanced or unbalanced, the instantaneous sum of the currents must be zero, i.e. $i_1 + i_2 + i_3 = 0$, or $i_1 + i_2 = -i_3$.

Therefore the sum of the instantaneous quantities measured by the wattmeters is given by

$$i_1 e_{\text{I}} + i_2 e_{\text{II}} + i_3 e_{\text{III}}$$

and is equal to the total power in the system.

With a *balanced* system

$$i_1 = I_m \sin(\omega t - \varphi), \qquad i_2 = I_m \sin(\omega t - \varphi - \tfrac{2}{3}\pi);$$

$$\begin{aligned} v_{1-3} &= E_m \sin \omega t - E_m \sin(\omega t - \tfrac{4}{3}\pi) \\ &= E_m[2 \cos \tfrac{1}{2}(2\omega t - \tfrac{4}{3}\pi).\sin(\tfrac{1}{2} \times \tfrac{4}{3}\pi)] \\ &= \sqrt{3}.E_m \sin(\omega t - \tfrac{1}{6}\pi) \\ v_{2-3} &= E_m \sin(\omega t - \tfrac{2}{3}\pi) - E_m \sin(\omega t - \tfrac{4}{3}\pi) \\ &= E_m[2 \cos \tfrac{1}{2}(2\omega t - \tfrac{6}{3}\pi).\sin(\tfrac{1}{2} \times \tfrac{2}{3}\pi)] \\ &= \sqrt{3}.E_m \sin(\omega t - \tfrac{1}{2}\pi) \end{aligned}$$

Hence the instantaneous power measured by wattmeter W_1 is given by

$$p_1 = i_1 v_{1-3} = \sqrt{3}\, E_m I_m \sin(\omega t - \tfrac{1}{6}\pi).\sin(\omega t - \varphi)$$
$$= \tfrac{1}{2}\sqrt{3}\, E_m I_m [\cos(\tfrac{1}{6}\pi - \varphi) - \cos(2\omega t - \tfrac{1}{6}\pi - \varphi)],$$

and the reading on this instrument is given by

$$P_1 = \tfrac{1}{2}\sqrt{3}\, E_m I_m \cos(\tfrac{1}{6}\pi - \varphi)$$
$$= \sqrt{3}\, EI \cos(30 - \varphi)^\circ$$
$$= VI \cos(30 - \varphi)^\circ, \text{ where } V \text{ is the line voltage of the system.}$$

Similarly the instantaneous power measured by wattmeter W_2 is given by

$$p_2 = i_2 v_{2-3} = \sqrt{3}\, E_m I_m \sin(\omega t - \tfrac{1}{2}\pi) \sin(\omega t - \tfrac{2}{3}\pi - \varphi)$$
$$= \tfrac{1}{2}\sqrt{3}\, E_m I_m [\cos(\tfrac{1}{6}\pi + \varphi) - \cos(2\omega t - \tfrac{7}{6}\pi - \varphi)]$$

and the reading on this instrument is given by

$$P_2 = \tfrac{1}{2}\sqrt{3}\, E_m I_m \cos(\tfrac{1}{6}\pi + \varphi)$$
$$= \sqrt{3}\, EI \cos(30 + \varphi)^\circ$$
$$= VI \cos(30 + \varphi)^\circ$$

[*Note.* If φ is leading instead of lagging, $P_1 = VI \cos(30 + \varphi)^\circ$, and $P_2 = VI \cos(30 - \varphi^\circ)$.]

Now the ratio of the readings of the wattmeters gives

$$\frac{P_2}{P_1} = \frac{VI \cos(30 + \varphi)^\circ}{VI \cos(30 - \varphi)^\circ} = \frac{\cos 30^\circ \cos \varphi - \sin 30^\circ \sin \varphi}{\cos 30^\circ \cos \varphi + \sin 30^\circ \sin \varphi}$$
$$= \frac{\tfrac{1}{2}\sqrt{3} \cos \varphi - \tfrac{1}{2} \sin \varphi}{\tfrac{1}{2}\sqrt{3} \cos \varphi + \tfrac{1}{2} \sin \varphi}$$

Cross multiplying, we have

$$P_2(\sqrt{3} \cos \varphi + \sin \varphi) = P_1(\sqrt{3} \cos \varphi - \sin \varphi).$$

Whence

$$(P_1 + P_2) \sin \varphi = \sqrt{3}(P_1 - P_2) \cos \varphi$$

Hence

$$\tan \varphi = \frac{\sin \varphi}{\cos \varphi} = \sqrt{3}\left(\frac{P_1 - P_2}{P_1 + P_2}\right) = \sqrt{3}\left(\frac{1 - P_2/P_1}{1 + P_2/P_1}\right)$$

and

$$\cos \varphi = \frac{1}{\sqrt{(1 + \tan^2 \varphi)}} = \frac{1 + P_2/P_1}{2\sqrt{(1 - P_2/P_1 + (P_2/P_1)^2)}} \quad . \quad . \quad . \quad (71)$$

Therefore the power factor of a balanced system may be obtained from the readings of the wattmeters without a knowledge of the volt-amperes.

A curve, calculated from equation (71), connecting power factor and the ratio of the wattmeter readings is given in Fig. 120.

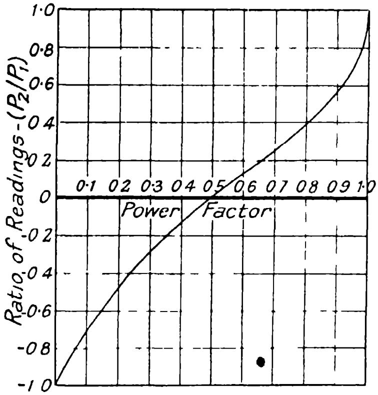
Variation of readings of wattmeters with variation of power factor. Another interesting feature of the two-wattmeter method of power measurement, when applied to balanced systems and when two wattmeters are employed, is the manner in which the readings vary when the power factor of the system is varied, the volt-amperes being constant. Thus, since one wattmeter, W_1, measures the product $VI \cos(30 - \varphi)^\circ$, and the other wattmeter, W_2, measures the product $VI \cos(30 + \varphi)^\circ$, equal readings will only be obtained on the instruments, with balanced loads, when $\cos(30 - \varphi)^\circ = \cos(30 + \varphi)^\circ$, i.e. when $\varphi = 0$.*

* If φ is leading instead of lagging, wattmeter W_1 measures the product $VI \cos(30 + \varphi)^\circ$, and wattmeter W_2 measures the product $VI \cos(30 - \varphi)^\circ$. In this case, assuming balanced loads, wattmeter W_2, Fig. 119, will show the larger reading (except in the special case when $\varphi = 0$).

Under these conditions,

$$P_1 = VI\cos 30° = 0.866\,VI\,;\quad P_2 = VI\cos 30° = 0.866\,VI\,;$$
$$P = P_1 + P_2 = 1.732\,VI.$$

For all other values of φ, except $\varphi = 90°$, the readings of the instruments will be unequal.



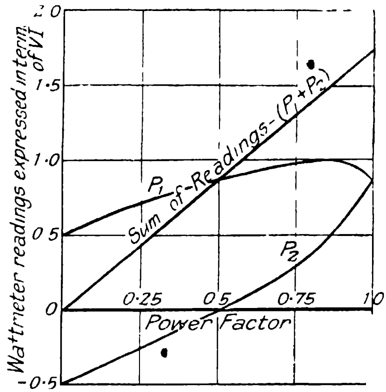


FIG. 120 FIG. 121

Variation of Wattmeter Readings (in Two-wattmeter Method) with Power Factor (Balanced Loads and Sinusoidal Current and Pressure)

For example,

when $\varphi = 30°$, i.e. the power factor $= \cos 30° = 0.866$,

$$\left.\begin{array}{l} P_1 = VI\cos(30° - 30°) = VI \\ P_2 = VI\cos(30° + 30°) = 0.5\,VI \end{array}\right\} P = P_1 + P_2 = 1.5VI (= 0.866 \times 1.732\,VI)$$

When $\varphi = 60°$, i.e. the power factor $= \cos 60° = 0.5$,

$$\left.\begin{array}{l} P_1 = VI\cos(30° - 60°) = 0.866\,VI \\ P_2 = VI\cos(30° + 60°) = 0 \end{array}\right\} P = P_1 + P_2 = 0.866\,VI (= 0.5 \times 1.723\,VI)$$

When $\varphi = 90°$, i.e. the power factor $= \cos 90° = 0$,

$$\left.\begin{array}{l} P_1 = VI\cos(30° - 90°) = 0.5\,VI \\ P_2 = VI\cos(30° + 90°) = -0.5\,VI \end{array}\right\} P = P_1 + P_2 = 0.$$

A curve showing the manner in which the readings of the wattmeters vary as the power factor is varied is shown in Fig. 121.

Hence at unity power factor equal readings, both positive, are obtained on the wattmeters; at a power factor of 86·6 per cent the reading on one instrument is double that on the other; at a power factor of 50 per cent one instrument reads zero; and at zero power factor equal readings are obtained, but one is now positive and the other is negative. Moreover, at power factors above 50 per cent both wattmeters are reading positively, and the total power is therefore obtained by adding the readings. But at power factors below 50 per cent one wattmeter is reading in the negative direction, and the connections of its pressure coil, or, alternatively, the connections of its current coil, must be reversed in order to obtain the reading. Hence, under these conditions, the total power is obtained by subtracting the readings.

Therefore, when using separate wattmeters for measuring power by the two-wattmeter method it is very important to know whether both wattmeters are reading positively, or whether one instrument is reading positively and the other is reading negatively, as otherwise the total power cannot be computed. In cases where the polarity of the current and potential terminals is marked on the instruments, or where the manufacturer's diagram showing the correct connections of the instrument is available, or where the power factor of the load is known to be above, or below, 50 per cent. there will be no difficulty in interpreting the readings of the wattmeters. But in other cases a test will have to be made to ascertain the connections which give a positive reading on each wattmeter.*

Measurement of power in a three-phase four-wire system. The two-wattmeter method may also be adapted to measure the power in a three-phase four-wire system, but in this case it is necessary to supply the current coils of the wattmeters from current transformers† inserted in the principal line wires in order to obtain the correct magnitudes and phase differences of the currents in the current coils of the wattmeters, since in the three-phase four-wire system the instantaneous sum of the currents in the principal line wires, or " outers," is not necessarily equal to zero, as is the case for a three-wire system. In general, in the four-wire system, the sum of the instantaneous currents in the " outers " is equal to the instantaneous current in the neutral wire, and the vector sum of the R.M.S. currents in the " outers " is equal to the R.M.S. current in the neutral wire.

The connections of the wattmeters and current transformers for measuring the power in a four-wire system by the two-wattmeter method is shown in Fig. 122.

Assuming, for simplicity, the current transformers to have a ratio of transformation of unity, and neglecting the small phase displacement between the currents in primary and secondary windings, the currents in the secondary windings, which supply the current coils of the wattmeters, will be equal in magnitude to the currents in the line wires to which the primary windings are connected, and

* For example, each instrument is connected to a single phase circuit. Alternatively, if it is permissible to open-circuit the line wires (1 and 2, Fig. 119) in which the current coils of the wattmeters are connected, single-phase power may be supplied to each instrument without disconnecting it from the circuit; e.g. single-phase power may be supplied to wattmeter W_1 by open-circuiting line wire 2, and single-phase power may be supplied to wattmeter W_2 by open-circuiting line 1.

† A current transformer is a special type of transformer designed to give a constant ratio between the currents in the primary and secondary windings when operated with a closed secondary circuit. The primary winding is connected in *series* with the main circuit, and the secondary winding is connected to the current coil of a wattmeter, ammeter, or other current-measuring instrument, which is therefore supplied with current proportional to that in the primary circuit. Instrument (current and potential) transformers are considered in detail in Chapter XIV.

the mutual phase difference between the secondary currents will be equal to the mutual phase difference between the primary currents. It is necessary to observe, however, that the secondary currents are reversed in direction relatively to the primary currents.

If the instantaneous values of the currents in the secondary windings of the current transformers are denoted by i_1, i_2, i_3, the current in the current coil of wattmeter W_1 is equal to $i_1 - i_2$, and that in the current coil of wattmeter W_2 is equal to $i_3 - i_2$. Hence if the phase voltages of the system are denoted by e_{I}, e_{II}, e_{III}, and

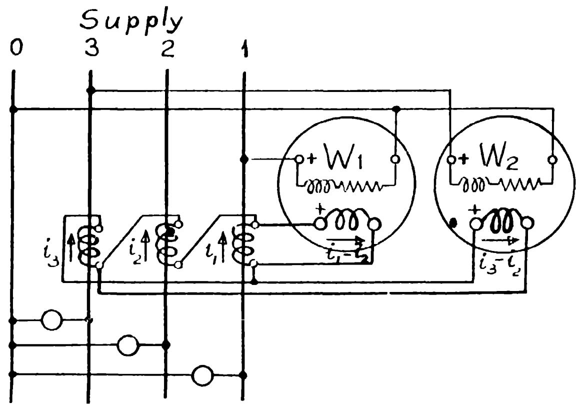


FIG. 122.- Circuit Diagram for Two-wattmeter Method of Measuring Power in Three-phase Four-wire System

the assumed positive directions are those shown by the arrows in Fig. 122, the instantaneous power measured by wattmeter W_1 is given by $p_1 = e_1(i_1 - i_2)$, and that measured by wattmeter W_2 is given by $p_2 = e_{\text{III}}\ (i_3 - i_2)$. The sum of these quantities is therefore given by

$$\begin{aligned} p = p_1 + p_2 &= e_{\text{I}}(i_1 - i_2) + e_{\text{III}}(i_3 - i_2) \\ &= e_{\text{I}}i_1 + e_{\text{III}}i_3 - i_2(e_{\text{I}} + e_{\text{III}}) \\ &= e_{\text{I}}i_1 + e_{\text{II}}i_2 + e_{\text{III}}i_3, \end{aligned}$$

since, in a symmetrical system, $e_{\text{I}} + e_{\text{III}} = -e_{\text{II}}$.

But the expression $(e_{\text{I}}i_1 + e_{\text{II}}i_2 + e_{\text{III}}i_3)$ represents the instantaneous power in the system. Therefore two wattmeters, or a polyphase wattmeter, connected as shown in Fig. 122, will measure the total power in a three-phase four-wire system whether the loads are balanced or unbalanced.

An extreme case of unbalanced loading occurs when a single-phase load is connected across two of the line wires. Thus, assume the

load to be connected across lines 1 and 2, as in Fig. 123(a), and let i denote the instantaneous value of the line current. Then the current in the current coil of wattmeter W_1 is equal to $2i$, and that in the current coil of wattmeter W_2 is equal to i. Hence the instantaneous power measured by wattmeter W_1 is given by $p_1 = 2ie_{\rm I}$, and that measured by wattmeter W_2 is given by $p_2 = ie_{\rm III}$. Whence the sum of these quantities gives

$$\begin{aligned} p = p_1 + p_2 = 2ie_{\rm I} + ie_{\rm III} &= i(2e_{\rm I} + e_{\rm III}) \\ &= i(e_{\rm I} + (e_{\rm I} + e_{\rm III})) \\ &= i(e_{\rm I} - e_{\rm II}) \\ &= iv_{1\text{-}2}, \end{aligned}$$

where $v_{1\text{-}2}$ is the voltage between the line wires, 1 and 2, across which the load is connected.

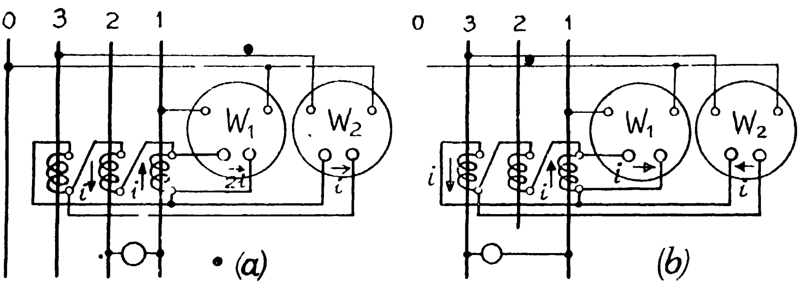


FIG. 123.—Circuit Diagrams showing Currents in Wattmeter Coils with Single-phase Loads

But the expression $iv_{1\text{-}2}$ represents the instantaneous power supplied to the load. Therefore the sum of the readings of the wattmeters, or the reading on the polyphase wattmeter, when such an instrument is used, gives the power supplied to the single-phase load.

Similarly, if a single-phase load is connected across lines 1 and 3, as in Fig. 123(b), and i denotes the instantaneous value of the line current, the current in the current coil of wattmeter W_1 is equal to i, and that in the current coil of wattmeter W_2 is equal to $-i$. Hence the instantaneous power measured by wattmeter W_1 is given by $p_1 = ie_{\rm I}$, and that measured by wattmeter W_2 is given by $p_2 = ie_{\rm III}$. Whence the sum of these quantities gives

$$\begin{aligned} p = p_1 + p_2 = ie_{\rm I} - ie_{\rm III} &= i(e_{\rm I} - e_{\rm III}) \\ &= iv_{1\text{-}3}, \end{aligned}$$

where $v_{1\cdot 3}$ is the voltage between the lines, 1 and 3, across which the load is connected. Therefore, in this case also, the sum of the readings of the wattmeters gives the power supplied to the single-phase load.

Again, if the load is connected between one of the principal line wires and the neutral wire (e.g. between lines 1 and 0), the current i circulates only in the current coil of wattmeter W_1. The power measured by this wattmeter is, therefore, given by $p = e_1 i$, and that measured by the other wattmeter (W_2) is zero.

When the load is connected between lines 2 and 0 the current i circulates (in the negative direction) in the coils of both wattmeters, and accordingly $p_1 = -ie_1$, $p_2 = -ie_3$; $p_1 + p_2 = -i(e_1 + e_3) = ie_2$.

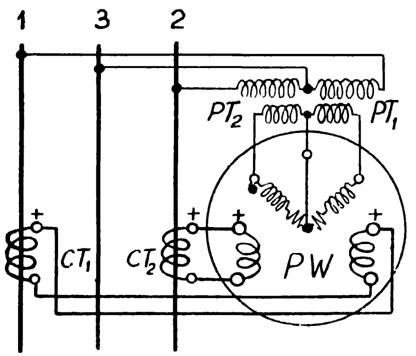


FIG. 124.—Circuit Diagram of Polyphase Wattmeter with Instrument Transformers

Hence in general two wattmeters, or a polyphase wattmeter connected according to Fig. 122, may be employed for measuring the power in any three-phase system, whether three-wire or four-wire, under any condition of loading. In the case of the three-phase three-wire system, however, the connections, when current transformers are necessary,* may be simplified to those shown in Fig. 124, for which only two current transformers are required.

Reactive power in three-phase circuits. In Chapter V, p. 79, the reactive power in a single-phase circuit was defined as: the product of the impressed E.M.F. and the component of the current which is perpendicular to it, i.e. the product $EI \sin\varphi$. This definition may be extended to include polyphase circuits under conditions of balanced loads. Thus the reactive power in a polyphase circuit of n phases is given by the product $nEI \sin\varphi$, where

* Current transformers are necessary with all high-voltage systems in order to avoid the direct connection of the instrument to the high-voltage circuit. In such cases the potential coils of the instruments are supplied from potential transformers. Current transformers are also necessary in cases where the line currents are larger than those for which the current coils of instruments can be conveniently wound. Further details of the uses of current transformers are given in Chapter XIV.

E, I, denote the E.M.Fs. and currents of each phase. Hence for a balanced three-phase system the reactive power is given by

$$P_x = 3EI \sin \varphi = \sqrt{3}.VI \sin \varphi.$$

As in the case of single-phase circuits, the reactive power in a polyphase circuit represents the rate at which energy must be supplied to the circuit to maintain the magnetic and electrostatic fields, i.e. the rate at which energy is stored in the circuit ; but in the polyphase circuit with balanced loads the stored energy is constant and does not surge between the generator and the circuit.

Measurement of reactive power. In a balanced three-phase circuit the reactive power may be measured directly on a single wattmeter by so connecting the current and potential coils that at unity power factor the currents in them have a phase difference of 90°. For example, if the current coil of the wattmeter is connected in one line wire, No. 1, and the potential coil is connected across the other line wires, Nos. 2 and 3, as shown in Fig. 125, the reading on the wattmeter will represent the product

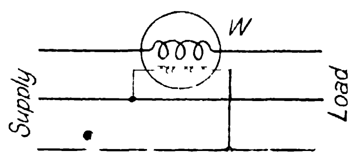


FIG. 125. Connections of a Wattmeter to Measure the Reactive Power in a Three phase System

$$VI \cos (90 - \varphi) = VI \sin \varphi,$$

and therefore the reactive power in the system will be given by $\sqrt{3}$ times the reading on the wattmeter.

When, however, the two-wattmeter method of power measurement is employed, and separate wattmeters are used, the reactive power may be obtained from the readings on the wattmeters. Thus, if P_1, P_2, are the readings on the wattmeters, then

$$P_1 = VI \cos (30 - \varphi)^\circ, \qquad P_2 = VI \cos (30 + \varphi)^\circ.$$

Whence,

$$\begin{aligned} P_1 - P_2 &= VI[\cos(30 - \varphi)^\circ - \cos(30 + \varphi)^\circ \\ &= VI[\cos 30^\circ . \cos \varphi + \sin 30^\circ . \sin \varphi - \cos 30^\circ . \cos \varphi \\ &\qquad + \sin 30^\circ \sin \varphi] \\ &= VI \sin \varphi \end{aligned}$$

Therefore the reactive power in the system is given by $\sqrt{3}$ times the difference of the readings on the wattmeters.

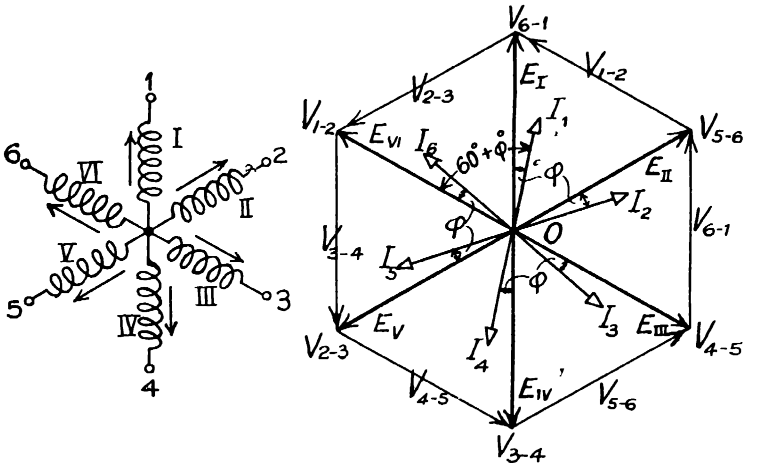
SIX-PHASE SYSTEM

In a symmetrical system the phase E.M.Fs. are represented by the equations

$$e_{\mathrm{I}} = E_m \sin \omega t,\quad e_{\mathrm{II}} = E_m \sin(\omega t - \tfrac{1}{3}\pi),\quad e_{\mathrm{III}} = E_m \sin(\omega t - \tfrac{2}{3}\pi),$$

$$e_{\mathrm{IV}} = E_m \sin(\omega t - \pi),\quad e_{\mathrm{V}} = E_m \sin(\omega t - \tfrac{4}{3}\pi),\quad e_{\mathrm{VI}} = E_m \sin(\omega t - $$

Star-connected system. With a star-connected system the line E.M.Fs. are equal to the phase E.M.Fs. (since the E.M.F. between any two adjacent line wires is equal to the vector difference of the E.M.Fs. in the phases to which these lines are connected), and the line E.M.F. vectors are displaced 60° (leading) with respect to the



(Circuit diagram) (Vector diagram)

FIG. 126.—Circuit and Vector Diagrams for Six-phase Star-connected System

[NOTE. The vectors OE_{I}, OE_{II}, . . . represent the phase E.M.Fs., and the vectors OV_{1-2}, OV_{2-3}, . . . (which are coincident with OE_{VI}, OE_{I}, . . .) represent the line E.M.Fs. Observe that the line E.M.Fs. may be drawn in the form of a regular hexagon.]

phase E.M.F. vectors, as shown by the vector diagram of Fig. 126, and also by the equations

$$\begin{aligned}
v_{1\,2} = e_{\mathrm{I}} - e_{\mathrm{II}} &= E_m \sin \omega t - E_m \sin(\omega t - \tfrac{1}{3}\pi) \\
&= 2E_m[\cos(\omega t - \tfrac{1}{6}\pi) \,.\, \sin \tfrac{1}{6}\pi] = E_m \sin(\omega t + \tfrac{1}{3}\pi) \\
v_{2\,3} = e_{\mathrm{II}} - e_{\mathrm{III}} &= E_m \sin(\omega t - \tfrac{1}{3}\pi) - E_m \sin(\omega t - \tfrac{2}{3}\pi) \\
&= 2E_m[\cos(\omega t - \tfrac{1}{2}\pi).\sin \tfrac{1}{6}\pi] = E_m \sin \omega t, \\
v_{3\,4} = e_{\mathrm{III}} - e_{\mathrm{IV}} &= E_m \sin(\omega t - \tfrac{2}{3}\pi) - E_m \sin(\omega t - \pi) \\
&= 2E_m[\cos(\omega t - \tfrac{5}{6}\pi).\, \sin \tfrac{1}{6}\pi] = E_m \sin(\omega t - \tfrac{1}{3}\pi),
\end{aligned}$$

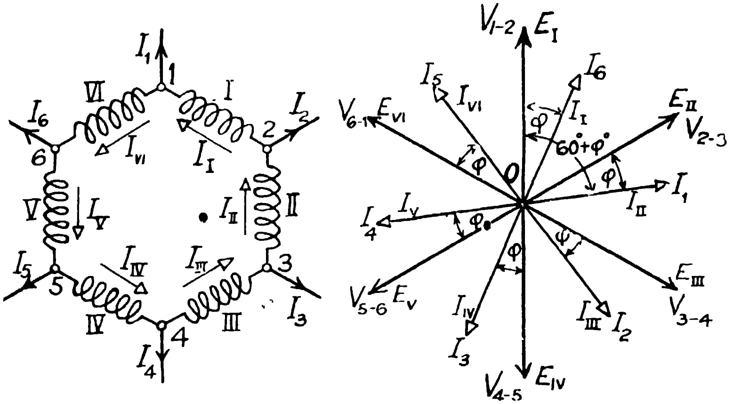
etc., etc.

The R.M.S. values of the line E.M.Fs. are equal to those of the phase E.M.Fs. Thus

$$V_{1\text{-}2} = E,\ V_{2\text{-}3} = E,\ V_{3\text{-}4} = E,\ V_{4\text{-}5} = E,\ V_{5\text{-}6} = E,$$

where E is the R.M.S. value of each of the phase E.M.Fs.

The line currents are equal to the phase currents, and the phase difference between the line-E.M.F. vectors and the line-current vectors is equal to $(60 + \varphi)^\circ$, lagging, where φ is the phase difference between phase E.M.Fs. and currents.



Circuit diagram *Vector diagram*

FIG. 127. Circuit and Vector Diagrams for Six-phase Mesh-connected System

Mesh, or ring-connected, system. With balanced loads the line currents are equal to the phase currents, and the line-current vectors are displaced 60 degrees, lagging, with respect to the phase-current vectors, since the current in any line is equal to the vector difference of the currents in the phases connected to that line, as shown in the vector diagram of Fig. 127. Thus the phase difference between the line E.M.Fs. and the line currents is equal to $(60 + \varphi)$ degrees, lagging, where φ is the phase difference between the phase E.M.Fs. and currents.

The six-phase system is not, at present,* used for power distribution as this system possesses no advantages over the three-phase system for power supply to motors, and has the great disadvantage

* A proposal has been made to use a six-phase system for extra high-voltage transmission by means of triple-concentric underground cables. (*Journ. I.E.E.*, 61, 220: Paper by Major A. M. Taylor on "The possibilities of transmission by underground cables at 100,000/150,000 volts.")

of requiring double the number of line wires as a three-phase system. For supplying rotary converters, however, the increased number of phases of the six-phase system is an advantage, as a larger output can be obtained from a given size of armature than when the supply is three-phase, due to the lower resultant I^2R losses in the former case, other conditions being equal.

In the majority of cases where the six-phase system is used in practice the six-phase current is obtained from a three-phase system by means of static transformers as shown below. But in special cases (e.g. when a large amount of power in the direct-

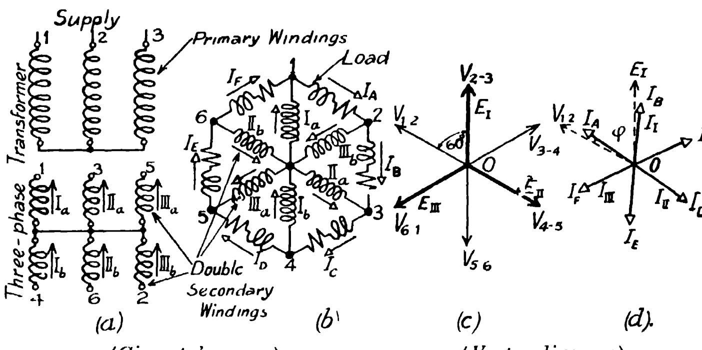


FIG. 128.—Circuit and Vector Diagrams for the Star Method of Supplying a Six-phase Load from a Three-phase Transformer

(a) Transformer Connections, (b) Circuit Diagram showing Directions of Currents in Secondary Windings and Load, (c) Vector Diagram of E.M.Fs., (d) Vector Diagram of Currents

current form is to be generated, using steam turbines as prime movers) the generators are wound to supply six-phase current directly to the rotary converters, which are located close to the generators, thus minimizing the disadvantages of a six-wire distribution.

Methods of obtaining a symmetrical six-phase system from a three-phase system. A symmetrical six-phase system may be obtained from a symmetrical three-phase system by means of a three-phase transformer, or, alternatively, three single-phase transformers, provided with double secondary windings. Both the star and mesh methods of interconnecting the secondary windings may be employed, the interconnections for the star connection being shown in Fig. 128, and those for the mesh connection being shown in Fig. 129. With both methods the line voltage is equal to the phase voltage (i.e. the voltage at the terminals of each of

the half-sections of the secondary windings), and the line current is equal to the phase current. Moreover, with balanced loads, the line-current vectors are displaced $(60 + \varphi)^\circ$, lagging, with respect to the line-voltage vectors.

With the *star connection* the E.M.F. between any pair of line wires is equal to the vector sum of the E.M.Fs., or the reversed E.M.Fs., as the case may be, in the two half-sections of the secondary windings to which these lines are connected. For example, if the E.M.Fs. in the double secondary windings are represented by the equations

$$e_{\mathrm{I}} = E_m \sin \omega t, \quad e_{\mathrm{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi), \quad e_{\mathrm{III}} = E_m \sin(\omega t - \tfrac{4}{3}\pi)$$

the line E.M.Fs. are given by the equations

$$v_{1\text{-}2} = e_{\mathrm{I}} + e_{\mathrm{III}} = E_m \sin(\omega t + \tfrac{1}{3}\pi), \quad v_{2\text{-}3} = -e_{\mathrm{II}} - e_{\mathrm{III}} = E_m \sin \omega t,$$
$$v_{3\text{-}4} = e_{\mathrm{I}} + e_{\mathrm{II}} = E_m \sin(\omega t - \tfrac{1}{3}\pi),$$
$$v_{4\text{-}5} = -e_{\mathrm{I}} - e_{\mathrm{III}} = E_m \sin(\omega t - \tfrac{2}{3}\pi), \quad v_{5\text{-}6} = e_{\mathrm{II}} + e_{\mathrm{III}} = E_m \sin(\omega t - \pi).$$
$$v_{6\text{-}1} = -e_{\mathrm{I}} - e_{\mathrm{II}} = E_m \sin(\omega t - \tfrac{4}{3}\pi).$$

The vector diagram is shown in Fig. 128 (*c*), in which the vectors OE_{I}. OE_{II}, OE_{III} represent the E.M.Fs. in the half-sections of the secondary windings, and the vectors $OV_{1\text{-}2}$, $OV_{2\text{-}3}$, . . . represent the E.M.Fs. between the line wires.

If the six-phase load is balanced, the currents in the secondary windings will be balanced, and may be represented by the vectors OI_{I}, OI_{II}, OI_{III} [Fig. 128 (*d*)] and the equations

$$i_{\mathrm{I}} = I_m \sin(\omega t - \varphi)\,;$$
$$i_{\mathrm{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi - \varphi)\,; \quad i_{\mathrm{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi)$$

The currents in the line wires 1, 2, 3 . . . are equal to i_{I}, $-i_{\mathrm{III}}$, i_{II}, $-i_{\mathrm{I}}$, i_{III}, $-i_{\mathrm{II}}$, as shown in the circuit diagram (*b*).

The currents in the branches of the load are given by

$$i_{\mathrm{A}} = i_{\mathrm{I}} + i_{\mathrm{III}} = I_m[\sin(\omega t - \varphi) + \sin(\omega t - \tfrac{4}{3}\pi - \varphi)]$$
$$= I_m \sin(\omega t + \tfrac{1}{3}\pi - \varphi)$$
$$i_{\mathrm{B}} = -i_{\mathrm{III}} - i_{\mathrm{II}} = -I_m[\sin(\omega t - \tfrac{4}{3}\pi - \varphi) + \sin(\omega t - \tfrac{2}{3}\pi - \varphi)]$$
$$= I_m \sin(\omega t - \varphi)$$

and so on. They are represented in the vector diagram, Fig. 128 (*d*). by the vectors OI_{A}, OI_{B}, . . .

With the *mesh connection* the line E.M.Fs. are equal to the E.M.Fs. in the half-sections of the secondary windings, and by

interconnecting these sections in the manner shown in Fig. 129 a phase difference of 60° is obtained between the E.M.Fs. of adjacent series-connected sections, as shown in the vector diagram of Fig. 130.

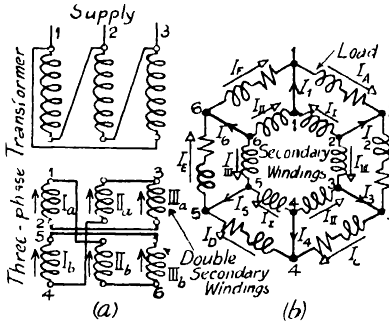


FIG. 129. Circuit Diagrams for the Mesh Method of Supplying a Six-phase Load from a Three-phase Transformer: (a) Transformer Connections, (b) Circuit Diagram showing Directions of Currents in Secondary Windings and Load

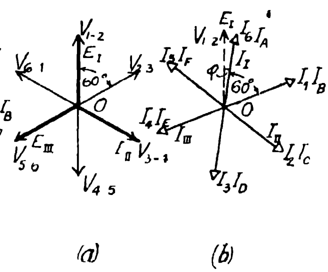


FIG. 130. Vector Diagrams [(a) E.M.Fs., (b) Currents] for Fig. 129

For example,

$$v_{1-2} = e_{\text{I}} = E_m \sin \omega t$$

$$v_{2-3} = -e_{\text{III}} = -E_m \sin(\omega t - \tfrac{4}{3}\pi) = E_m \sin(\omega t - \tfrac{1}{3}\pi)$$

$$v_{3-4} = e_{\text{II}} = E_m \sin(\omega t - \tfrac{2}{3}\pi)$$

and so on. These E.M.Fs. are represented in the vector diagram [Fig. 130 (a)] by the vectors OV_{1-2}, OV_{2-3}, OV_{3-4}, . . . , the E.M.Fs. in the secondary windings being represented by OE_{I}, OE_{II}, OE_{III}.

With a balanced system, i.e. with equal currents in each of the sections of the secondary winding, the current in any line wire is equal to the vector sum of the currents, or the reversed currents, as the case may be, in the sections to which that line is connected. For example, if the currents in the secondary windings are

$$i_{\text{I}} = I_m \sin \omega t, \quad i_{\text{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi), \quad i_{\text{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi),$$

the line currents are given by the equations

$$i_1 = i_{\text{I}} + i_{\text{II}} = I_m \sin(\omega t - \tfrac{1}{3}\pi), \quad i_2 = -i_{\text{I}} - i_{\text{III}} = I_m \sin(\omega t - \tfrac{2}{3}\pi)$$

$$i_3 = i_{\text{II}} + i_{\text{III}} = I_m \sin(\omega t - \pi),$$

$$i_4 = -i_{\text{I}} - i_{\text{II}} = I_m \sin(\omega t - \tfrac{4}{3}\pi), \quad i_5 = i_{\text{I}} + i_{\text{III}} = I_m \sin(\omega t - \tfrac{5}{3}\pi),$$

$$i_6 = -i_{\text{II}} - i_{\text{III}} = I_m \sin \omega t;$$

and the load currents are

$$i_A = i_1 - i_2 = I_m[\sin(\omega t - \tfrac{1}{3}\pi) - \sin(\omega t - \tfrac{2}{3}\pi)] = I_m \sin \omega t$$
$$i_B = i_2 - i_3 = I_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \pi)] = I_m \sin(\omega t - \tfrac{1}{3}\pi)$$

and so on.

Fig. 130 (b) is a vector diagram of currents. The vectors OI_{I}, OI_{II}, OI_{III} represent the currents in the secondary windings, and are shown lagging φ° with respect to the corresponding E.M.F. vectors in the diagram (a); the vectors OI_1, OI_2, . . . represent the line currents; and the vectors OI_A, OI_B, . . . represent the load currents.

Alternative connections. When the transformer windings supply a balanced mesh-connected load, as for example the armature of a

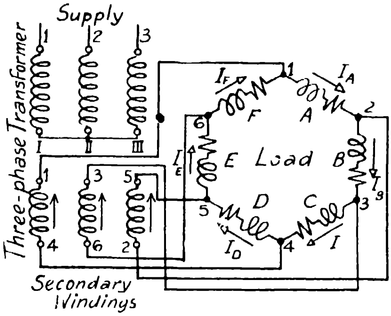


FIG. 131

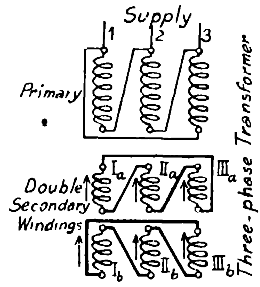


FIG. 132

Transformer Connections for the Diametrical and Double-delta Methods of Supplying a Six-phase Load from a Three phase Transformer

[NOTE. With the double-delta method (Fig 132) the line wires are connected to the top terminals of the upper and lower sections of the secondary windings in the order (left to right) 1, 3, 5 for the upper windings and 6, 2, 4 for the lower windings.]

rotary converter, the connections of Figs. 128, 129, may be replaced by those shown in Figs. 131, 132, respectively, which possess the advantage, over those of the former figures, of a higher line voltage and a smaller cross-section of line wire for a given power in the two cases.

The connection of Fig. 131 is called the "*diametrical*" *connection* and does not require double secondary windings on the transformers. With the load open circuited, the secondary windings are not interconnected and their E.M.Fs. are equivalent to three equal single-phase E.M.Fs. having a mutual phase difference of 120°. But when the windings are connected to the load the currents in the phases of the latter have a mutual phase difference of 60°, and a 60° phase difference exists between the E M.Fs. across successive pairs of line wires, so that a six-phase system is produced. These

conditions are represented in the circuit and vector diagrams of Fig. 133.

If the E.M.Fs. in the secondary windings are

$$e_{\text{I}} = 2E_m \sin \omega t,\ e_{\text{II}} = 2E_m \sin(\omega t - \tfrac{2}{3}\pi),\ e_{\text{III}} = 2E_m \sin(\omega t - \tfrac{4}{3}\pi)$$

the voltage across phase A of the (balanced) load is

$$v_{1-2} = \tfrac{1}{2}(e_{\text{I}} + e_{\text{III}}) = E_m[\sin \omega t + \sin(\omega t - \tfrac{4}{3}\pi)]$$
$$= E_m \sin(\omega t + \tfrac{1}{3}\pi)$$

that across phase B of the load is

$$v_{2-3} = -\tfrac{1}{2}(e_{\text{II}} + e_{\text{III}}) = -E_m[\sin(\omega t - \tfrac{2}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi)]$$
$$= E_m \sin \omega t$$

and so on.

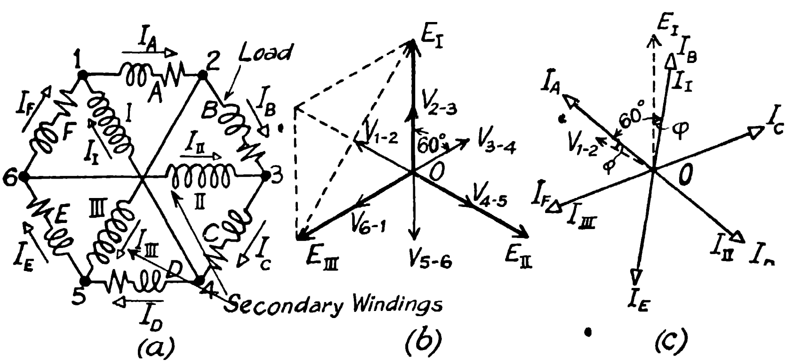


FIG. 133.—Circuit and Vector Diagrams for "Diametrical" Method of Supplying a Six-phase System

In Fig. 133 (b) the E.M.Fs. in the secondary windings are represented by the vectors OE_{I}, OE_{II}, OE_{III}, and the voltages across the load are represented by the vectors OV_{1-2}, OV_{2-3} . . .

If the currents in the secondary windings are represented by the equations

$$i_{\text{I}} = I_m \sin \omega t, \quad i_{\text{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi), \quad i_{\text{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi),$$

the current in phase A of the load is given by

$$i_{\text{A}} = i_{\text{I}} + i_{\text{III}} = I_m[\sin \omega t + \sin(\omega t - \tfrac{4}{3}\pi)] = I_m \sin(\omega t + \tfrac{1}{3}\pi).$$

Similarly, the current in phase B is given by

$$i_{\text{B}} = -i_{\text{II}} - i_{\text{III}} = -I_m[\sin \omega t - \tfrac{2}{3}\pi) + \sin(\omega t - \tfrac{4}{3}\pi)] = I_m \sin \omega t,$$

the current in phase C by

$$i_{\text{C}} = i_{\text{I}} + i_{\text{II}} = I_m\ [\sin \omega' + \sin(\omega t - \tfrac{2}{3}\pi) = I_m \sin(\omega t - \tfrac{1}{3}\pi),$$

and so on

In Fig. 133 (*c*) the vectors OI_{I}, OI_{II}, OI_{III} represent the currents in the secondary windings (which are shown lagging $\varphi°$ with respect to the corresponding E.M F. vectors in the diagram (*a*)), and the vectors OI_{A}, OI_{B}, . . . represent the load currents.

The connection of Fig. 132 is called the "*double delta*" *connection*: it requires double secondary windings on the transformers as in the six-phase connections of Figs. 128, 129. This connection, so far as the secondary windings of the transformer are concerned, is equivalent to two three-phase delta connections displaced 180° in relation to each other, as represented in the conventional diagram of Fig. 134.

If the currents in secondary windings are represented by the equations

$$i_{\text{I}} = I_m \sin \omega t, \quad i_{\text{II}} = I_m \sin(\omega t - \tfrac{2}{3}\pi), \quad i_{\text{III}} = I_m \sin(\omega t - \tfrac{4}{3}\pi),$$

the currents in the line wires are given by

$$i_1 = i_{\text{I}} - i_{\text{III}} = I_m[\sin \omega t - \sin(\omega t - \tfrac{4}{3}\pi)] = \sqrt{3}\, I_m \sin(\omega t - \tfrac{1}{6}\pi),$$

$$i_2 = i_{\text{II}} - i_{\text{III}} = I_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin(\omega t - \tfrac{4}{3}\pi)] = \sqrt{3}.I_m \sin(\omega t - \tfrac{1}{2}\pi),$$

$$i_3 = i_{\text{II}} - i_{\text{I}} = I_m[\sin(\omega t - \tfrac{2}{3}\pi) - \sin \omega t] = \sqrt{3}\, I_m \sin(\omega t - \tfrac{5}{6}\pi),$$

$$i_4 = i_{\text{III}} - i_{\text{I}} = I_m[\sin(\omega t - \tfrac{4}{3}\pi) - \sin \omega t] = \sqrt{3}.I_m \sin(\omega t - \tfrac{7}{6}\pi),$$

and so on.

The currents in the phases of the mesh-connected load are given by

$$i_{\text{A}} = i_1 - i_2 = \sqrt{3}\, I_m[\sin(\omega t - \tfrac{1}{6}\pi) - \sin(\omega t - \tfrac{1}{2}\pi)]$$
$$= \sqrt{3}\, I_m \sin(\omega t + \tfrac{1}{6}\pi),$$

$$i_{\text{B}} = i_2 - i_3 = \sqrt{3}.I_m[\sin(\omega t - \tfrac{1}{2}\pi) - \sin(\omega t - \tfrac{5}{6}\pi)]$$
$$= \sqrt{3}.I_m \sin(\omega t - \tfrac{1}{6}\pi),$$

$$i_{\text{C}} = i_3 - i_4 = \sqrt{3}\, I_m[\sin(\omega t - \tfrac{5}{6}\pi) - \sin(\omega t - \tfrac{7}{6}\pi)]$$
$$= \sqrt{3}\, I_m \sin(\omega t - \tfrac{1}{2}\pi),$$

and so on.

The vector diagrams are shown in Fig. 134. In the vector diagram of currents, the vectors OI_{I}, OI_{II}, OI_{III} represent the currents in the secondary windings—these vectors being shown lagging $\varphi°$ with respect to the corresponding E.M.F. vectors OE_{I}, OE_{II}, OE_{III} the vectors OI_1, OI_2, . . . represent the line currents, and the vectors OI_{A}, OI_{B}, . . . represent the load currents.

The determination of the voltage across each phase of the load is not so simple as the determination of the load currents, and is best effected by replacing the two delta-connected circuits by

equivalent star-connected circuits. Thus considering the delta circuit I_a, II_a, III_a, the E.M.Fs. of the secondary windings are represented by the vectors OE_{I}, OE_{II}, OE_{III} and by the vector triangle shown in full lines. The phase E.M.Fs. of the equivalent star-connected circuit are represented by the vectors OE_a, OE_b, OE_c, which have a phase difference of 30°, lagging, with respect to the vectors OE_{I}, OE_{II}, OE_{III}. Observe that in the upper diagram OE_a, OE_b, OE_c are drawn from the centre, O, of the triangle formed by the vectors E_{I}, E_{II}, E_{III}.

For the delta circuit I_b, II_b, III_b, the E.M.Fs. of the secondary windings are represented by the vectors OE_{I}, OE_{II}, OE_{III} and by the vector triangle (shown dotted) formed by the vectors E_{I}',

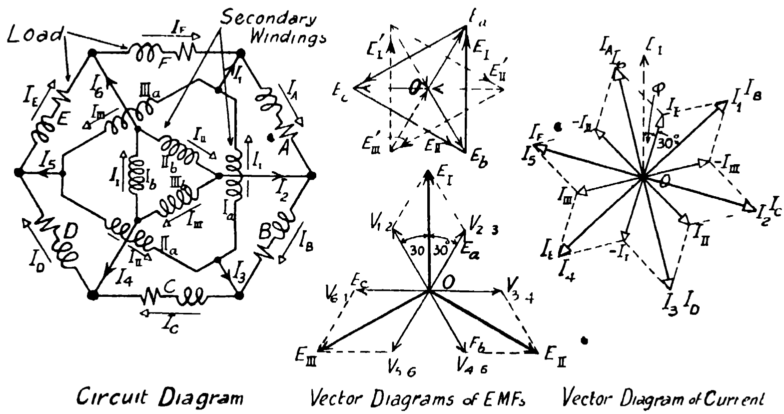


FIG. 134.—Circuit and Vector Diagrams for "Double-delta" Method of Supplying a Six-phase System

E_{II}', E_{III}'.* The phase E.M.Fs. of the equivalent star-connected circuit are represented by the vectors OE_a, OE_b, OE_c in the lower diagram and by the dotted vectors drawn from the corners to the centre, O, of the triangle E_{I}', E_{II}', E_{III}'.

Hence if the neutral points of the equivalent star circuits are connected together, the conditions are equivalent to Fig. 128. Thus the voltages across the phases of the load are equal in magnitude to the phase voltages of the star system, and are represented by the vectors OV_{1-2}, OV_{2-3}, . . . in Fig. 134. Observe that $\dot{V}_{1-2} + \dot{V}_{2-3} = \dot{E}_{I}$, $\dot{V}_{3-4} + \dot{V}_{4-5} = \dot{E}_{II}$, and so on.

* This triangle is reversed relatively to the triangle E_{I}, E_{II}, E_{III}, as the interconnections of the two sets of secondary windings are reversed relatively to each other.

To obtain analytical expressions for the load voltages, let the E.M.Fs. in the secondary windings be given by

$$e_I = E_m \sin \omega t,\ e_{II} = E_m \sin(\omega t - \tfrac{2}{3}\pi),\ e_{III} = E_m \sin(\omega t - \tfrac{4}{3}\pi)$$

Then if the E.M.Fs. of the equivalent star circuit are denoted by e_a, e_b, e_c, we have

$$e_a - e_b = e_I,\ e_b - e_c = e_{II},\ e_c - e_a = e_{III}$$

Hence, since $e_I + e_{II} + e_{III} = 0$, the solution of these equations gives

$$e_a = \tfrac{1}{3}(e_I - e_{III}),\ e_b = \tfrac{1}{3}(e_{II} - e_I),\ e_c = \tfrac{1}{3}(e_{III} - e_{II})$$

The voltage across phase A of the load is then given by

$$e_{12} = e_a + e_c = \tfrac{1}{3}(e_I - e_{II}) = \tfrac{1}{3}E_m[\sin \omega t - \sin(\omega t - \tfrac{2}{3}\pi)]$$
$$= \tfrac{1}{3}E_m \sin(\omega t + \tfrac{1}{6}\pi)$$

that across phase B is given by

$$e_{23} = e_b - e_c = \tfrac{1}{3}(e_I - e_{III}) = \tfrac{1}{3}E_m[\sin \omega t - \sin(\omega t - \tfrac{4}{3}\pi)]$$
$$= \tfrac{1}{3}E_m \sin(\omega t - \tfrac{1}{6}\pi)$$

and so on

The voltage, E_t, at the terminals of any half-section of the transformer winding for the double-delta connection is equal to the vector sum of the voltages across the two phases of the load to which that section is connected. Since the latter have a phase difference of 60°, therefore, $E_t = \sqrt{3}.E_l$, where E_l is the voltage across each phase of the load, the voltage drop in the line wires being neglected.

With the diametrical connection, Fig. 131, the voltage across each phase of the secondary winding of the transformer is equal to the vector sum of the voltages across three phases of the load, and since these have a mutual phase difference of 60°, the secondary voltage is equal to $2E_l$, where E_l is the voltage across each phase of the load and the voltage drop in the line wires is neglected. Hence, for equal power and voltage at the load in each case the diametrical connection requires line wires of smaller cross-section than those required for the double-delta connection.

The diametrical connection, therefore, possesses three advantages over the double-delta connection, viz. (1) single secondary windings are required on the transformers, (2) smaller cross-section of line wires, for the same load conditions, due to the voltage at the terminals of the secondary windings being 15·5[$= 100(2/\sqrt{3})$] per cent higher than that for the double-delta connection, (3) no interconnections are necessary between the secondary windings of the transformers.

Measurement of power in six-phase systems. Although very few cases occur in practice where the measurement of power in six-phase circuits is required, since this measurement is usually effected on the three-phase system from which the six-phase power is obtained, we shall consider briefly some methods by which the power in six-phase systems may be measured.

If the system is *balanced* the total power may be measured by a single wattmeter, as in a balanced three-phase system, by inserting the current coil in one line wire and supplying the potential coil with a voltage equivalent to the phase voltage of the system. For example, if the neutral point of the system is available the potential coil is connected between the neutral point and the line wire in which the current coil of the wattmeter is inserted

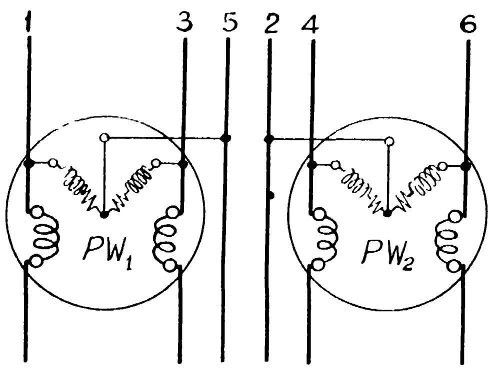


FIG. 135.—Double Two-wattmeter Method of Measuring Power in a Six-phase System (Polyphase Wattmeters)

If the neutral point is not available, a number of alternative connections are possible. For example, if the current coil is inserted in line No. 1, the potential coil may be connected across line wires 6 and 1, since the voltage between these lines is equal to, and is in phase with, the phase voltage of phase I (see vector diagram Fig. 126).

Alternatively, a star-connected potential circuit for the wattmeter, similar to that shown in Fig. 118, may be employed. In the present case the potential branch containing the potential coil must be connected to the line wire in which the current coil of the wattmeter is inserted, and the other branches must be connected to line wires which have a phase difference of 120° from that containing the current coil. Thus, it the current coil is connected in line 1, the end of the potential coil must be connected to this line and the other ends of the star-connected potential resistances must be connected to lines 3 and 5.

In all the above cases the wattmeter measures the power in one phase of the system, and the total power is therefore given by six times the reading of the wattmeter.

If the system is unbalanced the double two-wattmeter method, using two polyphase wattmeters connected as shown in Fig. 135, may be employed. The total power in the system is then given by the sum of the readings of the wattmeters. Thus, denoting instantaneous values of the phase voltages by

$$e_{\mathrm{I}},\ e_{\mathrm{II}},\ e_{\mathrm{III}},\ e_{\mathrm{IV}},\ e_{\mathrm{V}},\ e_{\mathrm{VI}},$$

line voltages by

$$v_{1\,2},\ v_{2\,3},\ v_{3\text{-}4},\ v_{4\,5},\ v_{5\,6},\ v_{6\,1},$$

and line currents by

$$i_1,\ i_2,\ i_3,\ i_4,\ i_5,\ i_6,$$

the instantaneous power measured by wattmeter PW_1 is given by

$$p_1 = i_1 v_{1\,5} + i_3 v_{3\,5},$$

and that measured by wattmeter PW_2 is given by

$$p_2 = i_4 v_{4\,2} + i_6 v_{6\,2}.$$

Now $v_{1\text{-}5} = e_{\mathrm{I}} - e_{\mathrm{V}},\ v_{3\,5} = e_{\mathrm{III}} - e_{\mathrm{V}},\ v_{4\,2} = e_{\mathrm{IV}} - e_{\mathrm{II}},\ v_{6\,2} = e_{\mathrm{VI}} - e_{\mathrm{II}}.$

Hence,

$$\begin{aligned} p_1 + p_2 &= i_1 v_{1\,5} + i_3 v_{3\text{-}5} + i_4 v_{4\,2} + i_6 v_{6\,2} \\ &= i_1(e_{\mathrm{I}} - e_{\mathrm{V}}) + i_3(e_{\mathrm{III}} - e_{\mathrm{V}}) + i_4(e_{\mathrm{IV}} - e_{\mathrm{II}}) + i_6(e_{\mathrm{VI}} - e_{\mathrm{II}}) \\ &= i_1 e_{\mathrm{I}} + i_3 e_{\mathrm{III}} + i_4 e_{\mathrm{IV}} + i_6 e_{\mathrm{VI}} - e_{\mathrm{V}}(i_1 + i_3) - e_{\mathrm{II}}(i_4 + i_6) \end{aligned}$$

But, since i_1, i_3, i_5 have a mutual phase difference of 120°

$$i_1 + i_3 = -i_5$$

and, since i_2, i_4, i_6 also have a mutual phase difference of 120°

$$i_4 + i_6 = -i_2$$

Therefore,

$$p_1 + p_2 = i_1 e_{\mathrm{I}} + i_2 e_{\mathrm{II}} + i_3 e_{\mathrm{III}} + i_4 e_{\mathrm{IV}} + i_5 e_{\mathrm{V}} + i_6 e_{\mathrm{VI}},$$

which is equal to the total power in the six-phase system.

Nine and Twelve-phase Systems

These systems are used with motor converters and also, in certain cases, with rotary converters. With a motor converter the nine or twelve-phase system is used in connection with the rotary-converter portion of the machine, and is obtained from the rotor of the induction-motor portion of the machine by a suitable winding and through the agency of the rotating magnetic field produced by the polyphase currents in the stator winding. The power supply to the stator is usually three phase, but the number

of phases for which the rotor may be wound is independent of the number of phases in the system from which the stator is supplied.

Nine and twelve-phase systems for supplying rotary converters must be obtained from a three-phase system by means of transformers, and the phase transformation is effected statically by suitable interconnection of the windings, as shown in the diagrams of Figs 136, 137.

For the nine-phase load, double secondary windings (giving unequal voltages) are required on the transformer. One set of

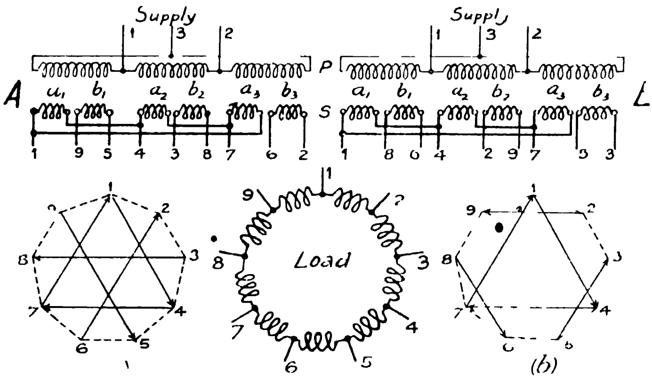


FIG. 136 – Methods of Supplying a Nine phase Load from a Three phase Transformer

windings, a_1, a_2, a_3, is connected in delta to points 1, 4, 7 of the load, and the other windings, b_1, b_2, b_3, are connected to the load points shown in Fig. 136. The relative voltages to be supplied by each winding can be easily determined from the geometry of the vector diagrams (a, b) of E.M Fs.

For the twelve-phase load, a number of alternative transformer connections are possible, involving double, triple, and quadruple secondary windings. Four of these methods are shown in Fig. 137.

Method A requires triple secondary windings, two sets of which are similar and are interconnected according to the double-delta method. The remaining set of windings is sometimes provided with mid-point tappings, which, when interconnected, form a neutral point to the mesh-connected load.

Method B requires quadruple secondary windings · two sets are connected in double delta, and the remaining two sets are connected in mesh. The windings forming the double-delta group are similar, and the windings forming the mesh-connected group are also similar, but the voltages of the two groups are unequal.

Methods *C* and *D* require only double secondary windings, the sections of which are not interconnected except through the load.

The relative voltages to be supplied by each section of the secondary windings for the several methods of connection can be easily determined from the geometry of the vector diagrams (*a*, *b*, *c*, *d*) of E.M.Fs.*

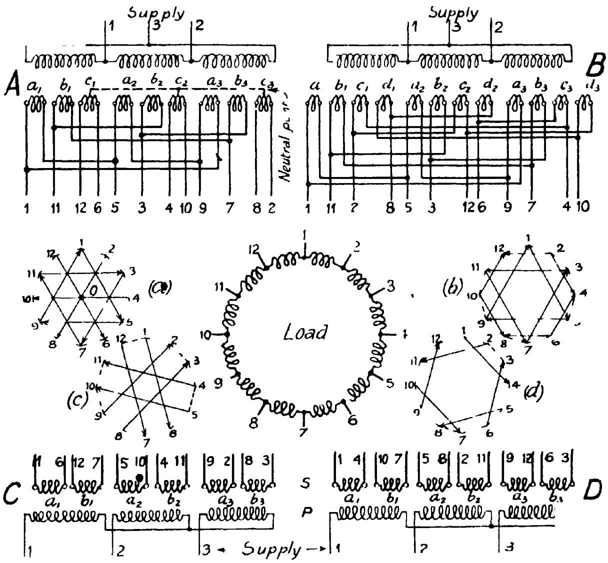


FIG. 137 Four Methods of Supplying a Twelve-phase Load from a Three-phase Transformer

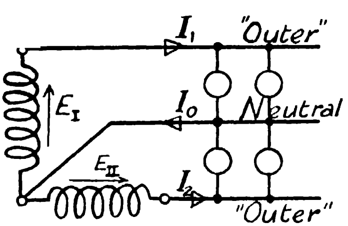
TWO-PHASE SYSTEM

Independent two-phase system. The independent, or four-wire, two-phase system has already been considered in connection with the simple polyphase alternator (p. 173). This system, however, may be obtained from a four-phase, star-connected, system by disconnecting the neutral point and connecting alternate phases in series, as shown in Fig. 145 (p. 229). In this manner two equal E.M.Fs., having a phase difference of 90°, are obtained. As each of the new phases is now independent, the new system is equivalent to two single-phase systems in which the phase difference between the E.M.Fs. and currents of each phase is 90°.

* Numerical examples are given in the collected examples at the end of the book.

Interconnected two-phase system. The two-phase system may be interconnected, so that only three line wires are necessary, by joining one end of each phase to a common line wire as shown in Fig. 138, this interconnected system being called the two-phase three-wire system. The common line wire is called the neutral wire and the other line wires are called the "outers."

The vector diagram for this system is shown in Fig 139, from which we observe that in a balanced system the E.M F across the "outers" is equal to the vector difference of the phase E M.Fs, and is numerically equal to $\sqrt{2}$ ($=1{\cdot}414$) times the phase E.M.F.



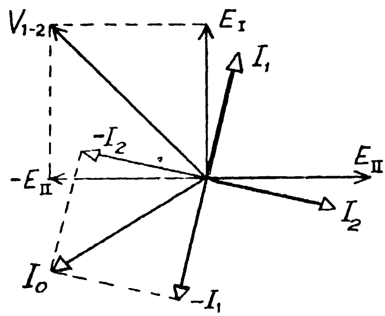


FIG 138 FIG. 139
Circuit and Vector Diagrams for Two phase Three wire System

Moreover, the E.M.F. across the "outers" has a phase difference of 45°, leading, with respect to the E M F of phase I.

The current in the neutral wire is equal to the reversed vector sum of the currents in the "outers," and with balanced loads the former is equal to $\sqrt{2}$ times the current in each "outer." Moreover, the current in the neutral has a phase difference of 135° with respect to the current in the "outers," being leading with respect to the current in one "outer," and lagging with respect to the current in the other "outer," as shown in Fig. 139.

Due to these large phase differences between the currents in the neutral and "outers" of a two-phase three-wire system the voltage drop in the neutral wire will have large phase differences with respect to the voltage drops in the "outers," and therefore when these voltage drops are compounded with the phase E.M.Fs. to obtain the voltages at the load, the latter will, in general, be unequal and will not have a phase difference of 90°. Thus, the two-phase three-wire system becomes unsymmetrical when loaded, and for this

reason the system does not possess much practical value for power transmission.

Comparison of two-phase and three-phase systems. The two-phase system is not so advantageous as the three-phase system for power supply, as four line wires are necessary to maintain a symmetrical two-phase system, whereas only three wires are necessary in the case of a three-phase system. The two-phase four-wire system, however, is used to some extent in this country and abroad for mixed lighting and power loads, the lighting load being single phase and the power load being two phase.

The majority of these installations were originally single-phase systems supplying lighting loads, and were converted into two-phase systems for the purpose of enabling a power load to be supplied in addition to the lighting load. For modern installations, however, the three-phase four-wire (star-connected) system is employed for mixed loads, as with this system the lighting portion of the load is supplied at the phase voltage of the system, and the power portion of the load is supplied at the "line" voltage of the system, which is $\sqrt{3}$ times the phase voltage. Moreover, a two-phase supply may be obtained from such a system by means of transformers connected in the manner shown in Fig. 140 and described below.

Method of obtaining a two-phase supply from a three-phase system. A two-phase supply may be obtained from a three-phase system by means of two single-phase static transformers connected in the manner shown in Fig. 140.*

Both transformers have similar magnetic circuits and may have similar secondary windings, in which case the primary winding of one transformer, B, must have only 86·6 per cent $(=\frac{1}{2}\sqrt{3})$ of the turns of the other transformer, A, and the latter must have its primary winding tapped at the mid-point. One end of the 86·6 per cent winding is connected to the mid-point of the primary winding of transformer, A, and the other end of the primary winding of B, as well as both ends of the primary winding of A, are connected to the three-phase line wires. The ends of the secondary windings are connected to the two-phase line wires.

Assuming the two-phase side to be unloaded, and the magnetizing currents, which are supplied from the three-phase side, to be balanced and sinusoidal, the resultant ampere-turns in transformer A are equal to the vector difference of the ampere-turns due to the currents in the half-sections of its primary winding. Hence, since these currents have a phase difference of 120°, the resultant ampere-turns are given by $F_A = \sqrt{3}.I_0 \times \frac{1}{2} N_1 = 0{\cdot}866 I_0 N_1$, where I_0 is the magnetizing current and N_1 the number of turns in the primary winding.

Similarly, the ampere-turns in transformer B are given by $F_B = 0{\cdot}866 I_0 N_1$, since the primary winding of this transformer has only 86·6 of the number of turns in the primary winding of transformer A. Moreover, the ampere-turns, F_B, have a phase difference of 90° with respect to F_A.

Hence the fluxes in the two transformers are equal and have a phase difference of 90°. Therefore the E.M.Fs., E_1, E_{11}, induced in the secondary windings are equal and have a phase difference of 90°.

* This connection is due to Prof. C. F. Scott, and is usually called the "Scott" connection.

Consider now the E.M.Fs. induced in the primary windings by these fluxes. Since the fluxes are equal, the E.M.Fs. will be proportional to the number of turns in each winding, and therefore the E.M.F., E_B, induced in B, will only be 86·6 per cent of that E_A, induced in the two half-sections of A. Moreover, these E.M.Fs. have a phase difference of 90°.

Taking the positive directions of these internal, or induced E.M.Fs. as those marked by the arrows in Fig. 140, and neglecting the resistance and reactance of the primary winding, the resultant internal E.M.Fs. between the terminals 1, 2, 3, of the three-phase side are

$$\dot{E}_{1\text{-}2} = \dot{E}_B - \tfrac{1}{2}\dot{E}_A, \qquad \dot{E}_{2\text{-}3} = \dot{E}_A, \qquad \dot{E}_{3\text{-}1} = -\tfrac{1}{2}\dot{E}_A - \dot{E}_B$$

Expressing these quantities symbolically, and taking E_B as the quantity of reference, we have

$$\dot{E}_B = E_B(1 + j0) = E_A(0{\cdot}866 + j0)\,; \quad \dot{E}_A = E_A(0 - j1)$$

$$\dot{E}_{1\text{-}2} = \dot{E}_B - \tfrac{1}{2}\dot{E}_A = E_A(0{\cdot}866 + j0{\cdot}5)$$

$$\dot{E}_{2\text{-}3} = E_A(0 - j1)$$

$$\dot{E}_{3\text{-}1} = -\tfrac{1}{2}\dot{E}_A - \dot{E}_B = E_A(-0{\cdot}866 - j0{\cdot}5)$$

Whence the magnitudes of these E.M.Fs. are given by

$$E_{1\text{-}2} = E_A\sqrt{(0{\cdot}5^2 + 0{\cdot}866^2)} = E_A,$$

$$E_{2\text{-}3} = E_A,$$

$$E_{3\text{-}1} = E_A\sqrt{(0{\cdot}5^2 + 0{\cdot}866^2)} = E_A\,;$$

and their phase differences with respect to E_B are given by

$$\varphi_{1\text{-}2} = \tan^{-1}(-0{\cdot}5/0{\cdot}866) = 30°,$$

$$\varphi_{2\text{-}3} = \tan^{-1} - 1/0 = -90°$$

$$\varphi_{3\text{-}1} = \tan^{-1}(-0{\cdot}5/0{\cdot}866) = 150°.$$

Therefore the internal E.M.Fs. between the terminals 1, 2, 3, of the three-phase side are equal to one another and have a mutual phase difference of 120°. Thus the symmetry and balance of the three-phase supply system are not affected.

A vector diagram showing the induced E.M.Fs. and fluxes is given in Fig. 140 (c), in which the vectors $OV_{1\text{-}2}$, $OV_{2\text{-}3}$, $OV_{3\text{-}1}$ represent the line voltages of the three-phase system; OI_{01}, OI_{02}, OI_{03}, the magnetizing currents (which are lagging 90° with respect to the corresponding phase voltages of the three-phase system); OF_A, OF_B, the ampere-turns supplied by the magnetizing currents; $O\Phi_A$, $O\Phi_B$, the fluxes due to the magnetizing ampere-turns; OE_{A1}, OE_{A2}, and OE_{B1}, OE_{B2}, the E.M.Fs. induced in the primary and secondary windings of the two transformers.

If the fluxes are to remain constant when the two-phase side is loaded—which will be the case if the three-phase supply voltage is constant and the resistance and reactance voltage drops in the primary windings are negligible—the magnetizing ampere-turns must remain constant, and therefore the ampere-turns due to the load currents in the secondary windings must be balanced by an equivalent number of ampere-turns in the primary windings, the vector sum of the ampere-turns in primary and secondary windings being, in all cases, equal to the magnetizing ampere-turns.

The conditions for balanced loads are represented in the vector diagram (*d*) Fig. 140, in which the magnetizing ampere-turns are represented by the vectors OF_A, OF_B; the load currents by OI_I, OI_{II}; the secondary ampere-turns by OF_{A2}, OF_{B2}; and the primary ampere-turns by OF_{A1}, OF_{B1}, which have a phase difference of 90°. The primary current of transformer B is, therefore, represented by the vector OI_1.

The primary ampere-turns of transformer A are due to the currents in

the half-sections of this winding, and these currents are supplied from lines 2 and 3 of the three-phase system. Hence if OF_{A1} is resolved into vectors Oe, Od, 60° apart and each equal to $\frac{1}{2}OF_{A1}$, then Od will represent the ampere-turns due to the current in line 2, and Oe reversed, i.e Of will represent those due to the current in line 3. Since the angle between OF_{A1} and OF_{B1} is 90°, the vectors Oe, Of will have a mutual phase difference of 120° with respect to the vector OF_{B1}. Hence if these ampere-turn vectors are converted into the current vectors OI_2, OI_3, then each of these vectors will be found to be equal in magnitude to the vector OI_1.

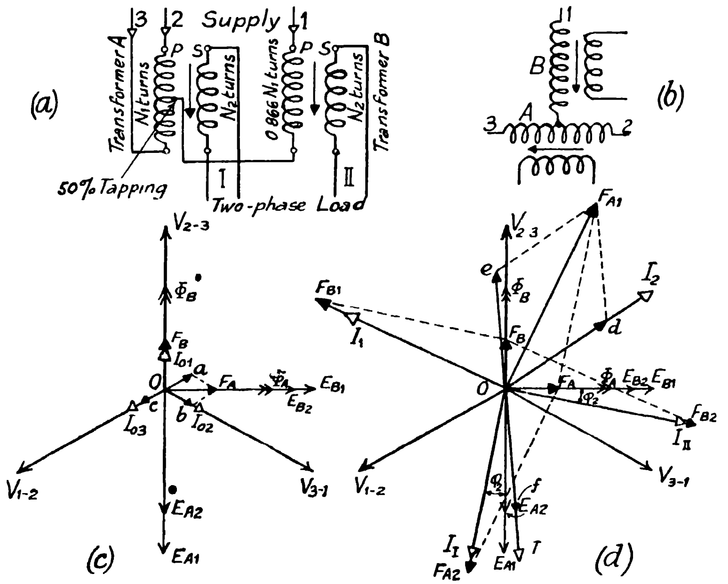


FIG. 140.—Circuit and Vector Diagrams for Scott's Method of Supplying a Two-phase Load from a Three-phase System : (a) Transformer Connections, (b) Conventional Circuit Diagram, (c) Vector Diagram for No-load, (d) Vector Diagram for Balanced Loads

[NOTE. The effects of losses and magnetic leakage are ignored in the vector diagrams.]

Therefore, with balanced loads on the two-phase side the currents on the three-phase side will also be balanced.

In order to maintain these balanced conditions in transformers of the "core" type, the coils forming the two half-sections of the primary winding of transformer A must be interlaced, or sandwiched, in order that each part of the magnetic circuit may be acted upon equally by the joint M.M.Fs. due to the two phases of the three-phase system which supplies this transformer.

Moreover, if the magnetizing ampere-turns are neglected, and if $N_2=N_1=N$, say—i.e. the voltage across each phase of the two-phase side is equal to the voltage between the line wires of the three-phase side, the effects of resistance and reactance of the windings being neglected—and I_1 is the current in the primary windings, we have

$$I_2N = 0{\cdot}866 I_1 N$$

whence,

$$I_1 = I_2/0{\cdot}866 = I_2(2/\sqrt{3}) = 1{\cdot}15 I_2.$$

This, then, is the numerical relationship between the currents in the primary and secondary windings when the loads are balanced and the magnetizing current is ignored.

Measurement of power in two-phase systems. With a balanced system the total power may be measured by means of a single wattmeter which is so connected as to measure the power in one phase of the system. The total power in the system is therefore given by twice the reading of the wattmeter. A diagram showing the connections for the four-wire system is given in Fig. 141. With unbalanced systems the "two-wattmeter" method must be employed.

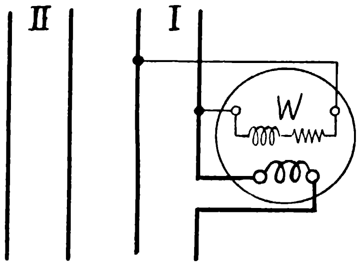


FIG. 141. Connections for the Measurement of Power in Balanced Two-phase Circuits

The connections of the two wattmeters for the four-wire system are given in Fig. 142, and it is apparent that, as each wattmeter (or each element in the case of a polyphase wattmeter) measures the power in one phase, the total power in the system is given by the sum of the readings of the wattmeters (or by the reading of the polyphase wattmeter when such instrument is employed).

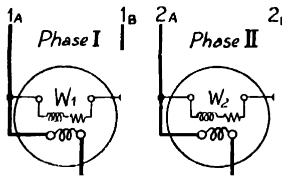


FIG. 142

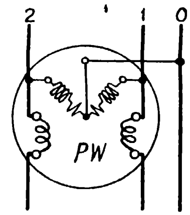


FIG. 143

Connections for the Measurement of Power in Unbalanced Two-phase Circuit

The connections of a polyphase wattmeter to a three-wire system are given in Fig. 143.

FOUR-PHASE SYSTEM

In a four-phase symmetrical system there are four equal E.M.Fs. 90° apart. The phase E.M.Fs. may therefore be represented by the equations

$$e_{\text{I}} = E_m \sin \omega t,\ e_{\text{II}} = E_m \sin(\omega t - \tfrac{1}{2}\pi),\ e_{\text{III}} = E_m \sin(\omega t - \pi),$$
$$e_{\text{IV}} = E_m \sin(\omega t - \tfrac{3}{2}\pi).$$

Star-connected system. The circuit diagram for a star-connected system is given in Fig. 144, in which the assumed positive directions for E.M.Fs. and currents are indicated by arrows. The instantaneous value of the line E.M.Fs. are given by the equations

$$v_{1\text{-}2} = e_{\text{I}} - e_{\text{II}} = E_m[\sin \omega t - \sin(\omega t - \tfrac{1}{2}\pi)] = \sqrt{2}.E_m \sin(\omega t + \tfrac{1}{4}\pi),$$

$$v_{2\text{-}3} = e_{\text{II}} - e_{\text{III}} = E_m[\sin(\omega t - \tfrac{1}{2}\pi) - \sin(\omega t - \pi)] = \sqrt{2}.E_m \sin(\omega t - \tfrac{1}{4}\pi)$$

$$v_{3\text{-}4} = e_{\text{III}} - e_{\text{IV}} = E_m[\sin(\omega t - \pi) - \sin(\omega t - \tfrac{3}{2}\pi)] = \sqrt{2}.E_m \sin(\omega t - \tfrac{3}{4}\pi),$$

$$v_{4\text{-}1} = e_{\text{IV}} - e_{\text{I}} = E_m[\sin(\omega t - \tfrac{3}{2}\pi) - \sin \omega t] = \sqrt{2}.E_m \sin(\omega t - \tfrac{5}{4}\pi).$$

Thus the line E.M.Fs. are equal to one another and have a mutual phase difference of 90°. Moreover, the line E.M.Fs. have a phase difference of 45°, leading, with respect to the phase E.M.Fs.

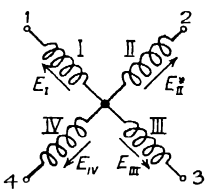


Fig. 144

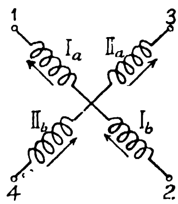


Fig. 145

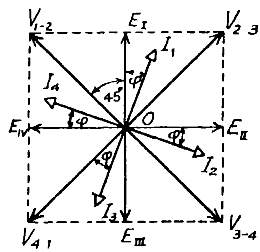


Fig. 146

Circuit and Vector Diagrams for Star-connected Four-phase System

The R.M.S. values of the line E.M.Fs. are

$$V_{1\text{-}2} = \sqrt{2}.E, \quad V_{2\ 3} = \sqrt{2}.E, \quad V_{3\text{-}4} = \sqrt{2}.E, \quad V_{4\ 1} = \sqrt{2}.E,$$

where E is the R.M.S. value of the phase E.M.Fs.

If the neutral point is disconnected and alternate phases are connected in series, as in Fig. 145, we obtain an independent two-phase system, the E.M.Fs. of which are given by

$$v_{1\ 3} = e_{\text{I}} - e_{\text{III}} = E_m[\sin \omega t - \sin(\omega t - \pi)] = 2E_m \sin \omega t$$

$$v_{2\text{-}4} = e_{\text{II}} - e_{\text{IV}} = E_m[\sin(\omega t - \tfrac{1}{2}\pi) - \sin(\omega t - \tfrac{3}{2}\pi)] = 2E_m \sin(\omega t - \tfrac{1}{2}\pi)$$

The *vector diagram* for a star-connected system with balanced loads is shown in Fig. 146, in which the phase E.M.Fs. are represented by the vectors OE_{I}, OE_{II}, OE_{III}, OE_{IV}; the line E.M.Fs. by $OV_{1\text{-}2}$, $OV_{2\text{-}3}$, $OV_{3\ 4}$, $OV_{4\ 1}$, and the line currents by OI_1, OI_2, OI_3, OI_4. Observe that the phase difference between the line-voltage vectors and the line-current vectors is $(45 + \varphi)°$, and is lagging with respect to the line voltage, φ being the phase difference between the phase E.M.F. and current.

Mesh-connected system. The circuit diagram for a four-phase mesh-connected system is shown in Fig. 147. If the instantaneous values of the phase currents are represented by the equations

$$i_{\mathrm{I}} = I_m \sin(\omega t - \varphi), \qquad i_{\mathrm{II}} = I_m \sin(\omega t - \tfrac{1}{2}\pi - \varphi),$$
$$i_{\mathrm{III}} = I_m \sin(\omega t - \pi - \varphi), \qquad i_{\mathrm{IV}} = I_m \sin(\omega t - \tfrac{3}{2}\pi - \varphi),$$

the line currents will be given by

$$\begin{aligned} i_1 = i_{\mathrm{I}} - i_{\mathrm{IV}} &= I_m[\sin(\omega t - \varphi) - \sin(\omega t - \tfrac{3}{2}\pi - \varphi)] \\ &= \sqrt{2}.I_m \sin(\omega t - \tfrac{1}{4}\pi - \varphi), \\ i_2 = i_{\mathrm{II}} - i_{\mathrm{I}} &= I_m[\sin(\omega t - \tfrac{1}{2}\pi - \varphi) - \sin(\omega t - \varphi)] \\ &= \sqrt{2}.I_m \sin(\omega t - \tfrac{3}{4}\pi - \varphi), \\ i_3 = i_{\mathrm{III}} - i_{\mathrm{II}} &= I_m[\sin(\omega t - \pi - \varphi) - \sin(\omega t - \tfrac{1}{2}\pi - \varphi)] \\ &= \sqrt{2}.I_m \sin(\omega t - \tfrac{5}{4}\pi - \varphi), \\ i_4 = i_{\mathrm{IV}} - i_{\mathrm{III}} &= I_m[\sin(\omega t - \tfrac{3}{2}\pi - \varphi) - \sin(\omega t - \pi - \varphi)] \\ &= \sqrt{2}.I_m \sin(\omega t - \tfrac{7}{4}\pi - \varphi). \end{aligned}$$

Thus the line currents have a mutual phase difference of 90°, and

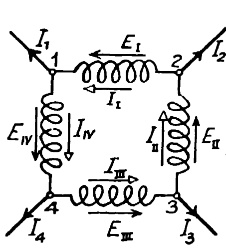


FIG. 147

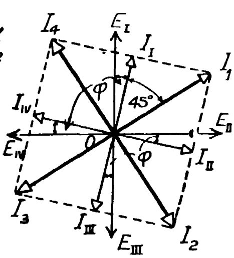


FIG. 148

Circuit and Vector Diagrams for Mesh-connected Four-phase System

a phase difference of $(45 + \varphi)°$ with respect to the line E.M.Fs., the latter being equal to the phase E.M.Fs.

The R.M.S. values of the line currents are

$$I_1 = \sqrt{2}.I, \quad I_2 = \sqrt{2}.I, \quad I_3 = \sqrt{2}.I, \quad I_4 = \sqrt{2}.I.$$

The vector diagram for a mesh-connected system is shown in Fig. 148.

Methods of obtaining a symmetrical four-phase system from a two-phase system. When four-phase current is required in practice for supplying a mesh-connected four-phase load, such as a four-phase rotary converter, it is obtained from a two-phase system

supply by means of static transformers, the methods being analagous to those adopted when six-phase current is to be supplied from a three-phase system.

If double secondary windings are provided on the transformers, the sections of these windings may be connected either in star or mesh, as in Fig. 149, to give a four-phase system.

If, however, only single secondary windings are provided on the transformers, the four-phase load may be supplied by adopting the connections shown in Fig. 150, which are analogous to the diametrical connections for the three-phase/six-phase system. These

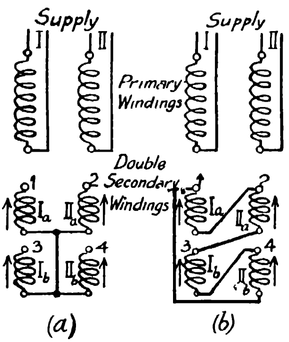


FIG. 149

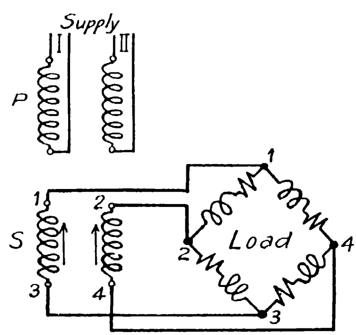


FIG. 150

Methods of Supplying a Four-phase Load (shown in Fig. 150) from a Two-phase Transformer

connections possess the following advantages over those of Fig. 149: (1) Single secondary windings are required on the transformers; (2) the four-phase line wires are of smaller cross section for the same load conditions due to the voltage at the terminals of the secondary windings being $\sqrt{2}$ times that for the connections of Fig. 149; (3) no interconnections are required between the transformers.

Measurement of power in four-phase system. Since, in practice, four-phase systems are almost always obtained from two-phase systems, the power in these cases may be measured on the two-phase side of the system. We shall, however, consider briefly the methods of measuring the power in a four-phase system, as this may be necessary when the system is supplied either from a four-phase generator or a two-phase/four-phase transformer.

With a balanced system the power may be measured by means of a single wattmeter which is so connected as to measure the power in one phase of the system

If the neutral point of the system is available the current coil of

the wattmeter is inserted in one of the line wires and the potential coil is connected between that line wire and the neutral point.

If the neutral point of the system is not available, a resistance equal in value to the potential-coil circuit of the wattmeter is connected in series with the potential coil, and the combination is connected between the line wire which contains the current coil and the alternate line wire, as shown in Fig. 151. In each case the total power in the system is equal to four times the power measured by the wattmeter.

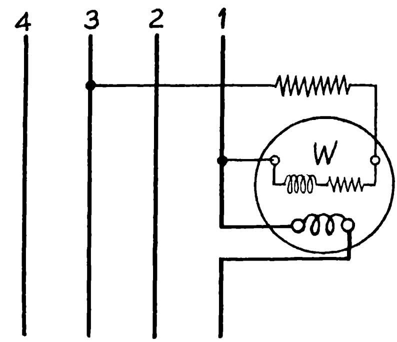


FIG. 151.—Connections for the Measurement of Power in Balanced Four-phase System

If the system is unbalanced either three or four wattmeters (or two polyphase wattmeters) must be employed, the connections being shown in Fig. 152.

When four wattmeters (or two polyphase wattmeters) are

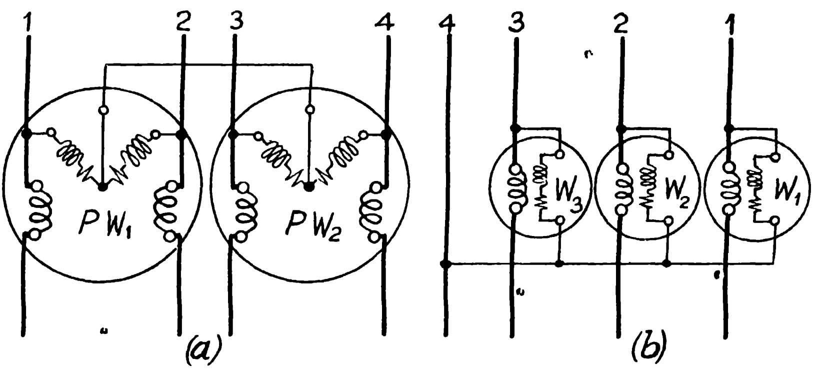


FIG. 152.—Connections for the Measurement of Power in Unbalanced Four-phase System

employed the potential coils may be connected to the neutral point of the system, if available, and each wattmeter, or each element of the polyphase wattmeters, then measures the power in the phase in which its current coil is connected. If the neutral point is not available and the potential coils are similar, the coils are connected

in star, as shown in Fig. 152 (*a*). In this case the total power is given by the sum of the readings of the wattmeters, although, as shown in Chapter IX (p. 249), the wattmeters do not measure the power in the respective phases unless the line voltages are equal and symmetrical.

When three wattmeters are employed they must be connected according to the diagram of Fig. 152 (*b*). In this case the algebraic sum of the readings gives the total power, but the readings on the wattmeters have no special significance.

The proof is as follows—

Let the instantaneous values of the line currents and voltages be denoted by i_1, i_2, i_3, i_4 ; $v_{1\,2}$, $v_{2\,3}$, $v_{3\,4}$, $v_{4\,1}$.

Then the power measured by the wattmeters is given by

$$p_1 = i_1 v_{1\text{-}4} = i_1(e_1 - e_4)$$
$$p_2 = i_2 v_{2\text{-}4} = i_2(e_2 - e_4)$$
$$p_3 = i_3 v_{3\text{-}4} = i_3(e_3 - e_4)$$

Hence the sum of the readings is given by

$$\begin{aligned} p_1 + p_2 + p_3 &= i_1(e_1 - e_4) + i_2(e_2 - e_4) + i_3(e_3 - e_4) \\ &= i_1 e_1 + i_2 e_2 + i_3(e_3 - e_4) + i_3(e_3 - e_4) \\ &= i_1 e_1 + i_2 e_2 + i_3 e_3 + i_4 e_4 \end{aligned}$$

which is equal to the total power in the circuit.

CHAPTER IX

CALCULATION OF POLYPHASE CIRCUITS

In this chapter the methods of calculating the currents and voltages in two-phase and three-phase circuits under various conditions of loading will be discussed. In all cases it is to be understood that the currents and E.M.Fs. vary sinusoidally with respect to time, so that vector diagrams and complex methods are applicable to solutions. Elementary cases of the calculation of currents in loads supplied at constant pressure will be considered first, and later the more general case, in which the pressure drop in the generator and line wires must be taken into account, will be discussed. As symbolic (complex algebraic) methods of solution are, in some cases, preferable to trigonometrical and graphical methods, we shall first deduce the complex expressions for the phase and line E.M.Fs. in two-phase and three-phase systems.

Symbolic representation of phase and line E.M.Fs. in a two-phase interconnected (three-wire) system. With a symmetrical system the no-load E.M.Fs. are represented by the symbolic expressions

$$\dot{E}_{\mathrm{I}} = E(1 + j0)$$
$$\dot{E}_{\mathrm{II}} = E(0 - j1)$$

Whence the E.M.Fs. between the line wires are

$$\dot{V}_{1\text{-}0} = \dot{E}_{\mathrm{I}} = E_{\mathrm{I}}(1 + j0)$$

or $\quad V_{1\text{-}0} = E$, between line 1 and the neutral

$$\dot{V}_{2\text{-}0} = \dot{E}_{\mathrm{II}} = E(0 - j1)$$

or $\quad V_{2\text{-}0} = E$, between line 2 and the neutral

$$\dot{V}_{1\text{-}2} = \dot{E}_{\mathrm{I}} - \dot{E}_{\mathrm{II}} = E(1 + j1)$$

or $\quad V_{1\text{-}2} = E\sqrt{2}$, between lines 1 and 2

Symbolic representation of phase and line E.M.Fs. in a three-phase, star-connected system. With a symmetrical system the phase E.M.Fs. may be represented by either of the symbolic expressions (72), (72*a*), according to the phase sequence, or phase rotation, of the generator. For example, if the armature of the simple alternator of Fig. 88 is rotated in the clockwise direction the phase sequence will be I, II, III, clockwise; but if the rotation is reversed,

the phase sequence will now be I, III, II, clockwise ; or I, II, III, counter-clockwise. These conditions are represented in Fig. 153, and it is apparent that a reversal of the direction of rotation of the armature causes a reversal of the phase sequence. The phase E.M.Fs. for clockwise rotation are given by equations (72) and by equations (72*a*) for counter-clockwise rotation. Thus,

Phase E.M.Fs. for clockwise phase rotation—

$$\left.\begin{aligned} \dot{E}_{\mathrm{I}} &= E(1 + j0) \\ \dot{E}_{\mathrm{II}} &= E(\cos 120^\circ - j\sin 120^\circ) = E(-0\cdot5 - j0\cdot866) \\ \dot{E}_{\mathrm{III}} &= E(\cos 240^\circ - j\sin 240^\circ) = E(-0\cdot5 + j0\cdot866) \end{aligned}\right\} \quad (72)$$

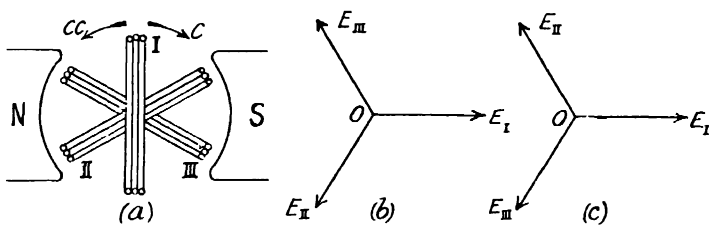


FIG. 153.– Illustrating Clockwise and Counter-clockwise Phase Sequence of a Three-phase System

Phase E.M.Fs. for counter-clockwise phase rotation—

$$\left.\begin{aligned} \dot{E}_{\mathrm{I}} &= E(1 + j0) \\ \dot{E}_{\mathrm{II}} &= E(\cos 120^\circ + j\sin 120^\circ) = E(-0\cdot5 + j0\cdot866) \\ \dot{E}_{\mathrm{III}} &= E(\cos 240^\circ + j\sin 240^\circ) = E(-0\cdot5 - j0\cdot866) \end{aligned}\right\} \quad (72a)$$

Hence the line E.M.Fs. are given by the expressions,

Clockwise phase rotation—

$$\begin{aligned} \dot{V}_{1\,2} &= \dot{E}_{\mathrm{I}} - \dot{E}_{\mathrm{II}} = E(1 + 0\cdot5 + j0\cdot866 = E(1\cdot5 + j0\cdot866) \\ \dot{V}_{2\,3} &= \dot{E}_{\mathrm{II}} - \dot{E}_{\mathrm{III}} = E(-0\cdot5 - j0\cdot866 + 0\cdot5 - j0\cdot866) = -j\sqrt{3}E \\ \dot{V}_{2\cdot1} &= \dot{E}_{\mathrm{III}} - \dot{E}_{\mathrm{I}} = E(-0\cdot5 + j0\cdot866 - 1 + j0) \\ &= E(-1\cdot5 + j0\cdot866) \end{aligned}$$

Counter-clockwise phase rotation—

$$\begin{aligned} \dot{V}_{1\cdot2} &= \dot{E}_{\mathrm{I}} - \dot{E}_{\mathrm{II}} = E(1 + 0\cdot5 - j0\cdot866) = E(1\cdot5 - j0\cdot866) \\ \dot{V}_{2\,3} &= \dot{E}_{\mathrm{II}} - \dot{E}_{\mathrm{III}} = E(-0\cdot5 + j0\cdot866 + 0\cdot5 + j0\cdot866) \\ &= j\sqrt{3}E \\ \dot{V}_{3\cdot1} &= \dot{E}_{\mathrm{III}} - \dot{E}_{\mathrm{I}} = E(-0\cdot5 - j0\cdot866 - 1 + j0) \\ &= E(-1\cdot5 - j0\cdot866) \end{aligned}$$

The absolute values and arguments of the line E.M.Fs. are,

Clockwise phase rotation—

$$V_{1\text{-}2} = E\sqrt{[(\tfrac{3}{2})^2 + (\tfrac{\sqrt{3}}{2})^2]} = E\sqrt{3}$$
$$V_{2\text{-}3} = E\sqrt{3}$$
$$V_{3\text{-}1} = E\sqrt{(\tfrac{3}{2})^2 + (\tfrac{\sqrt{3}}{2})^2]} = E\sqrt{3}$$
$$\theta_1 = \tan^{-1}(\tfrac{1}{2}\sqrt{3}/\tfrac{3}{2}) = \tan^{-1} 1/\sqrt{3} = 30°$$
$$\theta_2 = \tan^{-1} - \sqrt{3}/0 = \tan^{-1} - \infty = -90°$$
$$\theta_3 = \tan^{-1}(-\tfrac{1}{2}\sqrt{3}/\tfrac{3}{2}) = \tan^{-1} - (1/\sqrt{3}) = -210°$$

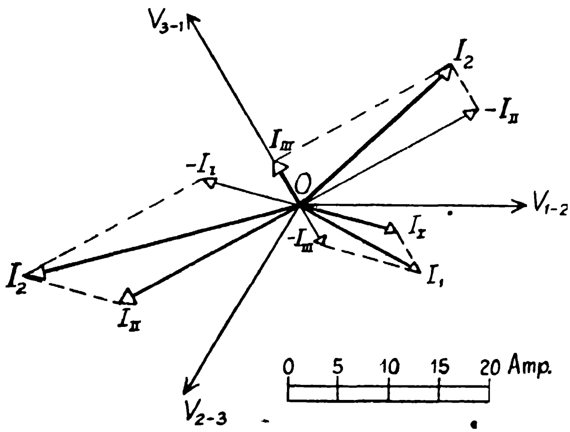


FIG. 154.—Graphic Solution for Currents in Delta-connected Load

Counter-clockwise phase rotation—

$$V_{1\text{-}2} = E\sqrt{[(\tfrac{3}{2})^2 + (\tfrac{\sqrt{3}}{2})^2]} = E\sqrt{3}$$
$$V_{2\text{-}3} = E\sqrt{3}$$
$$V_{3\text{-}1} = E\sqrt{[(\tfrac{3}{2})^2 + (\tfrac{\sqrt{3}}{2})^2]} = E\sqrt{3}$$
$$\theta_1 = \tan^{-1}(-\tfrac{1}{2}\sqrt{3}/\tfrac{3}{2}) = \tan^{-1} - 1/\sqrt{3} = -30°$$
$$\theta_2 = \tan^{-1} \sqrt{3}/0 = \tan^{-1} \infty = 90°$$
$$\theta_3 = \tan^{-1}(-\tfrac{1}{2}\sqrt{3}/-\tfrac{3}{2}) = \tan^{-1} - (1/\sqrt{3}) = 210°$$

The phase sequence of a polyphase system therefore affects the relative positions of the E.M.F. vectors of the system but has no effect upon the magnitudes of the E.M.Fs. when the system is symmetrical. Moreover, the phase sequence has no effect upon the magnitudes of the load and line currents when the system is balanced, but it may have a considerable effect upon these currents when the system is unbalanced, as is shown in the examples which follow.

Calculation of line and load currents for three-phase, delta-connected, unbalanced load supplied at constant voltage. The simplest solution is a graphic one and is shown in Fig. 154. The line voltages are represented by the vectors $OV_{1\,2}$, $OV_{2\,3}$, $OV_{3\,1}$. The currents in the branches of the load are calculated from the respective line voltages and impedances, and are represented by the vectors OI_{I}, OI_{II}, OI_{III}, which have phase differences of φ_1, φ_2, φ_3, respectively, with respect to the line voltages. The line currents are then obtained by determining the vector differences of these currents and are represented by the vectors OI_1, OI_2, OI_3; OI_1 being the vector difference of OI_{I} and OI_{III}, OI_2 the vector difference of OI_{II} and OI_{I}, and so on.

The problem, however, may be easily calculated throughout by

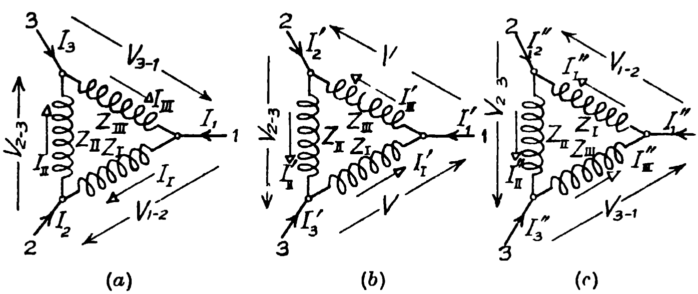


FIG. 155.—Circuit Diagrams for Delta connected Load

the symbolic method, the polar form described on p. 27 being preferable to the rectangular form. Thus, if the *phase rotation is clockwise*, the line E.M.Fs. are given by the expressions

$$\dot{V}_{1\text{-}2} = VJ^0, \quad \dot{V}_{2\text{-}3} = VJ^{\,120/90} \quad \dot{V}_{3\,1} = VJ^{-240/90} = VJ^{120/90}$$

and if the load impedances are arranged in the order Z_{I}, Z_{II}, Z_{III}, clockwise—the impedance Z_{I} being connected between lines 1 and 2, Z_{II} between lines 2 and 3, and Z_{III} between lines 3 and 1, as represented in Fig. 155*a*,—the load currents will be given by

$$\dot{I}_{\mathrm{I}} = \frac{\dot{V}_{1\,2}}{\dot{Z}_{\mathrm{I}}} = I_{\mathrm{I}}J^{-\varphi_1{}^{\circ}/90}$$

$$\dot{I}_{\mathrm{II}} = \frac{\dot{V}_{2\,3}}{\dot{Z}_{\mathrm{II}}} = I_{\mathrm{II}}J^{-(120+\varphi_2{}^{\circ})/90}$$

$$\dot{I}_{\mathrm{III}} = \frac{\dot{V}_{3\,1}}{\dot{Z}_{\mathrm{III}}} = I_{\mathrm{III}}J^{-(240+\varphi_3{}^{\circ})/90} = I_{\mathrm{III}}J^{(120-\varphi_3{}^{\circ})/90}$$

Hence the line currents are given by

$$\left.\begin{aligned}
\dot{I}_1 &= \dot{I}_{\mathrm{I}} - \dot{I}_{\mathrm{III}} = I_{\mathrm{I}} J^{\varphi_1^\circ/90} - I_{\mathrm{III}} J^{(120\ \varphi_3^\circ)/90} \\
&= I_{\mathrm{I}} J^{\varphi_1^\circ/90} + I_{\mathrm{III}} J^{(60+\varphi_3^\circ)/90} \\
\dot{I}_2 &= \dot{I}_{\mathrm{II}} - \dot{I}_{\mathrm{I}} = I_{\mathrm{II}} J^{(120+\varphi_2^\circ)/90} - I_{\mathrm{I}} J^{\varphi_1^\circ\ 90} \\
&= I_{\mathrm{II}} J^{(120+\varphi_2^\circ)/90} + I_{\mathrm{I}} J^{(180+\varphi_1^\circ)/90} \\
\dot{I}_3 &= \dot{I}_{\mathrm{III}} - \dot{I}_{\mathrm{II}} = I_{\mathrm{III}} J^{(120-\varphi_3^\circ)/90} - I_{\mathrm{II}} J^{(120+\varphi_2^\circ)/90} \\
&= I_{\mathrm{III}} J^{(120\ \varphi_3^\circ)/90} + I_{\mathrm{II}} J^{(60-\varphi_2^\circ)/90}
\end{aligned}\right\} \quad (73)$$

$$\left.\begin{aligned}
\text{or}\quad I_1 &= \sqrt{[I_{\mathrm{I}}^2 + I_{\mathrm{III}}^2 + 2I_{\mathrm{I}}I_{\mathrm{III}}\cos(60+\varphi_3^\circ-\varphi_1^\circ)]} \\
I_2 &= \sqrt{\{I_{\mathrm{II}}^2 + I_{\mathrm{I}}^2 + 2I_{\mathrm{I}}I_{\mathrm{II}}\cos[(180+\varphi_1^\circ)-(120+\varphi_2^\circ)]\}} \\
&= \sqrt{[I_{\mathrm{II}}^2 + I_{\mathrm{I}}^2 + 2I_{\mathrm{I}}I_{\mathrm{II}}\cos(60+\varphi_1^\circ-\varphi_2^\circ)]} \\
I_3 &= \sqrt{\{I_{\mathrm{III}}^2 + I_{\mathrm{II}}^2 + 2I_{\mathrm{II}}I_{\mathrm{III}}\cos[(120-\varphi_3^\circ)-(60-\varphi_2^\circ)]\}} \\
&= \sqrt{[I_{\mathrm{III}}^2 + I_{\mathrm{II}}^2 + 2I_{\mathrm{II}}I_{\mathrm{III}}\cos(60+\varphi_2^\circ-\varphi_3^\circ)]}
\end{aligned}\right\} \quad (74)$$

The phase differences between the line currents and the E.M.F. between lines 1 and 2 are given by

$$\left.\begin{aligned}
\alpha_1 &= \tan^{-1}\frac{I_{\mathrm{I}}\sin-\varphi_1^\circ + I_{\mathrm{III}}\sin-(60+\varphi_3^\circ)}{I_{\mathrm{I}}\cos-\varphi_1^\circ + I_{\mathrm{III}}\cos-(60+\varphi_3^\circ)} \\
\alpha_2 &= \tan^{-1}\frac{I_{\mathrm{II}}\sin-(120+\varphi_2^\circ) + I_{\mathrm{I}}\sin-(180+\varphi_1^\circ)}{I_{\mathrm{II}}\cos-(120+\varphi_2) + I_{\mathrm{I}}\cos-(180+\varphi_1^\circ)} \\
\alpha_3 &= \tan^{-1}\frac{I_{\mathrm{III}}\sin-(240+\varphi_3^\circ) + I_{\mathrm{II}}\sin-(300+\varphi_2^\circ)}{I_{\mathrm{III}}\cos-(240+\varphi_3^\circ) + I_{\mathrm{II}}\cos-(300+\varphi_2^\circ)}
\end{aligned}\right\} \quad (75)$$

If the *phase rotation of the system is reversed* and the arrangement of the load impedances is unaltered (i.e. the pressure across the impedance Z_{I} is now $V_{3\,1}$, that across Z_{III} is $V_{1\,2}$, and that across Z_{II} is $V_{2\,3}$, as represented in Fig. 155*b*), the line E.M.Fs. may be given by the expressions

$$\dot{V}_{1\,2} = VJ^0, \quad \dot{V}_{2\,3} = VJ^{120/90}, \quad \dot{V}_{3\,1} = VJ^{240/90} = VJ^{-120/90}$$

Hence the load currents are now given by

$$\dot{I}_{\mathrm{I}}' = \frac{\dot{V}_{3\,1}}{\dot{Z}_{\mathrm{I}}} = \frac{V}{Z_{\mathrm{I}}} J^{(120+\varphi_1^\circ)/90} = I_{\mathrm{I}}' J^{(120+\varphi_1^\circ)/90}$$

$$\dot{I}_{\mathrm{II}}' = \frac{\dot{V}_{2\cdot3}}{\dot{Z}_{\mathrm{II}}} = \frac{V}{Z_{\mathrm{II}}} J^{(120-\varphi_2^\circ)/90} = I_{\mathrm{II}}' J^{(120-\varphi_2^\circ)/90}$$

$$\dot{I}_{\mathrm{III}}' = \frac{\dot{V}_{1\cdot2}}{\dot{Z}_{\mathrm{III}}} = \frac{V}{Z_{\mathrm{III}}} J^{\varphi_3/90} = I_{\mathrm{III}}' J^{-\varphi_3^\circ/90}$$

where I_{I}' is the current in the impedance Z_{I}, I_{II}'' the current in the impedance Z_{II}, and I_{III}' the current in the impedance Z_{III}.

The line currents are given by

$$
\begin{aligned}
I_1' &= I_{III}' - I_I' = I_{III}' J^{\varphi_3°/90} - I_I' J^{(120+\varphi_2°)/90} \\
&= I_{III}' J^{-\varphi_3°/90} + I_I' J^{(60-\varphi_1°)/90} \\
I_2' &= I_{II}' - I_{III}' = I_{II}' J^{(120-\varphi_2°)/90} - I_{III}' J^{\varphi_3°/90} \\
&= I_{II}' J^{(120-\varphi_2°)/90} + I_{III}' J^{(180-\varphi_3°)/90} \\
I_3' &= I_I' - I_{II}' = I_I' J^{(120+\varphi_1°)/90} - I_{II}' J^{(120-\varphi_2°)/90} \\
&= I_I' J^{(120+\varphi_1°)/90} + I_{II}' J^{(60+\varphi_2°)/90}.
\end{aligned} \qquad (73)
$$

or

$$
\left.
\begin{aligned}
I_1' &= \sqrt{[I_{III}'^2 + I_I'^2 + 2I_{III}'I_I' \cos (60 - \varphi_1° + \varphi_3°)]}. \\
I_2' &= \sqrt{[I_{II}'^2 + I_{III}'^2 + 2I_{II}'I_{III}' \cos (60 - \varphi_3° + \varphi_2°)]}. \\
I_3' &= \sqrt{[I_I'^2 + I_{II}'^2 + 2I_I'I_{II}' \cos (60 - \varphi_2° + \varphi_1°)]}.
\end{aligned}
\right\} \qquad (74a)
$$

If, with the reversed phase rotation, the impedances Z_I and Z_{III} are interchanged so that the order is now Z_I, Z_{II}, Z_{III}, counter-clockwise (Fig. 155c), the pressure across the impedance Z_I is now V_{12}, and that across the impedance Z_{II} is V_{31}. Hence the load currents will be given by

$$
I_I'' = \frac{V_{12}}{Z_I} = I_I J^{\varphi_1°/90}
$$

$$
I_{II}'' = \frac{V_{23}}{Z_{II}} = I_{II} J^{(120-\varphi_2°)/90}
$$

$$
I_{III}'' = \frac{V_{31}}{Z_{III}} = I_{III} J^{(120+\varphi_3°)/90}
$$

where I_I'' is the current in the impedance Z_I, I_{II}'' the current in the impedance Z_{II}, and I_{III}'' the current in the impedance Z_{III}.

The line currents are

$$
\begin{aligned}
I_1'' &= I_I'' - I_{III}'' = I_I J^{\varphi_1°/90} - I_{III} J^{(120+\varphi_3°)/90} \\
&= I_I J^{\varphi_1°/90} + I_{III} J^{(60-\varphi_3°)/90} \\
I_2'' &= I_{II}'' - I_I'' = I_{II} J^{(120-\varphi_2°)/90} - I_I J^{\varphi_1°/90} \\
&= I_{II} J^{(120-\varphi_2°)/90} + I_I J^{(180-\varphi_1°)/90} \\
I_3'' &= I_{III}'' - I_{II}'' = I_{III} J^{(120+\varphi_3°)/90} - I_{II} J^{(120-\varphi_2°)/90} \\
&= I_{III} J^{(120+\varphi_3°)/90} + I_{II} J^{(60+\varphi_2°)/90}.
\end{aligned} \qquad (73b)
$$

or

$$
\left.
\begin{aligned}
I_1'' &= \sqrt{[I_I^2 + I_{III}^2 + 2I_I I_{III} \cos (60 - \varphi_3° + \varphi_1°)]}. \\
I_2'' &= \sqrt{[I_{II}^2 + I_I^2 + 2I_{II} I_I \cos (60 - \varphi_1° + \varphi_2°)]}. \\
I_3'' &= \sqrt{[I_{III}^2 + I_{II}^2 + 2I_{III} I_{II} \cos (60 - \varphi_2° + \varphi_3°)]}.
\end{aligned}
\right\} \qquad (74b)
$$

Comparing these equations with those (74) for clockwise phase rotation, we find that the two sets of equations are similar except

for the signs of the phase-angles φ_1, φ_2, φ_3. Hence in the special case where the branches of the unbalanced load have the same power-factor (i.e. $\varphi_1 = \varphi_2 = \varphi_3$) the magnitudes of the line currents will be unaffected by a change of phase rotation, provided that the load is so connected that a given branch of the load is always connected between the same line wires.

Example. An unbalanced delta-connected load, the branch-circuit impedances of which are $Z_I = 10\underline{/15°}$, $Z_{II} = 5\underline{/30°}$, $Z_{III} = 20\underline{/0°}$, is supplied from a symmetrical three-phase system in which the line pressure is 100 volts. Calculate the line currents.

Assuming the phase rotation to be clockwise and the load impedance to be connected in the order Z_I, Z_{II}, Z_{III}, clockwise, the impedance Z_I being connected between line wires 1 and 2; and denoting the load currents by I_I, I_{II}, I_{III}, and the line currents by I_1, I_2, I_3, we have, from equations (73), (74),

$$\dot{I}_I = \frac{100}{10} J^{15/90} = 10 J^{-15/90}$$

$$I_I = 10A$$

$$\dot{I}_{II} = \frac{100}{5} J^{-(120+30)/90} = 20 J^{150/90}$$

$$I_{II} = 20A$$

$$\dot{I}_{III} = \frac{100}{20} J^{-240/90} = 5 J^{-240/90}$$

$$I_{III} = 5A$$

$$I_1 = \sqrt{[10^2 + 5^2 + 2 \times 10 \times 5 \cos(60 + 0 - 15)°]} = 14A$$

$$I_2 = \sqrt{[20^2 + 10^2 + 2 \times 20 \times 10 \cos(60 + 15 - 30)°]} = 28A$$

$$I_3 = \sqrt{[5^2 + 20^2 + 2 \times 5 \times 20 \cos(60 + 30 - 0)°]} = 20{\cdot}6A.$$

A vector diagram drawn to scale is shown in Fig. 154.

If the phase rotation is reversed and the arrangement of the load impedances is unaltered, the load and line currents are now obtained from equations (73*a*), (74*a*), and are as follow.

$$\dot{I}'_I = \frac{100}{10} J^{-(120+15)/90} = 10 J^{-135/90}$$

$$I'_I = 10A.$$

$$\dot{I}'_{II} = \frac{100}{5} J^{-(120-30)/90} = 20 J^{-90/90}$$

$$I'_{II} = 20\ A.$$

$$\dot{I}'_{III} = \frac{100}{20} J^{0/90} = 5 J^0$$

$$I'_{III} = 5A.$$

$$I'_1 = \sqrt{[5^2 + 10^2 + 2 \times 5 \times 10 \cos(60 - 15 + 0)°]} = 14A.$$

$$I'_2 = \sqrt{[20^2 + 5^2 + 2 \times 20 \times 5 \cos(60 - 0 + 30)°]} = 20{\cdot}6A.$$

$$I'_3 = \sqrt{[10^2 + 20^2 + 2 \times 10 \times 20 \cos(60 - 30 + 15)°]} = 28A.$$

The solution of the corresponding problem for a star-connected load is not so simple, as the voltage impressed on each branch of

the load is not known, and the determination of these voltages involves a knowledge of the potential of the neutral point of the load. The problem, however, may be solved analytically without a knowledge of the potential of the neutral point by converting the star-connected load into an equivalent delta-connected load, and calculating the load and line currents as above. The line currents will, of course, be the same in both cases.

Conversion of an unbalanced star-connected load into an equivalent delta-connected load. In order that the delta-connected load may be the equivalent of the star-connected load, the joint impedances and admittances between corresponding terminals must be the same in both cases. Thus, if Z_1, Z_2, Z_3, denote the impedances of the branches of the star-connected load, and Z_a, Z_b, Z_c, denote the corresponding impedances of the branches of the equivalent delta-connected load (see Fig. 156), then we have

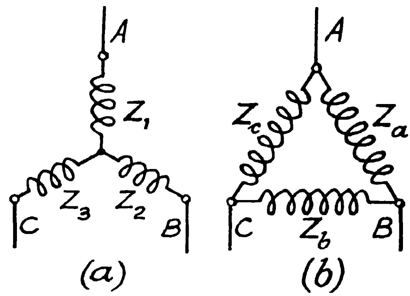


FIG. 156 – Circuit Diagrams for Equivalent Star and Delta Circuits

Joint impedance between terminals A and B of star-connected load

$$= Z_1 + Z_2$$

Joint impedance between terminals B and C of star-connected load

$$= Z_2 + Z_3$$

Joint impedance between terminals C and A of star-connected load

$$= Z_3 + Z_1$$

Joint impedance between terminals A and B of delta-connected load

$$= \frac{Z_a(Z_b + Z_c)}{Z_a + Z_b + Z_c}$$

Joint impedance between terminals B and C of delta-connected load

$$= \frac{Z_b(Z_a + Z_c)}{Z_a + Z_b + Z_c}$$

Joint impedance between terminals C and A of delta-connected load

$$= \frac{Z_c(Z_a + Z_b)}{Z_a + Z_b + Z_c}$$

Hence for the delta-connected load to be the equivalent of the star-connected load, we must have

$$Z_1 + Z_2 = \frac{Z_a(Z_b + Z_c)}{Z_a + Z_b + Z_c} = \frac{Z_b + Z_c}{1 + Z_b/Z_a + Z_c/Z_a}$$

$$Z_2 + Z_3 = \frac{Z_b(Z_a + Z_c)}{Z_a + Z_b + Z_c} = \frac{Z_a + Z_c}{1 + Z_a/Z_b + Z_c/Z_b}$$

$$Z_3 + Z_1 = \frac{Z_c(Z_a + Z_b)}{Z_a + Z_b + Z_c} = \frac{Z_a + Z_b}{1 + Z_a/Z_c + Z_b/Z_c}$$

Whence

$$\left.\begin{aligned} Z_a &= Z_1 + Z_2 + (Z_1Z_2/Z_3) \quad . \quad . \quad . \\ Z_b &= Z_2 + Z_3 + (Z_2Z_3/Z_1) \quad . \quad . \quad . \\ Z_c &= Z_3 + Z_1 + (Z_3Z_1/Z_2) \quad . \quad . \quad . \end{aligned}\right\} \quad (76$$

The method of obtaining the latter group of equations from the preceding group is as follows—

Dividing each equation by the numerator of its right-hand side, we have

$$\frac{Z_1 + Z_2}{Z_a(Z_b + Z_c)} = \frac{Z_2 + Z_3}{Z_b(Z_a + Z_c)} = \frac{Z_3 + Z_1}{Z_c(Z_a + Z_b)} = \frac{1}{Z_a + Z_b + Z_c}$$

from which we obtain

$$\frac{(Z_1 + Z_2) - (Z_2 + Z_3) + (Z_3 + Z_1)}{Z_a(Z_b + Z_c) - Z_b(Z_a + Z_c) + Z_c(Z_a + Z_b)} = \frac{2Z_1}{2Z_aZ_c} = \frac{1}{Z_a + Z_b + Z_c}$$

$$\frac{(Z_1 + Z_2) + (Z_2 + Z_3) - (Z_3 + Z_1)}{Z_a(Z_b + Z_c) + (Z_b(Z_a + Z_c) - Z_c(Z_a + Z_b)} = \frac{2Z_2}{2Z_aZ_b} = \frac{1}{Z_a + Z_b + Z_c}$$

$$\frac{-(Z_1 + Z_2) + (Z_2 + Z_3) + (Z_3 + Z_1)}{-Z_a(Z_b + Z_c) + Z_b(Z_a + Z_c) + Z_c(Z_a + Z_b)} = \frac{2Z_a}{2Z_bZ_c} = \frac{1}{Z_a + Z_b + Z_c}$$

$$\text{Hence} \quad \frac{Z_a}{Z_b} = \frac{Z_1}{Z_3}, \quad \frac{Z_b}{Z_c} = \frac{Z_2}{Z_1}, \quad \frac{Z_c}{Z_a} = \frac{Z_3}{Z_2}$$

Substituting in the original equations we have

$$Z_1 + Z_2 = \frac{Z_c(Z_2/Z_1) + Z_c}{1 + Z_3/Z_1 + Z_3/Z_2} = \frac{Z_cZ_2(Z_1 + Z_2)}{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}$$

whence

$$Z_c = Z_3 + Z_1 + (Z_1Z_3/Z_2)$$

Similarly

$$Z_b = Z_2 + Z_3 + (Z_2Z_3/Z_1)$$
$$Z_a = Z_1 + Z_2 + (Z_1Z_2/Z_3)$$

Denoting the *admittances* of the star-connected load by Y_1, Y_2, Y_3, and the corresponding admittances of the delta-connected load by Y_a, Y_b, Y_c, then for the two loads to be equivalent we must have

$$\frac{Y_1Y_2}{Y_1 + Y_2} = Y_a + \frac{Y_bY_c}{Y_b + Y_c} = \frac{Y_aY_b + Y_bY_c + Y_cY_a}{Y_b + Y_c}$$

$$\frac{\dot{Y}_2\dot{Y}_3}{\dot{Y}_2+\dot{Y}_3}=\dot{Y}_b+\frac{\dot{Y}_c\dot{Y}_a}{\dot{Y}_c+\dot{Y}_a}=\frac{\dot{Y}_a\dot{Y}_b+\dot{Y}_b\dot{Y}_c+\dot{Y}_c\dot{Y}_a}{\dot{Y}_c+\dot{Y}_a}$$

$$\frac{\dot{Y}_3\dot{Y}_1}{\dot{Y}_3+\dot{Y}_1}=\dot{Y}_c+\frac{\dot{Y}_a\dot{Y}_b}{\dot{Y}_a+\dot{Y}_b}=\frac{\dot{Y}_a\dot{Y}_b+\dot{Y}_b\dot{Y}_c+\dot{Y}_c\dot{Y}_a}{\dot{Y}_a+\dot{Y}_b}$$

Whence

$$\left.\begin{aligned}\dot{Y}_a&=\frac{\dot{Y}_1\dot{Y}_2}{\dot{Y}_1+\dot{Y}_2+\dot{Y}_3}\\ \dot{Y}_b&=\frac{\dot{Y}_2\dot{Y}_3}{\dot{Y}_1+\dot{Y}_2+\dot{Y}_3}\\ \dot{Y}_c&=\frac{\dot{Y}_3\dot{Y}_1}{\dot{Y}_1+\dot{Y}_2+\dot{Y}_3}\end{aligned}\right\} \qquad (77)$$

Graphical construction for obtaining the values of the equivalent impedances. As equation (76) shows the equivalent impedances

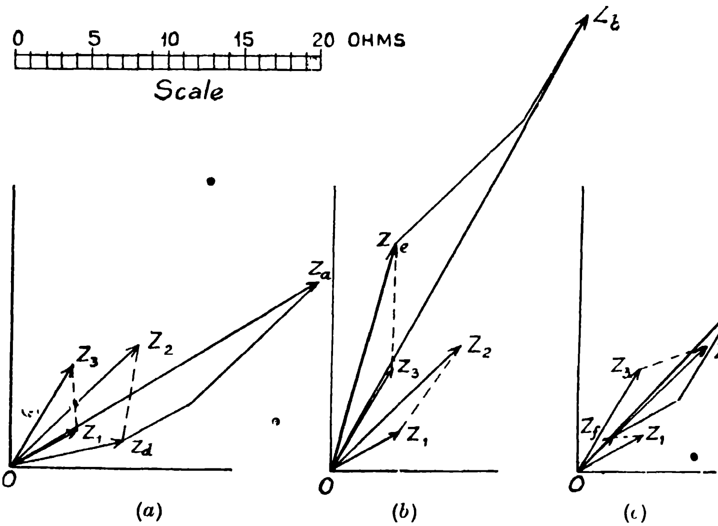


FIG. 157.—Graphic Construction for Equivalent Impedances

$\dot{Z}_a$, $\dot{Z}_b$. $\dot{Z}_c$, to be complex quantities, their calculation involves the application of the symbolic method. If desired, however, the quantities may be readily obtained by a simple graphical construction, which involves the simple geometric processes of addition and proportion.

Thus, let the complex quantities Z_1Z_2/Z_3, Z_2Z_3/Z_1, Z_3Z_1/Z_2, be denoted by Z_d, Z_e, Z_f, respectively. Then we have

$$\frac{Z_1}{Z_3} = \frac{Z_d}{Z_2} \qquad \frac{Z_2}{Z_1} = \frac{Z_e}{Z_3} \qquad \frac{Z_3}{Z_2} = \frac{Z_f}{Z_1}$$

Hence to determine, say, Z_d, vectors OZ_1, OZ_2, OZ_3, representing the star-connected impedances Z_1, Z_2, Z_3, are drawn from a common point, O (Fig. 157*a*), and a triangle OZ_2Z_d, similar to triangle OZ_3Z_1, is constructed upon OZ_2. Then OZ_d represents the quantity $Z_d = Z_1Z_2/Z_3$, since $OZ_d/OZ_2 = OZ_1/OZ_3$.

Therefore Z_a is given by the geometric sum of OZ_1, OZ_2, OZ_d, i.e. by OZ_a (Fig. 157*a*).

The quantities Z_b, Z_c, are determined in a similar manner, as shown in Fig. 157*b*, *c*.

Example. Determine the line currents in an unbalanced star-connected load when supplied from a symmetrical three-phase system at a line pressure of 100 V., the impedances of the branches of the load being $Z_1 = 5\underline{/30°}$, $Z_2 = 12\underline{/45°}$, and $Z_3 = 8\underline{/60°}$ ohms.

The graphic solution for the equivalent impedances of the delta-connected load is given in Fig. 157. The analytical solution is as follows—

The compound terms, $Z_1 Z_2/Z_3$, etc., in the equations (76) for the equivalent impedances Z_a, Z_b, Z_c, are first evaluated, using the polar form of symbolic notation, and the vector addition is then carried out, using the rectangular form of notation. Thus

$$\frac{Z_1Z_2}{Z_3} = \frac{5 \times 12}{8} j^{(30+45-60)/90} = 7{\cdot}5\, j^{15/90}$$

$$\frac{Z_2Z_3}{Z_1} = \frac{12 \times 8}{5} j^{(45+60-30)/90} = 19{\cdot}2\, j^{75/90}$$

$$\frac{Z_3Z_1}{Z_2} = \frac{8 \times 5}{12} (j^{60+30-45)/90} = 3{\cdot}33\, j^{45/90}$$

Hence

$$\begin{aligned} Z_a &= (5\cos 30° + j5\sin 30°) + (12\cos 45° + j12\sin 45°) \\ &\qquad + (7{\cdot}5\cos 15° + j7{\cdot}5\sin 15°) \\ &= (4{\cdot}33 + 8{\cdot}48 + 7{\cdot}24) + j(2{\cdot}5 + 8{\cdot}48 + 1{\cdot}94) \\ &= \boldsymbol{20{\cdot}05 + j12{\cdot}92} \end{aligned}$$

$$\begin{aligned} Z_b &= (12\cos 45° + j12\sin 45°) + (8\cos 60° + j8\sin 60°) \\ &\qquad + (19{\cdot}2\cos 75° + j19{\cdot}2\sin 75°) \\ &= (8{\cdot}48 + 4 + 4{\cdot}97) + j(8{\cdot}48 + 6{\cdot}93 + 18{\cdot}55) \\ &= \boldsymbol{17{\cdot}45 + j33{\cdot}96} \end{aligned}$$

$$\begin{aligned} Z_c &= (8\cos 60° + j8\sin 60°) + (5\cos 30° + j5\sin 30°) \\ &\qquad + (3{\cdot}33\cos 45° + j3{\cdot}33\sin 45°) \\ &= (4 + 4{\cdot}33 + 2{\cdot}36) + j(6{\cdot}93 + 2{\cdot}5 + 2{\cdot}36) \\ &= \boldsymbol{10{\cdot}69 + j11{\cdot}79} \end{aligned}$$

Whence

$$
\begin{aligned}
Z_a &= \sqrt{(20{\cdot}05^2 + 12{\cdot}92^2)} &&= 23{\cdot}83 \text{ O.}\\
\varphi_a &= \tan^{-1} 12{\cdot}92/20{\cdot}05 &&= 32{\cdot}8^\circ\\
Z_b &= \sqrt{(17{\cdot}45^2 + 33{\cdot}96^2)} &&= 38{\cdot}1 \text{ O.}\\
\varphi_b &= \tan^{-1} 33{\cdot}96/17{\cdot}45 &&= 62{\cdot}8^\circ\\
Z_c &= \sqrt{(10{\cdot}69^2 + 11{\cdot}79^2)} &&= 15{\cdot}9 \text{ O.}\\
\varphi_c &= \tan^{-1} 11{\cdot}79/10{\cdot}69 &&= 47{\cdot}8^\circ
\end{aligned}
$$

Assuming the phase rotation to be clockwise, and denoting the currents in the branches of the equivalent delta-connected load by I_a, I_b, I_c, and the corresponding line currents by I_1, I_2, I_3, we have

$$
\begin{aligned}
I_a &= \frac{V_{1-2}}{Z_a} = \frac{100}{23{\cdot}83}\,J^{-32.8/90} = 4{\cdot}2\,J^{32.8/90}\\
I_b &= \frac{V_{2-3}}{Z_b} = \frac{100}{38{\cdot}1}\,J^{-(120+62.8)/90} = 2{\cdot}62\,J^{-182.8/90}\\
I_c &= \frac{V_{3-1}}{Z_c} = \frac{100}{15{\cdot}9}\,J^{-(210+47.8)/90} = 6{\cdot}29\,J^{-287.8/90}
\end{aligned}
$$

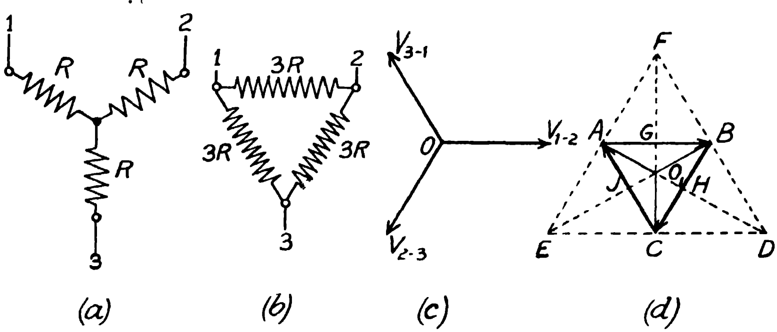


FIG. 158.—Method of Determining Potential of Neutral Point of a Balanced Star-connected Non-inductive Load

Therefore,

$$
\begin{aligned}
I_1 &= \sqrt{[I_a^2 + I_c^2 + 2I_aI_c\cos(60 + \varphi_c - \varphi_a)^\circ]}\\
&= \sqrt{[4{\cdot}2^2 + 6{\cdot}29^2 + 2 \times 4{\cdot}2 \times 6{\cdot}29\cos(60 + 47{\cdot}8 - 32{\cdot}8)^\circ]} = 8{\cdot}42\text{A.}\\
I_2 &= \sqrt{[I_b^2 + I_a^2 + 2I_bI_a\cos(60 + \varphi_a - \varphi_b)^\circ]}\\
&= \sqrt{[2{\cdot}62^2 + 4{\cdot}2^2 + 2 \times 2{\cdot}62 \times 4{\cdot}2\cos(60 + 32{\cdot}8 - 62{\cdot}8)^\circ]} = 6{\cdot}6\text{A.}\\
I_3 &= \sqrt{[I_c^2 + I_b^2 + 2I_cI_b\cos(60 + \varphi_b - \varphi_c)^\circ]}\\
&= \sqrt{[6{\cdot}29^2 + 2{\cdot}62^2 + 2 \times 6{\cdot}29 \times 2{\cdot}62\cos(60 + 62{\cdot}8 - 47{\cdot}8)^\circ]} = 7{\cdot}42\text{A.}
\end{aligned}
$$

Determination of the potential of the neutral point and the phase voltages for a balanced star-connected load supplied from a three-phase system. In this case the potential of the neutral point and the voltage across each branch of the load is easily determined graphically by drawing the line-voltage vectors in the form of a triangle and determining the centre of gravity of this triangular area. The point so obtained represents the potential of the neutral point, and vectors drawn from it to the corners of the triangle

represent the voltages across the branches of the load. The construction is shown in Fig. 158, in which the triangle ABC is drawn, having its sides AB, BC, CA, equal and parallel to the line-voltage vectors OV_{1-2}, OV_{2-3}, OV_{3-1} (Fig. 158*c*) respectively. The mid-points, G, H, J, of these sides are joined to the opposite corners. The common point, O_1, of intersection of these lines is the centre of gravity of the triangular area ABC, and represents the potential of the neutral point of the load. The voltages across the branches of the load are represented by the vectors O_1A, O_1B, O_1C (Fig. 158*d*).

Proof. Consider for simplicity a *non-inductive*, star-connected, balanced load (Fig. 158*a*). Let this be replaced by an equivalent delta-connected load (Fig. 158*b*). Then the currents in the branches of the latter will be proportional to, and in phase with, the line voltages: they may, therefore, be represented by the vectors OV_{1-2}, OV_{2-3}, OV_{3-1} (Fig. 158*c*), and by the triangle ABC (Fig. 158*d*). The current scale and the magnitudes of the currents may be easily calculated. Thus, if R is the resistance of each branch of the star-connected load, the resistance of each branch of the equivalent delta-connected load is equal to $3R$, and the currents in the branches of this load are given by

$$\dot{I}_{\mathrm{I}} = \dot{V}_{1\,2}/3R, \qquad \dot{I}_{\mathrm{II}} = \dot{V}_{2\,3}/3R, \qquad \dot{I}_{\mathrm{III}} = \dot{V}_{3\,1}/3R.$$

Hence the current scale of the vector diagram is equal to $1/3R$ times the voltage scale.

Now the line currents are equal to the vector differences of the currents in adjacent branches of the delta-connected load: they are, therefore, represented in the vector diagram by the diagonals AD, BE, CF, of the parallelograms, $ABDC$, $BCEA$, $CAFB$, respectively, described on the sides of the triangle ABC.

But the voltages across the branches of the star-connected load are proportional to, and in phase with, the line currents, and are, therefore, given by

$$\dot{E}_1 = R\left\{\frac{1}{3R}(\dot{V}_{1-2} - \dot{V}_{3-1})\right\} = \frac{1}{3}(\dot{V}_{1-2} - \dot{V}_{3-1})$$

$$\dot{E}_2 = R\left\{\frac{1}{3R}(\dot{V}_{2-3} - \dot{V}_{1-2})\right\} = \frac{1}{3}(\dot{V}_{2-3} - \dot{V}_{1-2})$$

$$\dot{E}_3 = R\left\{\frac{1}{3R}(\dot{V}_{3-1} - \dot{V}_{2-3})\right\} = \frac{1}{3}(\dot{V}_{3-1} - \dot{V}_{2-3})$$

Hence AD, BE, CF, represent the voltages across the branches of the star-connected load to a scale three times the original voltage scale of the diagram.

Since the diagonals AD, BE, CF, bisect the sides BC, CA, AB, of the triangle ABC, the former intersect one another at a common point, O_1, and the distances O_1G, O_1H, O_1J, are equal to one-third of the distances GC, HA, JB, respectively. Therefore, O_1A, O_1B, O_1C, represent the voltages across the branches of the star-connected load to the original voltage scale, and the point O_1 represents the potential of the neutral point of the load.

The construction for the proof in the case of a balanced *inductive* load differs slightly from that for the case when the load is non-inductive, as the currents in the branches of the load are not in phase with the voltages across those branches. Thus, in Fig. 159*d*, the triangle ABC represents the vector triangle for the line voltages. From the corners of this triangle are drawn the vectors BG, CL, AQ, representing the currents in the branches of the

equivalent delta-connected load. If these vectors are made equal to the vectors *AB*, *BC*, *CA*, respectively, the current scale will be equal to $1/3Z$ times the voltage scale of the diagram, where Z denotes the impedance of each branch of the star-connected load.

The line-current vectors, *BJ*, *CN*, *AS*, are then determined by constructing the parallelograms *BGJH*, *CLNM*, *AQST*; and each parallelogram is rotated so as to bring the current vectors *BG*, *CL*, *AQ*, in line with the corresponding voltage vectors, *AB*, *BC*, *CA*. Then the new positions of the line-current vectors will represent the voltages across the branches of the star-connected load to a scale three times the original voltage scale of the diagram. Hence,

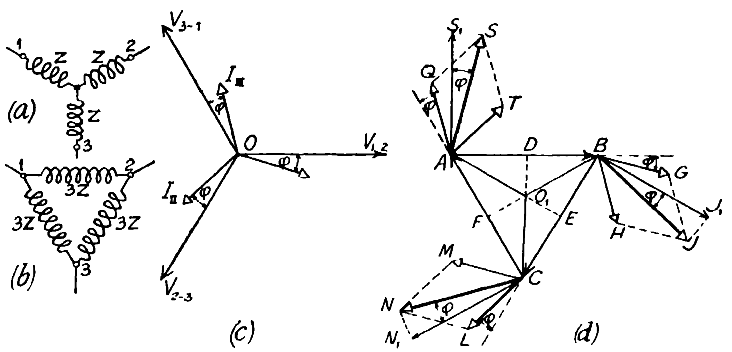


FIG. 159.—Method of Determining Potential of Neutral Point of a Balanced Star-connected Inductive Load: (*a*, *b*) Equivalent Circuit Diagrams, (*c*) Vector Diagram for (*b*), (*d*) Construction for Obtaining Potential of Neutral Point

if from the points *A B C* the lines *AE*, *BF*, *CD*, are drawn parallel to BJ_1, CN_1, AS_1, respectively, the former will intersect at a common point, O_1, which represents the potential of the neutral point of the load. Moreover, since the points *D*, *E*, *F*, are the mid-points of the sides *AB*, *BC*, *CA*, respectively, the point O_1 is the centre of gravity of the triangular area *ABC*.

Hence, in all cases of balanced star-connected loads, the potential of the neutral point of the load may be obtained by determining the centre of gravity of the triangular area formed by the vector triangle of the line voltages.

Potential difference between neutral points of generator and balanced star-connected load. When the supply system is symmetrical the neutral point of a balanced star-connected load is at the same potential as that of the generator, since the line-voltage vectors then form either an equilateral triangle (when the number of phases is equal to three) or a regular polygon (when the number of phases is greater than three), and the potential of the neutral point of the system is represented by the centre of gravity of this triangle or polygon, which also represents the potential of the neutral point of the load. Hence the voltages across each branch of the load are equal to, and in phase with, the voltage across the

corresponding phase of the generator, assuming the voltage drop in the connecting wires to be negligible.

But when the system is unsymmetrical, the potential of the neutral point of the system is not generally represented by the centre of gravity of the line-voltage vector triangle or polygon. In this case the voltages across the branches of the load are neither equal to, nor in phase with, the corresponding phase voltages of the generator. For example, consider an unsymmetrical three-phase system in which the phase voltages are represented by the vectors OE_{I}, OE_{II}, OE_{III}, Fig. 160. The line voltages are then represented by the sides of the triangle, E_{I}, E_{II}, E_{III}, formed by joining the extremities of these vectors, and the potential of the neutral point is represented by O. The centre of gravity of this triangle is at O_1, which therefore represents the potential of the neutral point of a balanced star-connected load supplied from the system. The voltages across the branches of the load are represented by vectors drawn from O_1 to the points E_{I}, E_{II}, E_{III}.

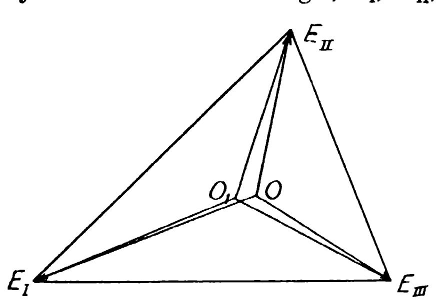


FIG. 160.—Showing Difference of Potential between Neutral Points of an Unsymmetrical System and of a Balanced Star-connected Load

The potential difference between the two neutral points is represented, to the voltage scale of the diagram, by the distance OO_1.

Hence, when a non-inductive star-connected balanced potential circuit is employed in connection with the three- and four-wattmeter methods of measuring power in three- and four-phase unbalanced systems (p. 201), the voltages across the branches of the potential circuit are not necessarily equal to, nor in phase with, the phase voltages of the system. Under these conditions, the readings of the separate wattmeters do not represent the power in the phases of the system, although the sum of the readings is equal to the total power.

For example, in a three-phase unsymmetrical system the voltages across the branches of the potential circuits may be represented by O_1E_{I}, O_1E_{II}, O_1E_{III}, Fig. 160, and the phase voltages of the system by OE_{I}, OE_{II}, OE_{III}. Now O_1E_{I} is equal to the vector difference of OE_{I} and OO_1; O_1E_{II} is equal to the vector difference of OE_{II} and OO_1; O_1E_{III} is equal to the vector difference of OE_{III}

and OO_1. Taking instantaneous values, and denoting the potential difference, OO_1, between the neutral points by v_o, the line currents by i_1, i_2, i_3, and the phase voltages of the system by e_{I} e_{II}, e_{III}, the power measured by the separate wattmeters is given by

$$p_1 = i_1(e_{\mathrm{I}} - v_o) = i_1 e_{\mathrm{I}} - i_1 v_o$$

$$p_2 = i_2(e_{\mathrm{II}} - v_o) = i_2 e_{\mathrm{II}} - i_2 v_o$$

$$p_3 = i_3(e_{\mathrm{III}} - v_o) = i_3 e_{\mathrm{III}} - i_3 v_o$$

Whence $$p_1 + p_2 + p_3 = i_1 e_{\mathrm{I}} + i_2 e_{\mathrm{II}} + i_3 e_{\mathrm{III}} - v_o(i_1 + i_2 + i_3)$$

$$= i_1 e_{\mathrm{I}} + i_2 e_{\mathrm{II}} + i_3 e_{\mathrm{III}}$$

since $i_1 + i_2 + i_3 = 0$.

Variation of the neutral point potential of a star-connected, non-inductive circuit when the resistance of one branch is varied. Consider a star-connected, non-inductive circuit, of which one branch is variable, as in Fig 161, to be supplied from a symmetrical three-phase system at constant voltage. The variation of the resistance of one branch will then cause a variation in magnitude and phase of the currents in all the branches, as well as a variation of the potential of the neutral point. If the neutral point of the system is assumed to be at zero potential, the potential of the neutral point of the load may, according to the relative values of the resistances of the variable and fixed branches, have values between zero and the phase voltage of the system. For example, when the resistance of the variable branch is equal to that of each of the fixed branches, the potential of the neutral point is zero, and when the resistance of the variable branch is zero, the potential of the neutral point is equal to the phase voltage of the system. Again, when the resistance of the variable branch is infinite, the potential of the neutral point is now reversed and is equal to one-half of the phase voltage of the system.

The variation of the potential of the neutral point and the phase difference between the currents in the branches of fixed resistance for various ratios of variable resistance/fixed resistance is shown in the curves of Fig. 162.

Since the supply system is symmetrical, the currents in the branches of fixed resistance, and the voltages across these branches, must be always equal to each other, and the vector difference of these voltages must be equal to the (constant) voltage between the line wires to which the fixed branches are connected. Hence, if the line-voltage vectors are drawn in the form of an equilateral

triangle, ABC (Fig. 161b), and the side BC represents the voltage between the line wires to which the fixed branches are connected, the potential of the neutral point of the load is represented by a point in the line AD, where D is the mid-point of the side BC.

When the resistance of the variable branch is equal to that of each of the fixed branches, the potential of the neutral point of the load is represented by the point O (the distance OD being one-third of AD), and is zero, assuming the neutral point of the system to be at zero potential. When the resistance of the variable branch is decreased, the potential of the neutral point of the load increases, and is represented by a point, such as O_1 in OA. In the

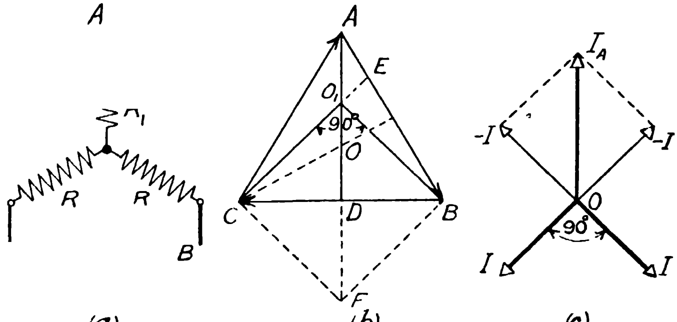


FIG. 161.—Circuit and Vector Diagrams for Star-connected Non-inductive Load with One Branch Variable

extreme case, when the resistance of this branch is zero, the potential of the neutral point of the load is represented by A, and the difference of potential between the two neutral points is equal to the phase voltage of the system, OA. In the other extreme case, when the resistance of the variable branch is infinite, the potential of the load neutral point is represented by D, and since OD is equal to $\frac{1}{2}OA$, the potential difference between the two neutral points is now equal to one half of the phase voltage of the system, and is reversed in direction.

The currents in the branches of fixed resistance, and the voltages across these branches, are represented, to different scales, by O_1C and O_1B, and the phase difference between the currents, or voltages, is represented by the angle BO_1C.

The current in the branch of variable resistance is given by the reversed vector sum of the currents in the branches of fixed resistance, and is represented by OI_A (Fig. 161c). This current is in

phase with the voltage across the variable branch, which is represented by O_1A (Fig. 161*b*). Thus, from the vector diagram, or the curves of Fig. 162, the value of the resistance of the variable branch, expressed in terms of the resistance of the fixed branches, necessary to obtain a given phase difference between the currents in the fixed branches may readily be obtained.

For example, suppose the currents in the fixed branches are to have a phase difference of 90°, the point O_1 (Fig. 161*b*) is determined such that the angle BO_1C is 90°. The value of the resistance of the variable branch may then be obtained either by calculation or

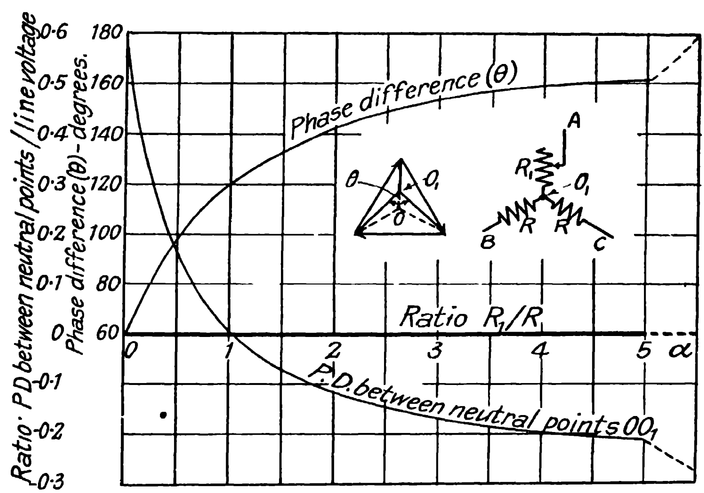


FIG. 162.—Curves relating to Fig. 161

from measurements on the vector diagram. Thus, if V denotes the line voltage, the voltages across the branches of the load are, from the geometry of Fig. 161*b*, equal to $V.\sqrt{2}/2$; $V.\sqrt{2}/2$; $V(\sqrt{3}-1)/2$. Hence if R denotes the resistance of each of the fixed branches, the current, I, in these branches is equal to $V.\sqrt{2}/2R$, and the current, I_A, in the variable branch is numerically equal to the vector sum of the currents in the fixed branches, i.e. $I_A = \sqrt{2}\,.\,I = V/R$. Hence the resistance of the variable branch is given by $R_1 = [V(\sqrt{3}-1)/2]/(V/R) = R(\sqrt{3}-1)/2 = 0{\cdot}366R$.

To obtain the value of R_1 from measurements on the vector diagram, produce CO_1 (Fig. 161*b*) to cut the side AB at E, and measure AE and BE. Then $AE/BE = R_1/R$.

Conversely, if the side AB be divided at E such that $AE/EB = R_1/R$ and the point E is joined to the opposite corner C, then the point,

O_1, of intersection of CE and AD gives the potential of the neutral point of the load.

Proof. From B and C (Fig. 161*b*) draw BF and CF parallel to O_1C and O_1B respectively, and produce AD to the point of intersection, F, of BF and CF. Then triangles AO_1E, AFB, are similar, and therefore

$$AO_1/AF = AE/AB.$$

Whence $$AO_1/O_1F = AE/EB.$$

Now OA_1 represents, to the voltage scale of the diagram, the voltage across the variable branch of the load, and O_1F represents, to the current scale, the current in this branch, since, by construction, O_1F is the vector sum of O_1B and O_1C, and the latter represent the currents in the fixed branches of the

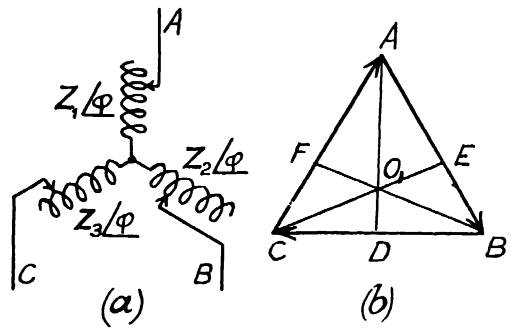


FIG. 163.—Circuit and Vector Diagrams for Star-connected Variable Load of Constant Power Factor

load. Hence, if the diagram is drawn for a voltage scale of 1 cm. = q volts, the current scale will be 1 cm. = q/R amp.

Whence $$R_1 = \frac{q \,.\, AO_1}{(q/R)O_1F} = R\,\frac{AO_1}{O_1F} = R\,\frac{AE}{EB}$$

Now, when $\angle BO_1C = 90°$, $AO_1 = AB(\sqrt{3}-1)/2$,

$$O_1F = O_1B \,.\, \sqrt{2} = \sqrt{2}(\sqrt{2} \,.\, AB/2) = AB.$$

Therefore $$\frac{AO_1}{O_1F} = \frac{AB[(\sqrt{3}-1)/2]}{AB} = \frac{\sqrt{3}-1}{2} = 0{\cdot}366,$$

i.e. $R_1 = 0{\cdot}366\,R$,

and generally $R_1/R = AE/EB$.

This graphical method of determining the potential of the neutral point may be extended to unsymmetrical systems, and to the more general cases, when all branches of the load are variable and the loads are inductive, *provided that the power factor of each branch of the load has the same value*

In the general case, where the branches of the load have the impedances Z_1, Z_2, Z_3, and the line voltages of the system are represented by the vector triangle ABC (Fig. 163*b*)—in which the side AB represents the voltage between the lines to which the

impedances Z_1, Z_2, are connected; the side BC represents the voltage between the lines to which the impedances Z_2, Z_3, are connected, and so on—the side AB is divided at E such that $AE/EB = Z_1/Z_2$; BC is divided at D such that $BD/DC = Z_2/Z_3$, CA is divided at F such that $CF/FA = Z_3/Z_1$. Then the lines joining the points A, D; B, F; C, E will intersect at a common point, O_1, which represents the potential of the neutral point of the load.

"Floating" neutral point. When the neutral point of the load is isolated from the neutral point of the generator, the potential of the former is subject to variations according to the unbalance of the load, and under certain conditions of loading a considerable difference of potential may exist between the two neutral points. Such an isolated neutral point is called a "floating" neutral point.

All star-connected loads supplied from polyphase systems without neutral wires have floating neutral points, and any unbalancing of the load causes variations not only of the potential of the neutral point but also of the voltages across the several branches of the load. Hence, when single-phase electric lighting loads are to be supplied from three-phase three-wire systems, the former must be delta-connected in order that the voltages across the branches of the load may not be appreciably affected by a slight unbalancing of the loads. If a star connection of the load is desired, then the four-wire system must be employed, and the neutral points of load and generator must be connected together.

Determination of the potential of the neutral point and the phase voltages for an unbalanced, star-connected, inductive load supplied from a three-phase system. The solution will be obtained for the general case where the power-factors of the branches of the load may all be unequal. The line voltages of the system supplying the load are assumed to be known, as will generally be the case in practice. The simplest solution is a graphical one, the construction being shown in Fig. 164. Before the construction is commenced, however, the impedances $Z_a\underline{/\varphi_a}$, $Z_b\underline{/\varphi_b}$, $Z_c\underline{/\varphi_c}$, of the equivalent delta-connected load must be determined, either by calculation or graphically. Having obtained these quantities, the vector triangle ABC (Fig. 164) is drawn to represent the line voltages of the supply system. From the corners B, C, A of this triangle the vectors BD, CG, AK, are drawn to represent the currents in the branches of the equivalent delta-connected load; BD representing the current in the branch supplied at the voltage represented by AB; CG representing the current in the branch supplied at the voltage represented by BC; and so on. The line currents

are then determined by constructing the parallelograms $BDEF$, $CGHJ$, $AKLM$, and are represented by the vectors BF, CJ, AM. If each of these vectors is multiplied by the impedance of the branch of the star-connected load through which the current (which is represented by the vector under consideration) passes, the resulting vectors will represent the voltages across the branches of the original star-connected load. This multiplication is carried out by rotating the vectors through the appropriate phase angles, φ_1, φ_2, φ_3, of the branches of the star-connected load, and at the same time changing the scale. The calculation of the new scale for the voltage may be avoided by employing the direct construction shown in Fig. 164, by means of which the load-voltage vectors are determined to the same scale as the line-voltage vectors. Thus the line-current vectors are rotated through the appropriate angles, φ_1, φ_2, φ_3, to the positions BF_1, CJ_1, AM_1, and parallels AO_1, BO_1, CO_1, are drawn from the corners A, B, C of the line-voltage vector triangle. These lines meet at a common point, O_1, which represents the potential of the neutral point of the load. Hence the vectors, AO_1, BO_1, CO_1, represent the voltages across the branches of the load to the same scale as the vectors AB, BC, CA, represent the line voltages.

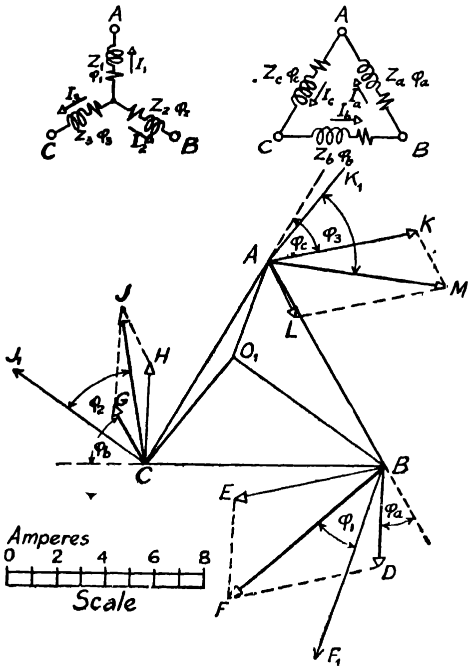


FIG. 164.—Graphic Solution for Determining Potential of Neutral Point of Unbalanced Star-connected Load (any Power Factor)

Proof. Let the impedances of the star-connected load be denoted by $Z_1\underline{/\varphi_1}$, $Z_2\underline{/\varphi_2}$, $Z_3\underline{/\varphi_3}$, and those of the equivalent delta-connected load by $Z_a\underline{/\varphi_a}$, $Z_b\underline{/\varphi_b}$, $Z_c\underline{/\varphi_c}$ Then if the line voltages are denoted by V_{1-2}, V_{2-3},

V_{3-1}, the currents in the branches of the delta-connected load are given by

$$I_a = V_{1-2}/Z_a, \quad I_b = V_{2-3}/Z_b, \quad I_c = V_{3-1}/Z_c$$

Hence the line currents are given by

$$I_1 = I_a - I_c, \quad I_2 = I_b - I_a, \quad I_3 = I_c - I_b$$

Therefore the voltages across the branches of the load are given by

$$V_{1-0} = I_1Z_1 = Z_1(V_{1-2}/Z_a - V_{3-1}/Z_c)$$
$$V_{2-0} = I_2Z_2 = Z_2(V_{2-3}/Z_b - V_{1-2}/Z_a)$$
$$V_{3-0} = I_3Z_3 = Z_3(V_{3-1}/Z_c - V_{2-3}/Z_b)$$

Whence

$$\begin{aligned}
V_{1-0} - V_{2-0} &= V_{1-2}\left(\frac{Z_1 + Z_2}{Z_a}\right) - V_{2-3}\frac{Z_2}{Z_b} - V_{3-1}\frac{Z_1}{Z_c} \\
&= V_{1-2}\left(\frac{Z_3Z_1 + Z_2Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) - V_{2-3}\left(\frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) \\
&\qquad - V_{3-1}\left(\frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) \\
&= V_{1-2}\left(\frac{Z_3Z_1 + Z_2Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) - (V_{2-3} + V_{3-1}) \\
&\qquad \left(\frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) \\
&= V_{1-2}\left(\frac{Z_3Z_1 + Z_2Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) + V_{1-2}\left(\frac{Z_1Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}\right) \\
&= V_{1-2}
\end{aligned}$$

Similarly $V_{2-0} - V_{3-0} = V_{2-3}$,

and $V_{3-0} - V_{1-0} = V_{3-1}$.

Thus the voltages across the branches of an unbalanced star-connected load supplied from a three-phase system may be represented by vectors drawn from a particular point, inside the vector triangle for the line voltages, to the corners of this triangle, and both quantities are represented to the same scale. Hence, in the construction of Fig. 164, since the lines AO_1, BO_1, CO_1, were drawn parallel to the vectors representing the quantities I_1Z_1, I_2Z_2, I_3Z_3, their common point of intersection represents the potential of the neutral point of the load, and their lengths represent the magnitudes of the voltages across the branches of the load.

Determination of the generator line voltage for three-phase, three-wire, systems, the voltage at the load being known. In many cases of the supply of electrical energy for power purposes the position of the "load," or place where the energy is utilized, is at a considerable distance from the generator, and therefore the impedance of the line wires will affect the voltages at the generator and load. As the load must usually be supplied at a definite voltage we must show how the voltage at the generator may be determined. We shall assume the load to be concentrated at a single point and to be supplied from the generator through a single-circuit transmission line, as these conditions are representative of the practical case of a sub-station, or a distributing station, being supplied from a central generating station.

Case I. Star-connected load. The line currents and the potential

of the neutral point of the load are first determined. The pressure drop in each line wire is then calculated and is added vectorially to the voltage across the corresponding branch of the load. The quantities so obtained represent the voltages between each terminal of the generator and the neutral point of the load. The voltages between the terminals of the generator are therefore determined.

The vector diagram is shown in Fig. 165, in which ABC represents the vector triangle for the known line voltages at the load; O_1 represents the potential of the neutral point of the load; and O_1A, O_1B, O_1C, represent the voltages across the branches of the load.

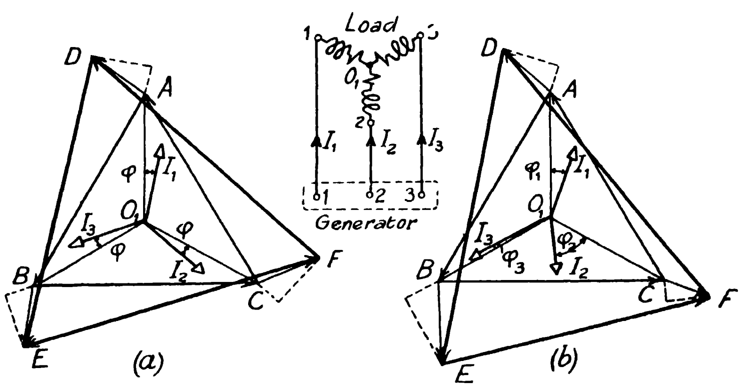


FIG. 165.—Graphic Method of Determining Generator Voltage for Star-connected System: (a) Balanced Load, (b) Unbalanced Load

The line currents are represented by the vectors O_1I_1, O_1I_2, O_1I_3. The pressure drops in the line wires are represented by the vectors AD, BE, CF, these vectors being drawn in the positions shown for convenience of carrying out the vector addition. [*Note.* AD represents the pressure drop in line 1, CF the pressure drop in line 2, and BE the pressure drop in line 3.] By adding, vectorially, the pressure drop in any line wire to the veltage across the corresponding branch of the load we obtain the voltage between the neutral point of the load and the terminal of the generator. These voltages are represented by the vectors O_1D, O_1E, O_1F. Therefore the triangle DEF is the vector triangle for the line voltages at the terminals of the generator.

Observe that when the loads are balanced and the line wires have equal impedances, the vector triangle for the generator voltages is of similar shape to that for the load voltages (Fig. 165*a*), but that when the loads are unbalanced, one of the voltage triangles is distorted relatively to the other (Fig. 165*b*).

Case II. Delta-connected load. The vector diagram for this case is shown in Fig. 166. The known (line) voltages across the branches of the load are represented by the sides AB, BC, CA, of the vector triangle ABC, and the currents in the line wires are represented by the vectors AD, BE, CF, these vectors being obtained from the vectors AI_{I}, BI_{II}, CI_{III}, representing the currents in the branches of the load. Observe that the vectors AD, BE, CF are so drawn in relation to the vectors for the load voltages that if BA represents

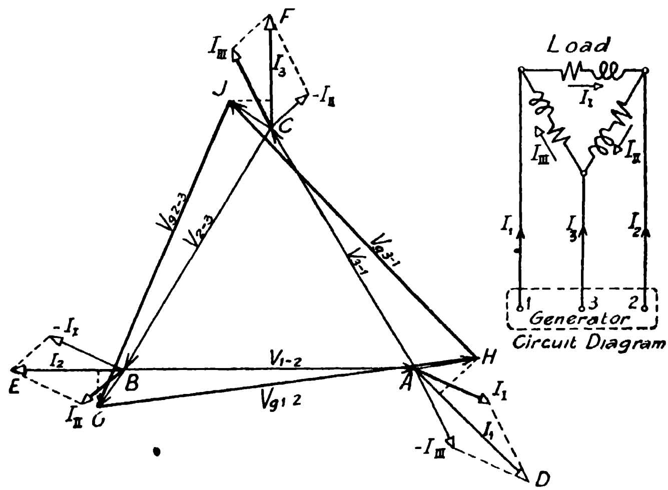


FIG. 166.— Graphic Method of Determining Generator Voltage for Delta-connected Load

the voltage between lines 1 and 2, AD will represent the current in line 1; if AC represents the voltage between lines 3 and 1, CF will represent the current in line 3; and so on.

The vectors BG, AH, CJ, representing the pressure drop in each of the line wires, are now added at the appropriate corners of the voltage triangle ABC (BG representing the pressure drop in line 2, CJ the pressure drop in line 3, and AH the pressure drop in line 1), and by joining the free ends, G, H, J, we obtain the vector triangle, GHJ, for the generator voltages.

The above observations regarding the nature of the load and the shapes of the vector triangles for the generator and load voltages apply equally well to the present case.

Determination of load currents and voltages for three-phase, three-wire, systems, the generator E.M.F. being known. In the

preceding section we determined the voltage necessary at the terminals of the generator in order to give a definite voltage at the load. When the load voltages are symmetrical and the load is balanced, the voltages at the generator will also be symmetrical. But if the load voltages are to be symmetrical and equal when the load is unbalanced, the voltages between the terminals of the generator must be unsymmetrical and unequal, due to the pressure drop in the line wires. Under these conditions there may be difficulty in obtaining the required voltages. It will, therefore, be of interest to consider the converse of the above case, viz. the determination of the voltages at the load when the internal E.M.Fs. of the generator are known.

When a three-phase generator is unsymmetrically loaded, the voltages at its terminals will, in general, be unsymmetrical, although the E.M.Fs. generated in the phases may be symmetrical. Since the generated E.M.F. may be considered as equivalent to the no-load E.M.F., we may determine the terminal voltages when the generator is loaded by compounding the no-load E.M.Fs. of the several phases with the pressure drop in these phases due to impedance and armature reaction. The currents in the load may therefore be determined from the no-load generator E.M.Fs. and the impedances of the load, line wires and generator.

Case I. Generator and load star connected. The currents may be determined by two methods, one involving the calculation of the equivalent delta-connected circuit, and the other involving the determination of the difference of potential between the neutral points of the generator and the load. In the first method the total impedance of each phase of the star-connected circuit (i.e. the sum of the impedances of each phase of the generator, load, and connecting wire) is calculated, and the impedance of the equivalent delta-connected circuit is determined. This delta-circuit is then assumed to be supplied at the no-load generator line voltage, and the branch circuit and line currents are calculated. The line currents will be the same as those in the original star-connected circuit.

If the potential difference between the neutral points of the generator and load is known, the line currents may be obtained by dividing the total impedance of each phase into the difference between the no-load E.M.F. of the appropriate phase of the generator and the potential difference between the two neutral points. Thus if the no-load phase E.M.Fs. of the generator are represented in Fig. 167 by the vectors OE_{I}, OE_{II}, OE_{III}, and O_1 represents the potential of the neutral point of the load, the current in phase I of the system is given by O_1E_{I}/(sum of impedances of phase I of

generator, load, and connecting line wire). The currents in phases II and III of the system are obtained by dividing the E.M.Fs. represented by O_1E_{II}, O_1E_{III}, by the appropriate impedances. The currents are represented by the vectors OI_1, OI_2, OI_3.

The voltages across the branches of the load are obtained by multiplying the line currents by the appropriate load impedances. These voltages are represented in the vector diagram of Fig. 167 by the vectors OV_1, OV_2, OV_3.

The voltages at the terminals of the generator may be obtained

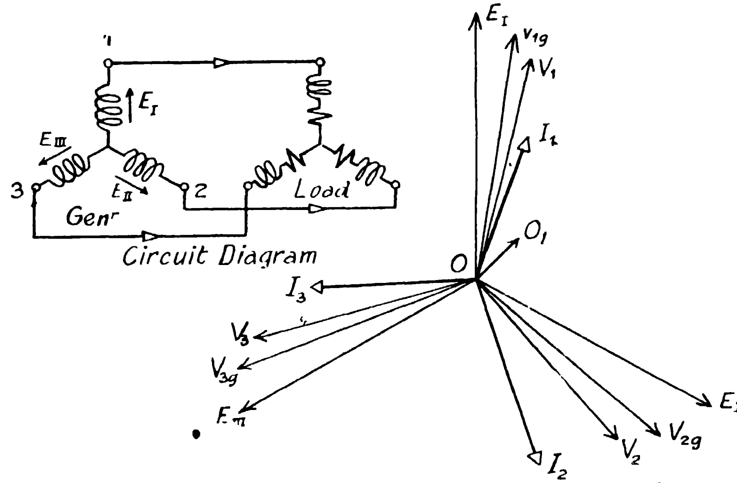


FIG. 167.—Graphic Method of Determining Load Voltage for Star connected System

by compounding the voltages across the branches of the load with the pressure drops in the line wires. The latter are represented by the vectors V_1V_{1g}, V_2V_{2g}, V_3V_{3g}. Thus the phase voltages of the generator when loaded are represented by the vectors OV_{1g}, OV_{2g}, OV_{3g}, and the pressure drops in the phases of the generator are represented by the vectors E_IV_{1g}, $E_{II}V_{2g}$, $E_{III}V_{3g}$.

Case II. Generator and load delta-connected. In this case we replace the delta connections of the generator and load by equivalent star-connected circuits. The solution is then obtained in the same manner as for the preceding case.

Determination of load currents and voltages for two-phase, three-wire systems, i.e. two-phase systems with neutral wire. The no-load phase E.M.Fs. of the generator are assumed to be known; they will be considered to be equal to each other and to have a phase difference of 90°.

Case I. Pressure drop in neutral wire ignored. If the pressure drop in the neutral wire is ignored the solution for the load currents and load voltages is easily obtained, since the neutral point of the load has the same potential as that of the generator, and therefore the E.M.Fs. and impedances are known for each phase of the system. The vector diagram is shown in Fig. 168*a*, in which the no-load E.M.Fs. of the generator are represented by OE_{I}, OE_{II}, and the currents in the load are represented by OI_1, OI_2. These currents are calculated from the no-load phase E.M Fs. and the total phase impedances.

The current in the neutral wire is represented by OI_o, which is the reversed vector sum of OI_1 and OI_2.

The voltages across the branches of the load are represented by OV_1, OV_2, and are obtained by subtracting from the no-load E.M.Fs. the pressure drop in the phases of the generator and the "outer" line wires.

The vector diagram has been drawn for the case of balanced loads and equal impedances in each phase of the generator and the connecting line wire, but zero impedance in the neutral wire. Under these conditions the diagram shows that the voltages across the branches of the load are equal and have a phase difference of 90°. Thus in this particular case the system is symmetrical when loaded.

Case II. Pressure drop in neutral wire considered. The direct analytical solution for this case involves the application of Kirchoff's laws and is given on p. 273.

An alternative analytical solution may be obtained very simply by employing the principle of super-position of electric currents. Thus, since the resultant E.M.F. in each phase, taken from the neutral point of the generator to the neutral point of the load, is equal to the vector difference of the no-load phase E.M.F. of the generator and the potential difference between the two neutral points, the actual current in each circuit may be obtained by super-imposing the fictitious currents due to (1) the no-load generator E.M.F. acting alone, (2) the potential difference between the neutral points acting alone. These conditions are represented in the vector diagram of Fig. 168*b*, in which OE_{I}, OE_{II} represent the no-load E.M.Fs. of the generator, and OO_1 represents the difference of potential between the two neutral points, O representing the potential of the neutral point of the generator and O_1 representing that of the load. The resultant E.M.F. in each phase of the system, taken from the neutral point of the generator to the neutral point of the load *via* the generator, "outer" line wire, and load is

represented by O_1E_{I}, O_1E_{II}. The phase currents are therefore obtained by dividing these E.M.Fs. by the appropriate impedances, and are represented by OI_1, OI_2. These currents are equal to the vector difference of the fictitious currents OI_1', OI_2'; OI_1'', OI_2''; the former being obtained by dividing the phase impedances into the phase E.M.Fs., and the latter being obtained by dividing the phase impedances into the potential difference between the two neutral points.

Thus, let the no-load E.M Fs of the generator be denoted by

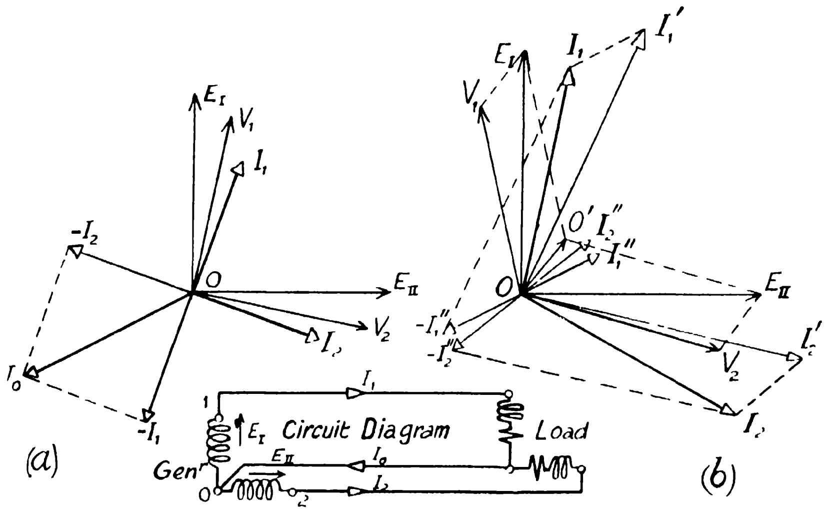


FIG. 168.—Graphic Method of Determining Load Voltage for Two-phase Three-wire System : (a) Pressure Drop in Neutral Wire Ignored, (b) Pressure Drop in Neutral Wire Considered

E_{I}, E_{II}; the impedances of the generator, line wires, and load by Z_{1g}, Z_{2g}; Z_{1l}, Z_{2l}, Z_{0l}; Z_1, Z_2, respectively; the potential difference between the neutral points of the generator and load by V_0; and the actual currents in the load and neutral wire by I_1, I_2, I_0, respectively. Then

$$I_0 = -(I_1 + I_2), \qquad V_0 = -I_0 Z_{0l}$$

$$I_1 = \frac{E_{\text{I}} - V_0}{Z_{1g} + Z_{1l} + Z_1} = \frac{E_{\text{I}} - V_0}{Z_{1t}} = \frac{E_{\text{I}}}{Z_{1t}} - \frac{V_0}{Z_{1t}} \quad . \tag{78}$$

$$= I_1' - I_1'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{78a}$$

$$I_2 = \frac{E_{\text{II}} - V_0}{Z_{2g} + Z_{2l} + Z_2} = \frac{E_{\text{II}} - V_0}{Z_{2t}} = \frac{E_{\text{II}}}{Z_{2t}} - \frac{V_0}{Z_{2t}} \quad . \tag{79}$$

$$= I_1' - I_2'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{79a}$$

where $Z_{1t} = Z_{1g} + Z_{1l} + Z_1$; $Z_{2t} = Z_{2g} + Z_{2l} + Z_2$; I_1', I_2', denote the currents which would be obtained in the phases of the generator and load if the two neutral points were at the same potential, i.e. if the impedance of the neutral wire were zero; and I_1'', I_2'', denote the phase currents which would be obtained if the generator E.M.Fs. were zero and the potential difference V_o existed between the neutral points

The potential difference, V_o, between the neutral points of generator and load under normal conditions may be calculated as follows—

Since $V_o = -I_o Z_{ol}$, and $I_o = -(I_1 + I_2)$, we have, by substitution from equations (78), (79),

$$V_o = Z_{ol}(I_1 + I_2) = Z_{ol}\left(\frac{E_{\text{I}}}{Z_{1t}} - \frac{V_o}{Z_{1t}} + \frac{E_{\text{II}}}{Z_{2t}} - \frac{V_o}{Z_{2t}}\right)$$
$$= Z_{ol}[(I_1' + I_2') - (V_o/Z_{1t} + V_o/Z_{2t})]$$

Whence $$V_o\left(\frac{1}{Z_{ol}} + \frac{1}{Z_{1t}} + \frac{1}{Z_{2t}}\right) = I_1' + I_2' \qquad (80)$$
$$= I_o'/(Y_{ol} + Y_{1t} + Y_{2t}) \qquad (80a)$$

where $I_o'(= I_1' + I_2')$ denotes the reversed neutral current corresponding to zero impedance in the neutral wire, and Y_{ol}, Y_{1t}, Y_{2t}, denote the admittances of the neutral wire and each phase respectively, the phase impedance including the impedance of the generator, connecting line wire, and load, but not the impedance of the neutral wire.

When V_o has been determined, the load currents I_1, I_2, may readily be calculated from equations (78), (79)

Instead of obtaining the load currents in this manner, we may determine them by calculating the fictitious currents I_1', I_1'', I_2', I_2'', and applying equations (78*a*), (79*a*). The currents I_1', I_2', are readily obtained, e.g.

$$I_1' = E_{\text{I}}/Z_{1t}, \qquad I_2' = E_{\text{II}}/Z_{2t},$$

but the currents I_1'', I_2'' must be obtained by combining equations (78), (79), (80). Thus

$$I_o = -(I_1 + I_2) = -[(I_1' - I_1'') + (I_2' - I_2'')]$$
$$= -(I_1' + I_2') + (I_1'' \quad I_2'')$$
$$= -I_o' + (I_1'' + I_2'')$$

or $$I_o' = -I_o + I_1'' + I_2''$$
$$= V_o/Z_{ol} + V_o/Z_{1t} + V_o/Z_{2t}$$
$$= V_o(Y_{ol} + Y_{1t} + Y_{2t})$$

Hence $I_1'' = I_o' \left(\frac{Y_{1t}}{Y_{ol} + Y_{1t} + Y_{2t}} \right)$

and $I_2'' = I_o' \left(\frac{Y_{2t}}{Y_{ol} + Y_{1t} + Y_{2t}} \right)$

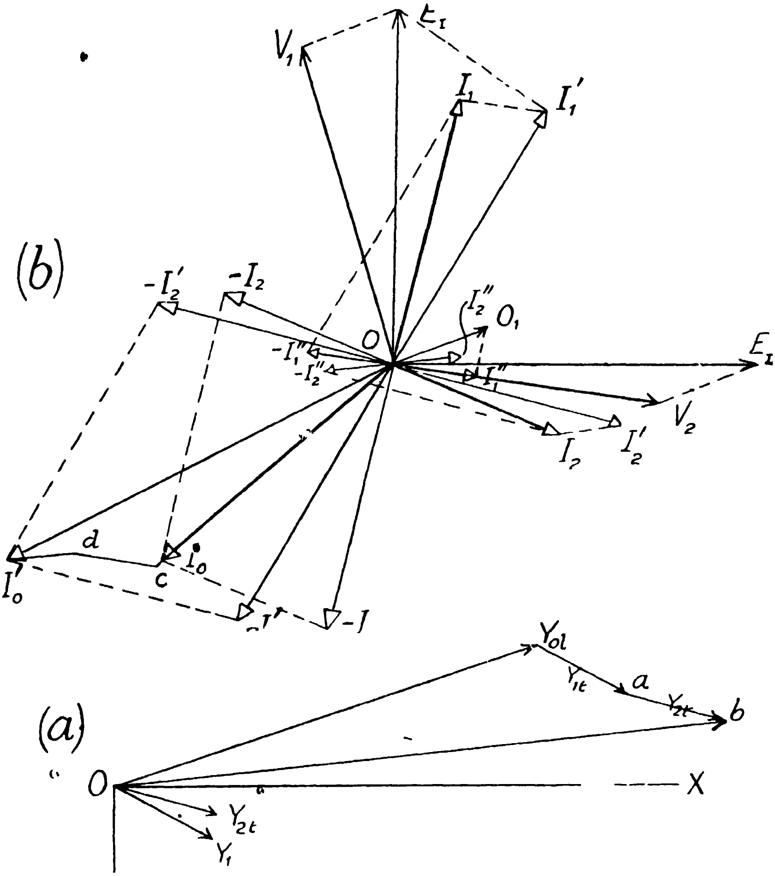


FIG. 169.—Graphic Method of Determining Load Voltage, Potential of Neutral Point of Load, and Fictitious Currents I_1'', I_2'', for a Two-phase Three-wire System

Whence $I_1'' : I_o' : : Y_{1t} : (Y_{ol} + Y_{1t} + Y_{2t})$

and $I_2'' : I_o' : : Y_{2t} : (Y_{ol} + Y_{1t} + Y_{2t})$

The currents I_1'', I_2'', may therefore be easily determined by a simple geometrical construction, which is shown in Fig. 169. In

the diagram of Fig. 169*a* the admittances Y_{1t}, Y_{ol}, Y_{2t}, are added together, thus giving the polygon $OY_{ol}ab$. In the diagram of Fig. 169*b* the currents I_1', I_2', are reversed and added together, thus giving the current vector $O-I_o'$. Upon this vector is constructed a polygon similar to the admittance polygon in Fig. 169*a*. Then if the sides Oc, cd, dI_o' of the polygon in Fig. 169*b* are proportional to the admittance vectors Y_{ol}, Y_{1t}, Y_{2t}, respectively, the sides cd and dI_o' will represent the currents I_1'', I_2'', respectively.

An extension of this construction enables the potential of the neutral point of the load to be determined graphically. Thus, since $I_1' = E_1/Z_{1t}$ and $I_1'' = V_o/Z_{1t}$, we have $I_1' : I_1'' :: E_1 : V_o$, or $I_1' : E_1 :: I_1'' : V_o$. Hence if we construct upon the vector OI_1'' (Fig. 169*b*) a triangle OO_1I_1'' similar to triangle OE_1I_1', then the side OO_1 represents the difference of potential between the neutral points.

The voltages across the branches of the load are obtained by subtracting from the no-load E.M.Fs. the pressure drop in the generator, " outer " line wire, and neutral wire. They are represented in the vector diagram (Fig. 168*b*) by OV_1, OV_2. It is apparent from an inspection of this diagram, that with balanced loads the two-phase, three-wire system is unsymmetrical, as the load voltages are not only unequal but their phase difference is greater than 90°. The dissymmetry in an actual case is shown quantitatively in the worked example which follows.

Example. A two-phase balanced load, each branch of which has an impedance of $5\underline{/25°}$ O., is supplied from a two-phase generator through a three-wire transmission line, the impedance of each " outer " line wire being $0{\cdot}6\underline{/48°}$ O., and that of the neutral wire being $0{\cdot}36\underline{/34°}$ O. The no-load phase E.M.Fs. of the generator are each equal to 1,150 V., and have a phase difference of 90° with respect to each other. The effective impedance of each branch of the load is $0{\cdot}7\underline{/78°}$. Determine the load currents and the voltages. Also the terminal voltages of the generator.

Since each phase of the system is balanced, the total impedance of each phase is given by

$$Z_t = Z_g + Z_l + Z = 0{\cdot}7\underline{/78°} + 0{\cdot}6\underline{/48°} + 5\underline{/25}$$
$$= (0{\cdot}7 \cos 78° + 0{\cdot}6 \cos 48° + 5 \cos 25°) + j(0{\cdot}7 \sin 78° + 0{\cdot}6 \sin 48° + 5 \sin 25°)$$
$$= (0{\cdot}1456 + 0{\cdot}402 + 4{\cdot}53) + j(0{\cdot}685 + 0{\cdot}446 + 2{\cdot}11)$$
$$= \boldsymbol{5{\cdot}08 + j3{\cdot}24}$$

Whence
$$Y_t = \frac{5{\cdot}08}{5{\cdot}08^2 + 3{\cdot}24^2} - j\frac{3{\cdot}24}{5{\cdot}08^2 + 3{\cdot}24^2} = \boldsymbol{0{\cdot}14 - j0{\cdot}089}$$

$$Z_{ol} = 0{\cdot}36 \cos 34° + j0{\cdot}36 \sin 34°$$
$$= 0{\cdot}36 \times 0{\cdot}829 + j0{\cdot}36 \times 0{\cdot}559$$
$$= \boldsymbol{0{\cdot}3 + j0{\cdot}2}$$

$$Y_{ol} = \frac{0 \cdot 3}{0 \cdot 3^2 + 0 \cdot 2^2} - j\frac{0 \cdot 2}{0 \cdot 3^2 + 0 \cdot 2^2} = 2 \cdot 31 - j1 \cdot 54$$

$$2Y_t + Y_{ol} = 2(0 \cdot 14 - j0 \cdot 089) + 2 \cdot 31 - j1 \cdot 54)$$
$$= 2 \cdot 59 - j1 \cdot 715$$

$$\frac{1}{2Y_t + Y_{ol}} = \frac{2 \cdot 59}{2 \cdot 59^2 + 1 \cdot 715^2} + j\frac{1 \cdot 715}{2 \cdot 59^2 + 1 \cdot 715^2} = 0 \cdot 268 + j0 \cdot 178$$

Taking the E.M.F. of phase I as the quantity of reference, we calculate the fictitious currents I_1', I_2', and determine the potential difference, V_o, between the neutral points of the generator and load. Thus

$$I_1' = E_I Y_t = 1150(1 + j0)(0 \cdot 14 - j0 \cdot 089) = 161 - j102 \cdot 4$$
$$I_2' = E_{II} Y_t = 1150(0 - j1)(0 \cdot 14 - j0 \cdot 089) = -102 \cdot 4 - j161$$
$$I_o' = I_1' + I_2' = 58 \cdot 6 - j263 \cdot 4$$
$$V_o = I_o'/(2Y_t + Y_{ol}) = (58 \cdot 6 - j263 \cdot 4)(0 \cdot 268 + j0 \cdot 178)$$
$$= 62 \cdot 6 - j60 \cdot 3$$

Whence $V_o = \sqrt{(62 \cdot 6^2 + 60 \cdot 3^2)} = 86 \cdot 9$ V.

Phase difference between E_I and $V_o = \tan^{-1} - 60 \cdot 3/62 \cdot 6 = -43 \cdot 9°$

When V_o is known, the load currents I_1, I_2, and the voltages, V_1, V_2, across the branches of the load are easily calculated. Thus

$$I_1 = (E_I - V_o)Y_t = (1150 - 62 \cdot 6 + j60 \cdot 3)(0 \cdot 14 - j0 \cdot 089)$$
$$= 156 \cdot 3 - j87 \cdot 6$$
$$I_1 = \sqrt{(156 \cdot 3^2 + 87 \cdot 6)^2} = 179 \text{ A.}$$

Phase difference between E_I and $I_1 = \tan^{-1} - 87 \cdot 6/156 \cdot 3 = -29 \cdot 3°$

$$V_1 = I_1 Z_1 = (156 \cdot 3 - j87 \cdot 6)(4 \cdot 53 + j2 \cdot 11)$$
$$= 895 - j68$$
$$V_1 = \sqrt{(895^2 + 68^2)} = 899 \text{ V.}$$

Phase difference between E_I and $V_1 = \tan^{-1} - 68/895 = -4 \cdot 3°$.

Phase difference between V_1 and $I_1 = -(29 \cdot 3 - 4 \cdot 3) = -25°$, which agrees with the given phase angle of the load impedance.

$$I_2 = (E_{II} - V_o)Y_t = (-j1150 - 62 \cdot 6 + j60 \cdot 3)(0 \cdot 14 - j0 \cdot 089)$$
$$= -105 \cdot 8 - j147$$
$$I = \sqrt{(105 \cdot 8^2 + 147^2)} = 181 \text{ A.}$$

Phase difference between E_I and $I_2 = \tan^{-1} - 147/-105 \cdot 8 = -(180 - 54 \cdot 3)°$
$= -125 \cdot 7°$

Phase difference between I_1 and $I_2 = -(125 \cdot 7 - 29 \cdot 3)° = -96 \cdot 4°$

$$V_2 = I_2 Z = (-105 \cdot 8 - j147)(4 \cdot 53 + j2 \cdot 11)$$
$$= -169 - j891$$
$$V_2 = \sqrt{(169^2 + 891^2)} = 907 \text{ V.}$$

Phase difference between E_I and $V_2 = \tan^{-1} - 891/-169 = -(180 - 79 \cdot 3)°$
$= -100 \cdot 7°$

Phase difference between V_1 and $V_2 = -(100 \cdot 7 - 4 \cdot 3)° = -96 \cdot 4°$

Phase difference between V_2 and $I_2 = -(125 \cdot 7 - 100 \cdot 7)° = -25°$, which agrees with the given phase angle of the load impedance.

$$I_o = -(I_1 + I_2) = -[(156 \cdot 3 - j87 \cdot 6) + (-105 \cdot 8 - j147)]$$
$$= -50 \cdot 5 + j234 \cdot 6$$
$$I_o = \sqrt{(50 \cdot 5^2 + 234 \cdot 6^2)} = 240 \text{ A}$$

Phase difference between E_I and $I_0 = \tan^{-1} 234{\cdot}6/-50{\cdot}5 = (180 - 77{\cdot}9)^\circ = 102{\cdot}1^\circ$

Phase difference between V_0 and $-I_0 = -(77{\cdot}9 - 43{\cdot}9)^\circ = -34^\circ$, which agrees with the phase angle of the neutral wire impedance.

The numerical value of V_0 may now be checked by calculating the product $I_0 Z_{0l}$, thus

$$I_0 Z_{0l} = 240 \times 0{\cdot}36 = 86{\cdot}4 \text{ V.}$$

The generator terminal voltages per phase are

$$V_{Ig} = E_I - I_1 Z_g = 1150 - [(156{\cdot}3 - j87{\cdot}6)(0{\cdot}1456 + j0{\cdot}685)]$$
$$= 1066 - j95{\cdot}1$$

$$V_{Ig} = \sqrt{(1066^2 + 95{\cdot}1^2)} = 1071 \text{ V.}$$

Phase difference between E_I and $V_{Ig} = \tan^{-1} - 95{\cdot}1/1066 = -5{\cdot}1^\circ$

$$V_{IIg} = E_{II} - I_2 Z_g = -j1150 - [-105{\cdot}8 - j147)(0{\cdot}1456 + j0{\cdot}685)]$$
$$= -85{\cdot}4 - j1056$$

$$V_{IIg} = \sqrt{(85{\cdot}4^2 + 1056^2)} = 1061 \text{ V.}$$

Phase difference between E_I and $V_{IIg} = \tan^{-1} - 1056/-85{\cdot}4 = -(180 - 85{\cdot}4)^\circ = -94{\cdot}6^\circ$

Phase difference between V_{Ig} and $V_{IIg} = -(94{\cdot}6 - 5{\cdot}1)^\circ = -89{\cdot}5^\circ$

Determination of load currents and voltages for three-phase, four-wire systems, i.e. three-phase systems with star-connected loads and neutral wire. *Case I. Pressure drop in line wires ignored.* In this case the currents in the several branches of the load are easily calculated when the phase voltages of the system and the impedances of the load are known. The current in the neutral wire is equal to the vector sum of the currents in the branches of the load, and may be determined either graphically, by means of a vector diagram. or analytically, using the symbolic method.

Example. An unbalanced star-connected load—the branches of which have the following impedances—$Z_1 = 2{\cdot}5\underline{/10^\circ}$, $Z_2 = 3{\cdot}0\underline{/15^\circ}$, $Z_3 = 3{\cdot}5\underline{/5^\circ}$ ohms, is supplied from a three-phase, four-wire symmetrical system, the line voltage being 400 V., and the phase rotation clockwise. Determine the currents in each line wire, ignoring the pressure drop in all line wires.

The phase voltage of the system $= 400/\sqrt{3} = 231$ V. Hence, if the line currents are denoted by I_1, I_2, I_3, we have

$$I_1 = 231/2{\cdot}5 = 92{\cdot}4 \text{ A.}$$
$$I_2 = 231/3 = 77 \text{ A.}$$
$$I_3 = 231/3{\cdot}5 = 66 \text{ A.}$$

These currents are lagging with respect to the phase voltages by the angles 10°, 15°, and 5°, respectively.

The *graphical solution* for the current in the neutral wire is shown in Fig. 170, the vector diagram being drawn to a scale of 1 cm. = 10 A. By measurement, the vector OI_0, representing the sum of the vectors OI_1, OI_2, OI_3, is 1·38 cm., and therefore the current in the neutral wire is $1{\cdot}38 \times 10 = 13{\cdot}8$ A. The angle between OI_0 and OI_1 is 41°.

The *analytical solution* for the current in the neutral wire may be affected either by the symbolic method or by resolving the phase currents into components along two axes perpendicular to each other, and determining

the resultant algebraically. We shall adopt the latter method, and shall take one axis along the direction of the vector OI_1 (Fig. 170).

The components along this axis are therefore

$$I_1,\ I_2 \cos(120 + 15 - 10)^\circ \text{ and } I_3 \cos(240 + 5 - 10)^\circ\,;$$

their sum is given by

$$I_x = 92{\cdot}4 - 77 \times 0{\cdot}5736 - 66 \times 0{\cdot}5736 = 10{\cdot}4$$

The components along the perpendicular axis are

$$I_2 \sin(120 + 5 - 10)^\circ \text{ and } I_3 \sin(240 + 5 - 10)^\circ,$$

their sum is given by

$$I_y = -77 \times 0{\cdot}819 + 66 \times 0{\cdot}819 = -9{\cdot}1$$

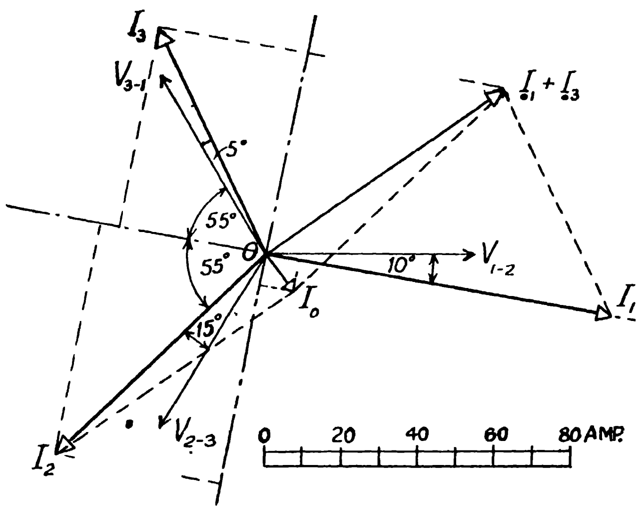


FIG. 170.—Graphic Solution for Current in Neutral Wire of Three-phase Four-wire System

Whence the current in the neutral wire is given by

$$I_0 = \sqrt{(I_x^2 + I_y^2)} = \sqrt{(10{\cdot}4^2 + 9{\cdot}1^2)} = 13{\cdot}8\ \text{A}$$

and the phase difference of this current with respect to I_1 is given by

$$\varphi_0 = \tan^{-1} I_y/I_x = \tan^{-1} - 9{\cdot}1/10{\cdot}4 = -41{\cdot}2^\circ$$

Case II. General Case—pressure drop in all parts of circuit considered. The direct analytical solution to this case involves the application of Kirchoff's laws and is given on p. 290.

An alternative analytical solution may be obtained by a similar method to that given previously, on p. 260, for the two-phase, three-wire system. In the present case, let the no-load phase E.M.Fs. of the generator be denoted by $E_{\text{I}}, E_{\text{II}}, E_{\text{III}}$; the impedances of each phase of the system (including the impedances of the generator, connecting line wire and load, but not the impedance of

the neutral wire) by Z_{1t}, Z_{2t}, Z_{3t}; the impedance of the neutral wire by Z_{0l}; the actual currents in the neutral wire and the branches of the load by I_0, I_1, I_2, I_3, respectively, and the potential difference between the two neutral points by V_0. Then

$$I_0 = -(I_1 + I_2 + I_3), \qquad V_0 = -I_0 Z_{0l}$$

$$I_1 = \frac{E_{\mathrm{I}} - V_0}{Z_{1t}} = \frac{E_{\mathrm{I}}}{Z_{1t}} - \frac{V_0}{Z_{1t}} = I_1' - I_1'' \quad . \quad . \quad . \quad (81)$$

$$I_2 = \frac{E_{\mathrm{II}} - V_0}{Z_{2t}} = \frac{E_{\mathrm{II}}}{Z_{2t}} - \frac{V_0}{Z_{2t}} = I_2' - I_2'' \quad . \quad . \quad . \quad (82)$$

$$I_3 = \frac{E_{\mathrm{III}} - V_0}{Z_{3t}} = \frac{E_{\mathrm{III}}}{Z_{3t}} - \frac{V_0}{Z_{3t}} = I_3' - I_3'' \quad . \quad . \quad . \quad (83)$$

where I_1', I_2', I_3', denote the phase currents which would be obtained if the two neutral points were at the same potential, and I_1'', I_2'', I_3'', denote the phase currents which would be obtained if the generator E.M.Fs. were zero and the potential V_0 existed between the neutral points.

Hence
$$\begin{aligned} I_0 &= -[(I_1' - I_1'') + (I_2' - I_2'') + (I_3' - I_3'')] \\ &= -(I_1' + I_2' + I_3') + (I_1'' + I_2'' + I_3'') \\ &= -I'_0 + (I_1'' + I_2'' + I_3'') \end{aligned}$$

or
$$\begin{aligned} I_0' &= -I_0 + I_1'' + I_2'' + I_3'' \\ &= V_0/Z_{0l} + V_0/Z_{2t} + V_0/Z_{2t} + V_0/Z_{3t} \\ &= V_0(Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t}) \end{aligned}$$

where I_0' denotes the current which would be obtained in the neutral wire if the impedance of this wire were zero, and Y_{0l}, Y_{1t}, Y_{2t}, Y_{3t}, denote the admittances of the neutral wire and the several phases, respectively.

The potential difference, V_0, between the neutral points of generator and load under normal conditions is given by

$$V_0 = I_0'/(Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t}) \quad . \quad . \quad . \quad . \quad (84)$$

this equation being obtained in the same manner as the corresponding equation (80) for the two-phase case.

The load currents I_1, I_2, I_3, may be determined either directly from the E.M.Fs. and impedances or, indirectly, by the super position of the fictitious currents I_1', I_1''; I_2', I_2''; I_3', I_3''. In the latter method the procedure is similar to the two-phase case previously considered. Thus

$$I_0' = V_0(Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t})$$
$$I_1'' = V_0 Y_{1t}; \quad I_2'' = V_0 Y_{2t}; \quad I_3'' = V_0 Y_{3t}$$

Whence $I_1'' : I_0' :: Y_{1t} : (Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t})$

$I_2'' : I_0' :: Y_{2t} : (Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t})$

$I_3'' : I_0' :: Y_{3t} : (Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t})$

The geometrical construction for determining the currents I_1'', I_2'', I_3'', is as follows—

A vector polygon is constructed to represent the admittances

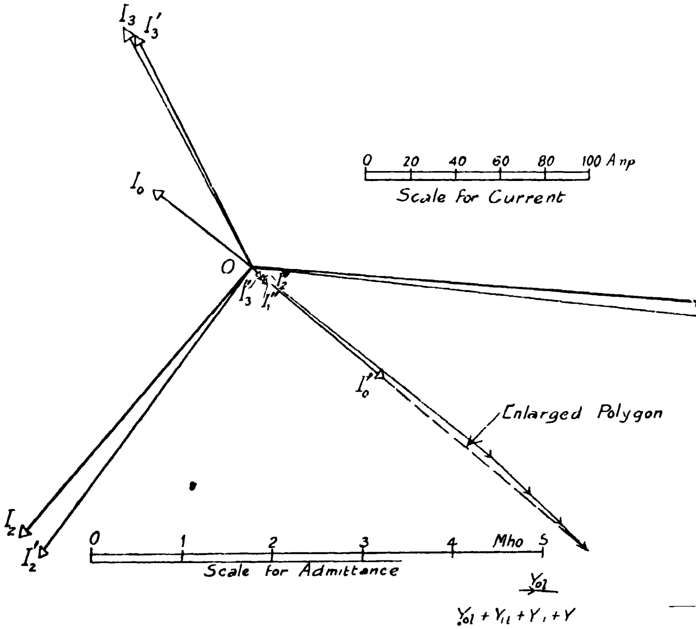


FIG. 171.—Graphic Method of Determining the Fictitious Currents I_1'', I_2'', I_3'' for a Three-phase Four-wire System

Y_{1t}, Y_{2t}, Y_{3t}, Y_{0l}, $(Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t})$; the currents I_1', I_2', I_3', are calculated; their vector sum (I_0') is determined graphically, and upon the vector representing this quantity is constructed a polygon similar to the admittance vector polygon, the vector I_0' corresponding to the side of the latter which represents the quantity $(Y_{0l} + Y_{1t} + Y_{2t} + Y_{3t})$. The currents I_1'', I_2'', I_3'', in the admittances Y_{1t}, Y_{2t}, Y_{3t}, are then represented by the sides of the current polygon which are similar to the sides representing the latter quantities in the admittance polygon.

The geometrical construction in Fig. 171 refers to the worked example which follows.

Example. An unbalanced star-connected load of unity power factor—the branches of which have resistances of 1·0, 1·3, 1·85 ohms—is supplied from a symmetrical three-phase star-connected generator through a four-core cable 1000 yd. long. The cross sections of the cores are 0·25, 0·25, 0·25, 0·125 sq. in., and the core of smallest cross section forms the neutral conductor. The impedance of each phase of the generator is $0{\cdot}13\angle 76°$ ohm. The no-load E.M.F. of the generator per phase is 240 V. and the phase rotation is clockwise. Determine the currents and voltages at the load.

The resistance of the cable connecting the load and generator is 0·1 Ω. for the principal conductors and 0·2 Ω. for the neutral, the inductance being negligible in each case.

The impedance per phase of the generator is

$$Z_g = 0{\cdot}13 \cos 76° + j\,0{\cdot}13 \sin 76° = \boldsymbol{0{\cdot}0314 + j0{\cdot}1262}$$

The total impedances per phase are therefore

$$Z_{1t} = 0{\cdot}0314 + j0{\cdot}1262 + 0{\cdot}1 + 1{\cdot}0 = 1{\cdot}1314 + j0{\cdot}1262$$
$$Z_{2t} = 0{\cdot}0314 + j0{\cdot}1262 + 0{\cdot}1 + 1{\cdot}3 = 1{\cdot}4314 + j0{\cdot}1262$$
$$Z_{3t} = 0{\cdot}0314 + j0{\cdot}1262 + 0{\cdot}1 + 1{\cdot}85 = 1{\cdot}9814 + j0{\cdot}1262$$

and the impedance of the neutral wire is

$$Z_{0l} = 0{\cdot}2 + j0$$

Whence the admittances per phase are

$$Y_{1t} = \frac{1{\cdot}1314}{1{\cdot}1314^2 + 0{\cdot}1262^2} - j\frac{0{\cdot}1262}{1{\cdot}1314^2 + 0{\cdot}1262^2} = 0{\cdot}872 - j0{\cdot}0973$$
$$Y_{2t} = \frac{1{\cdot}4314}{1{\cdot}4314^2 + 0{\cdot}1262^2} - j\frac{0{\cdot}1262}{1{\cdot}4314^2 + 0{\cdot}1262^2} = 0{\cdot}693 - j0{\cdot}0611$$
$$Y_{3t} = \frac{1{\cdot}9814}{1{\cdot}9814^2 + 0{\cdot}1262^2} - j\frac{0{\cdot}1262}{1{\cdot}9814^2 + 0{\cdot}1262^2} = 0{\cdot}502 - j0{\cdot}032$$
$$Y_{0l} = 5{\cdot}0 + j0$$
$$\therefore\ Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l} = 7{\cdot}067 - j0{\cdot}1904$$
$$\frac{1}{Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l}} = \frac{7{\cdot}067}{7{\cdot}067^2 + 0{\cdot}1904^2} + j\frac{0{\cdot}1904}{7{\cdot}067^2 + 0{\cdot}1904^2} = 0{\cdot}1412 + j0{\cdot}0038$$

Since the no-load phase E.M.Fs. are given by the expressions

$$E_{\mathrm{I}} = 240(1 + j0), \quad E_{\mathrm{II}} = 240(-0{\cdot}5 - j0{\cdot}866) = -120 - j208,$$
$$E_{\mathrm{III}} = 240(-0{\cdot}5 + j0{\cdot}866) = -120 + j208,$$

the fictitious currents I_1', I_2', I_3', are

$$I_1' = E_{\mathrm{I}} Y_{1t} = 240(0{\cdot}872 - j0{\cdot}0973) = 209{\cdot}3 - j23{\cdot}4$$
$$I_2' = E_{\mathrm{II}} Y_{2t} = (-120 - j208)(0{\cdot}693 - j0{\cdot}061) = -95{\cdot}9 - j136{\cdot}9$$
$$I_3' = E_{\mathrm{III}} Y_{3t} = (-120 + j208)(0{\cdot}502 - j0{\cdot}032) = -53{\cdot}5 + j108{\cdot}3$$

Whence $I_0' = I_1' + I_2' + I_3' = 59{\cdot}9 - j52$

The fictitious currents I_1'', I_2'', I_3'', are now obtained thus

$$I_1'' = I_0' Y_{1t}/(Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l})$$
$$= (59{\cdot}9 - j52)\,(0{\cdot}1412 + j0{\cdot}0038)\,(0{\cdot}872 - j0{\cdot}0973)$$
$$= 6{\cdot}867 - j7{\cdot}053$$
$$I_2'' = I_0' Y_{2t}/(Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l})$$
$$= (59{\cdot}9 - j52)\,(0{\cdot}1412 + j0{\cdot}0038)\,(0{\cdot}693 - j0{\cdot}061)$$
$$= 5{\cdot}565 - j5{\cdot}458$$
$$I_3'' = I_0' Y_{3t}/(Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l})$$
$$= (59{\cdot}9 - j52)\,(0{\cdot}1412 + j0{\cdot}0038)\,(0{\cdot}502 - j0{\cdot}032)$$

Hence $I_1 = I_1' - I_1'' = \mathbf{209{\cdot}3 - j23{\cdot}4 - (6{\cdot}87 - j7{\cdot}05) = 202{\cdot}4 - j16{\cdot}35}$

$$I_1 = \sqrt{(202{\cdot}4^2 + 16{\cdot}35^2)} = 203 \text{ A.}$$

$$I_2 = I_2' - I_2'' = \mathbf{-95{\cdot}9 - j136{\cdot}9 - (5{\cdot}56 - j5{\cdot}46) = -101{\cdot}5 - j131{\cdot}4}$$

$$I_2 = \sqrt{(101{\cdot}5^2 + 131{\cdot}4^2)} = 166 \text{ A.}$$

$$I_3 = I_3' - I_3'' = \mathbf{-53{\cdot}5 + j108{\cdot}3 - (4{\cdot}11 - j3{\cdot}85) = -57{\cdot}6 + j112{\cdot}1}$$

$$I_3 = \sqrt{(57{\cdot}6^2 + 112{\cdot}1^2)} = 126 \text{ A.}$$

$$I_0 = -(I_1 + I_2 + I_3) = -[\mathbf{(202{\cdot}4 - j16{\cdot}35) + (-101{\cdot}5 - j131{\cdot}4)} \\ \mathbf{+ (-57{\cdot}6 + j112{\cdot}1)}] = \mathbf{-43{\cdot}5 + j35{\cdot}6}$$

$$I_0 = \sqrt{(43{\cdot}5^2 + 35{\cdot}6^2)} = 56{\cdot}2 \text{ A.}$$

Phase difference between I_1 and $E_1 = \tan^{-1} - 16{\cdot}35/202{\cdot}4 = -5°$

„ „ „ I_2 „ $E_1 = \tan^{-1} - 131{\cdot}4/-101{\cdot}5 = -(180 - 52{\cdot}3)° = -127{\cdot}7°$

„ „ „ I_3 „ $E_1 = \tan^{-1} 112{\cdot}1/-57{\cdot}6 = -(180 + 62{\cdot}8)° = -242{\cdot}8°$

„ „ „ I_0 „ $E_1 = \tan^{-1} 35{\cdot}6/-43{\cdot}5 = -(180 + 39{\cdot}3)° = -219{\cdot}3°$

The pressures across the branches of the load are in phase with the respective currents, and their magnitudes are

$$V_1 = 1{\cdot}0 \times I_1 = 203 \text{ V.}$$
$$V_2 = 1{\cdot}3 \times I_2 = 215{\cdot}8 \text{ V.}$$
$$V_3 = 1{\cdot}85 \times I_3 = 233 \text{ V.}$$

The pressure drop in the neutral wire is

$$V_0 = 0{\cdot}2 \times I_0 = 11{\cdot}2 \text{ V.}$$

As a check on the above calculations we may calculate the potential difference between the neutral points by means of equation (84), thus

$$V_0 = I_0'/(Y_{1t} + Y_{2t} + Y_{3t} + Y_{0l}) = \mathbf{(59{\cdot}9 - j52)(0{\cdot}1412 + j0{\cdot}0038)} \\ = \mathbf{8{\cdot}58 - j7{\cdot}06}$$

$$V_0 = \sqrt{(8{\cdot}58^2 + 7{\cdot}06^2)} = 11{\cdot}12 \text{ V.}$$

Phase difference between E_1 and $V_0 = \tan^{-1} - 7{\cdot}06/8{\cdot}58 = -39{\cdot}5°$

These values are sufficiently close to the previous values for all practical purposes.

It will be of interest to calculate the pressure drop in the principal line wires and the terminal pressure at the generator.

Pressure drop in the "outer" line wires

$$V_{1l} = 0{\cdot}1 \times I_1 = 20{\cdot}3 \text{ V.}$$
$$V_{2l} = 0{\cdot}1 \times I_2 = 16{\cdot}6 \text{ V.}$$
$$V_{3l} = 0{\cdot}1 \times I_3 = 12{\cdot}6 \text{ V.}$$

The pressures between the terminals of the generator and the neutral point of the load are

$$V_{1g-0} = 203 + 20{\cdot}3 = 223{\cdot}3 \text{ V.}$$
$$V_{2g-0} = 215{\cdot}8 + 16{\cdot}6 = 232{\cdot}4 \text{ V.}$$
$$V_{3g-0} = 233 + 12{\cdot}6 = 245{\cdot}6 \text{ V.}$$

The generator terminal pressures per phase (i.e. the pressure between the terminals of the generator and the neutral point of the generator) are

$$V_{\mathrm{I}} = E_{\mathrm{I}} - I_1 Z_g = (240 + j0) - (202{\cdot}4 - j16{\cdot}35)\ (0{\cdot}0314 + j0{\cdot}1262)$$
$$= 231{\cdot}6 - j25{\cdot}4$$
$$V_{\mathrm{I}} = \sqrt{(231{\cdot}6^2 + 25{\cdot}4^2)} = 232{\cdot}9 \text{ V.}$$
$$V_{\mathrm{II}} = E_{\mathrm{II}} - I_2 Z_g = (-120 - j208) - (-101{\cdot}5 - j131{\cdot}4)(0{\cdot}0314 + j0{\cdot}1262)$$
$$= -133{\cdot}4 - j191$$
$$V_{\mathrm{II}} = \sqrt{(133{\cdot}4^2 + 191^2)} = 233 \text{ V.}$$
$$V_{\mathrm{III}} = E_{\mathrm{III}} - I_3 Z_g = (-120 + j208) - (-57{\cdot}6 + j112{\cdot}1)(0{\cdot}0314 + j0{\cdot}1262)$$
$$= -104 + j211{\cdot}7$$
$$V_{\mathrm{III}} = \sqrt{(104^2 + 211{\cdot}7^2)} = 235{\cdot}9 \text{ V.}$$

The line voltages at the terminals of the generator are

$$V_{g1\text{-}2} = V_{\mathrm{I}} - V_{\mathrm{II}} = 231{\cdot}6 - j25{\cdot}4 - (-133{\cdot}4 - j191) = 365 + j165{\cdot}6$$
$$V_{g1\text{-}2} = \sqrt{(365^2 + 165{\cdot}6^2)} = 401 \text{ V.}$$
$$V_{g2\text{-}3} = V_{\mathrm{II}} - V_{\mathrm{III}} = -133{\cdot}4 - j191) - (-104 + j211{\cdot}7) = -29{\cdot}4 - j402{\cdot}7$$
$$V_{g2\text{-}3} = \sqrt{(29{\cdot}4^2 + 402{\cdot}7^2)} = 403{\cdot}5 \text{ V.}$$
$$V_{g3\text{-}1} = V_{\mathrm{III}} - V_{\mathrm{I}} = -104 + j211{\cdot}7 - (231{\cdot}6 - j25{\cdot}4) = -335{\cdot}6 + j237$$
$$V_{g3\text{-}1} = \sqrt{(335{\cdot}6^2 + 237^2)} = 411 \text{ V.}$$

and the line voltages at the load are

$$V_{1\text{-}2} = V_1 - V_2 = 202{\cdot}4 - j16{\cdot}4 - (-132 - j170{\cdot}8) = 334{\cdot}4 + j154{\cdot}4$$
$$V_{1\text{-}2} = \sqrt{(334{\cdot}4^2 + 154{\cdot}4^2)} = 368{\cdot}5 \text{ V.}$$
$$V_{2\text{-}3} = V_2 - V_3 = -132 - j170{\cdot}8 - (-106{\cdot}5 + j207{\cdot}3) = -25{\cdot}5 - j378$$
$$V_{2\text{-}3} = \sqrt{(25{\cdot}5^2 + 378^2)} = 379 \text{ V.}$$
$$V_{3\text{-}1} = V_3 - V_1 = -106{\cdot}5 + j207{\cdot}3 - (202{\cdot}4 - j16{\cdot}4) = -308{\cdot}9 + j223{\cdot}7$$
$$V_{3\text{-}1} = \sqrt{(308{\cdot}9^2 + 223{\cdot}7^2)} = 381{\cdot}5 \text{ V.}$$

[NOTE. In this example the resistances of the line wires have been chosen much higher than the values which would be adopted in practice.]

Application of Kirchoff's laws to polyphase circuits. The analytical solution of general problems connected with polyphase circuits is effected by the application of Kirchoff's laws relating to electric networks. These laws may be stated thus—

1. At every junction of two or more branches of an electric circuit the algebraic sum of all currents is zero, i.e. the sum of the currents flowing towards the junction must equal the sum of the currents flowing away from the junction.

2. In every closed electric circuit carrying a current the algebraic sum of all E.M.Fs. taken in order round the circuit is zero, i.e. the E.M.Fs. due to the current—viz. the induced E.M.Fs. due to self, or mutual, inductance and the potential drop due to resistance, etc.—must balance the impressed E.M.F. (or the E.M.F. generated, or induced, in the circuit by external means).

In applying these laws to alternating-current circuits we may

consider either the instantaneous values of the E.M.Fs. and currents or the R.M.S. values of these quantities. When dealing with instantaneous values only the magnitudes and directions of the E.M.Fs. and currents need be considered. The calculations, therefore, are carried out by ordinary algebraic methods.

But when R.M.S. values are employed the relative phase differences of the currents and E.M.Fs. must be considered as well as their magnitudes. Hence, in this case, the calculations must be carried out by the symbolic method, as we are dealing with complex quantities.

The *method of calculation* in the case of complex circuits or networks is as follows—

The complex circuit is reduced to a number of closed circuits, or meshes, in each of which a current is assumed to circulate in a definite direction independently of the currents in the adjacent meshes. The E.M.Fs. in each mesh are then calculated and are equated to zero in accordance with Kirchoff's second law, the number of equations so obtained being equal to the number of meshes. The values of the fictitious circulating currents are then obtained by solving these equations, and when these currents are known, the currents in, and the potential difference across, all parts of the network may easily be obtained.

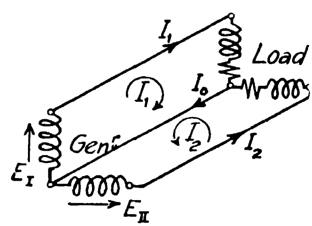


FIG. 172.—Circuit Diagram for Two-phase Three-wire System

The process of obtaining the general equations for the meshes is quite simple, but the reduction of these equations to obtain the equations for the several currents is usually complicated when the number of separate currents exceeds three.

In order to illustrate the method of procedure, we shall consider a number of cases of two and three-phase systems.

Case I. Calculation of currents in two-phase, three-wire system. Let the no-load phase E.M.Fs. of the generator be denoted by E_I, E_{II}; the impedances per phase of the generator and load by Z_{1g}, Z_{2g}, Z_1, Z_2, respectively; the impedances of the line wires by Z_{1l}, Z_{2l}, Z_{0l}; and the currents in the line wires by I_1, I_2, I_0, the subscript $_0$ referring to the neutral wire in all cases.

The system may then be considered as equivalent to two meshes in which the circulating currents are I_1, I_2, and the externally-produced E.M.Fs. are E_I, E_{II}, Fig. 172.

The general equations for the meshes are—

$$\mathbf{E}_{\text{I}} - \mathbf{I}_1(\mathbf{Z}_{1g} + \mathbf{Z}_{1l} + \mathbf{Z}_1) - (\mathbf{I}_1 + \mathbf{I}_2)\mathbf{Z}_{0l} = 0$$

$$\mathbf{E}_{\text{II}} - \mathbf{I}_2(\mathbf{Z}_{2g} + \mathbf{Z}_{2l} + \mathbf{Z}_2) - (\mathbf{I}_1 + \mathbf{I}_2)\mathbf{Z}_{0l} = 0$$

Re-arranging terms, we have

$$\begin{aligned} &\mathbf{I}_1(\mathbf{Z}_{1g} + \mathbf{Z}_{1l} + \mathbf{Z}_1 + \mathbf{Z}_{0l}) + \mathbf{I}_2\mathbf{Z}_{0l} &&= \mathbf{E}_{\text{I}} \\ &\mathbf{I}_1\mathbf{Z}_{0l} + \mathbf{I}_2(\mathbf{Z}_{2g} + \mathbf{Z}_{2l} + \mathbf{Z}_2 + \mathbf{Z}_{0l}) &&= \mathbf{E}_{\text{II}} \end{aligned}$$

or

$$\mathbf{I}_1(\mathbf{Z}_{1t} + \mathbf{Z}_{0l}) + \mathbf{I}_2\mathbf{Z}_{0l} = \mathbf{E}_{\text{I}} \quad . \quad . \quad (\alpha)$$

$$\mathbf{I}_1\mathbf{Z}_{0l} + \mathbf{I}_2(\mathbf{Z}_{2t} + \mathbf{Z}_{0l}) = \mathbf{E}_{\text{II}} \quad . \quad (\beta)$$

where $\mathbf{Z}_{1t} = \mathbf{Z}_{1g} + \mathbf{Z}_{1l} + \mathbf{Z}_1$, $\mathbf{Z}_{2t} = \mathbf{Z}_{2g} + \mathbf{Z}_{2l} + \mathbf{Z}_2$

Hence

$$\mathbf{I}_1 = \frac{(\mathbf{E}_{\text{I}} - \mathbf{E}_{\text{II}})\mathbf{Z}_{0l} + \mathbf{E}_{\text{I}}\mathbf{Z}_{2t}}{\mathbf{Z}_{1t}\mathbf{Z}_{2t} + \mathbf{Z}_{0l}(\mathbf{Z}_{1t} + \mathbf{Z}_{2t})} \quad . \quad . \quad . \quad . \quad (85)$$

$$= \frac{\mathbf{E}_{\text{I}} - \mathbf{E}_{\text{II}}}{\mathbf{Z}_{1t} + \mathbf{Z}_{2t} + \mathbf{Z}_{1t}\mathbf{Z}_{2t}/\mathbf{Z}_{0l}} + \frac{\mathbf{E}_{\text{I}}}{\mathbf{Z}_{1t} + \mathbf{Z}_{0l} + \mathbf{Z}_{1t}\mathbf{Z}_{0l}/\mathbf{Z}_{2t}} \quad . \quad (85a)$$

$$\mathbf{I}_2 = \frac{(\mathbf{E}_{\text{II}} - \mathbf{E}_{\text{I}})\mathbf{Z}_{0l} + \mathbf{E}_{\text{II}}\mathbf{Z}_{1t}}{\mathbf{Z}_{1t}\mathbf{Z}_{2t} + \mathbf{Z}_{0l}(\mathbf{Z}_{1t} + \mathbf{Z}_{2t})} \quad . \quad . \quad . \quad . \quad (86)$$

$$= \frac{\mathbf{E}_{\text{II}} - \mathbf{E}_{\text{I}}}{\mathbf{Z}_{1t} + \mathbf{Z}_{2t} + \mathbf{Z}_{1t}\mathbf{Z}_{2t}/\mathbf{Z}_{0l}} + \frac{\mathbf{E}_{\text{II}}}{\mathbf{Z}_{2t} + \mathbf{Z}_{0l} + \mathbf{Z}_{2t}\mathbf{Z}_{0l}/\mathbf{Z}_{1t}} \quad . \quad (86a)$$

In the special case of balanced loads, when $\mathbf{Z}_{1t} = \mathbf{Z}_{2t} = \mathbf{Z}_t$, we have

$$\mathbf{I}_1 = \frac{\mathbf{E}_{\text{I}} - \mathbf{E}_{\text{II}}}{2\mathbf{Z}_t + \mathbf{Z}_t^2/\mathbf{Z}_{0l}} + \frac{\mathbf{E}_{\text{I}}}{\mathbf{Z}_t + 2\mathbf{Z}_{0l}} \quad . \quad . \quad . \quad (85b)$$

$$\mathbf{I}_2 = \frac{\mathbf{E}_{\text{II}} - \mathbf{E}_{\text{I}}}{2\mathbf{Z}_t + \mathbf{Z}_t^2/\mathbf{Z}_{0l}} + \frac{\mathbf{E}_{\text{II}}}{\mathbf{Z}_t + 2\mathbf{Z}_{0l}} \quad . \quad . \quad . \quad (86b)$$

Applying Kirchoff's first law to the neutral point of the generator we have

$$\mathbf{I}_0 = -(\mathbf{I}_1 + \mathbf{I}_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

Example. Calculation of the load currents for the two-phase three-wire system given in the example on p. 264.

Data from pp. 264, 265 are as follows—

Generator no load E.M.F. per phase = 1,150 V.

Impedances per phase in ohms—Generator, $0{\cdot}7\underline{/78^\circ}$; "Outer" line wires, $0{\cdot}6\underline{/48^\circ}$; Neutral wire, $0{\cdot}36\underline{/34^\circ}$: Load, $5\underline{/25^\circ}$.

Denoting the total impedance per phase (i.e. the sum of the impedances of generator, load, and connecting line wire, but not the neutral wire) by $\mathbf{Z}_t$, and the impedance of the neutral wire by $\mathbf{Z}_{0l}$, we have, from p. 264,

$$\mathbf{Z}_t = 5{\cdot}08 + j3{\cdot}24, \qquad \mathbf{Z}_{0l} = 0{\cdot}3 + j0{\cdot}2$$

The line currents, I_1 and I_2, are given by equations (85*b*), (86*b*), thus

$$I_1 = \frac{E_{\mathrm{I}} - E_{\mathrm{II}}}{2Z_t + Z_t^2/Z_{0l}} + \frac{E_{\mathrm{I}}}{Z_t + 2Z_{0l}}$$

$$I_2 = \frac{E_{\mathrm{II}} - E_{\mathrm{I}}}{2Z_t + Z_t^2/Z_{0l}} + \frac{E_{\mathrm{II}}}{Z_t + 2Z_{0l}}$$

In evaluating these equations, it is best to evaluate the denominators first—

$$\frac{Z_t^2}{Z_{0l}} = \frac{(5\cdot08^2 - 3\cdot24^2 + j2 \times 5\cdot08 \times 3\cdot24)\,(0\cdot3 - j0\cdot2)}{0\cdot3^2 + 0\cdot2^2} = 86\cdot4 + j52\cdot1$$

$$2Z_t + Z_t^2/Z_{0l} = 2(5\cdot08 + j3\cdot24) + (86\cdot4 + j52\cdot1) = 96\cdot56 + j58\cdot58$$

$$Z_t + 2Z_{0l} = (5\cdot08 + j3\cdot24) + 2(0\cdot3 + j0\cdot2) = 5\cdot677 + j3\cdot642$$

Hence,

$$I_1 = \frac{(1150 + j0) - (0 - j1150)}{96\cdot56 + j58\cdot58} + \frac{1150 + j0}{5\cdot677 + j3\cdot642}$$

$$= 157\cdot6 - j88\cdot7$$

$$I_1 = \sqrt{(157\cdot6^2 + 88\cdot7^2)} = 180\cdot7 \text{ A.}$$

$$I_2 = \frac{(0 - j1150) - (1150 + j0)}{96\cdot56 + j58\cdot58} + \frac{0 - j1150}{5\cdot677 + j3\cdot642}$$

$$= -106\cdot4 - j146\cdot8$$

$$I_2 = \sqrt{(106\cdot4^2 + 146\cdot8^2)} = 181\cdot1 \text{ A.}$$

[NOTE. These values check very closely with those contained on pp. 265, 266, the slight differences being due to the use of the slide rule in making the computations.]

Case II. Calculation of currents in a three-phase, three-wire system with star-connected generator and load. Let the no-load phase E.M.Fs.

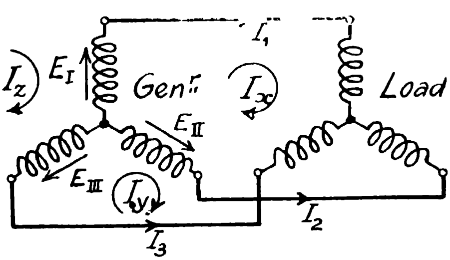


FIG. 173.—Circuit Diagram for Star/Star Three-phase System

of the generator be denoted by E_{I}, E_{II}, E_{III}; the line currents by I_1, I_2, I_3; and the total impedances per phase (i.e. the sum of the impedances per phase of the generator, load, and connecting line wire) by Z_{1t}, Z_{2t}, Z_{3t}. Then the system is equivalent to three meshes (Fig. 173), in which the externally-produced E.M.Fs. are $E_{\mathrm{I}} - E_{\mathrm{II}}$, $E_{\mathrm{II}} - E_{\mathrm{III}}$, $E_{\mathrm{III}} - E_{\mathrm{I}}$; and the fictitious circulating currents are I_x, I_y, I_z.

Hence the general E.M.F. equations for the meshes are—

$$E_{\mathrm{I}} - (I_x - I_z)Z_{1t} - (I_x - I_y)Z_{2t} - E_{\mathrm{II}} = 0$$
$$E_{\mathrm{II}} - (I_y - I_x)Z_{2t} - (I_y - I_z)Z_{3t} - E_{\mathrm{III}} = 0,$$
$$E_{\mathrm{III}} - (I_z - I_y)Z_{3t} - (I_z - I_x)Z_{1t} - E_{\mathrm{I}} = 0$$

Since $I_x - I_z = I_1$, $I_y - I_x = I_2$, $I_z - I_y = I_3$, the preceding equations may be written in the form

$$I_1Z_{1t} - I_2Z_{2t} = E_{\mathrm{I}} - E_{\mathrm{II}} \quad . \quad . \quad . \quad (\alpha)$$
$$I_2Z_{2t} - I_3Z_{3t} = E_{\mathrm{II}} - E_{\mathrm{III}} \quad . \quad . \quad . \quad (\beta)$$
$$-I_1Z_{1t} + I_3Z_{3t} = E_{\mathrm{III}} - E_{\mathrm{I}} \quad . \quad . \quad . \quad (\gamma)$$

Applying Kirchoff's first law to the neutral point of the generator we have

$$I_1 + I_2 + I_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

The expressions for the line currents may be readily obtained from these four equations. For example, to obtain I_1, eliminate I_2 and I_3 from equations (α), (γ), (δ), thus

$$I_1Z_{1t}/Z_{2t} - I_2 = (E_{\mathrm{I}} - E_{\mathrm{II}})/Z_{2t} \quad . \quad . \quad . \quad (\alpha')$$
$$I_1Z_{1t}/Z_{3t} - I_3 = (E_{\mathrm{I}} - E_{\mathrm{III}})/Z_{3t} \quad . \quad . \quad . \quad (\gamma')$$
$$I_1 + I_2 + I_3 = 0 \quad . \quad . \quad . \quad . \quad (\delta')$$

Adding, we have

$$I_1\left(1 + \frac{Z_{1t}}{Z_{2t}} + \frac{Z_{1t}}{Z_{3t}}\right) = \frac{E_{\mathrm{I}} - E_{\mathrm{II}}}{Z_{2t}} + \frac{E_{\mathrm{I}} - E_{\mathrm{III}}}{Z_{3t}}$$

or
$$I_1Z_{1t}\left(\frac{1}{Z_{1t}} + \frac{1}{Z_{2t}} + \frac{1}{Z_{3t}}\right) = \frac{E_{\mathrm{I}} - E_{\mathrm{II}}}{Z_{2t}} + \frac{E_{\mathrm{I}} - E_{\mathrm{III}}}{Z_{3t}}$$

Whence

$$I_1 = \frac{Y_{1t}Y_{2t}(E_{\mathrm{I}} - E_{\mathrm{II}})}{Y_{1t} + Y_{2t} + Y_{3t}} + \frac{Y_{1t}Y_{3t}(E_{\mathrm{I}} - E_{\mathrm{III}})}{Y_{1t} + Y_{2t} + Y_{3t}} \quad . \quad . \quad (87)$$
$$= \frac{E_{\mathrm{I}} - E_{\mathrm{II}}}{Z_{1t} + Z_{2t} + Z_{1t}Z_{2t}/Z_{3t}} + \frac{E_{\mathrm{I}} - E_{\mathrm{III}}}{Z_{3t} + Z_{1t} + Z_{3t}Z_{1t}/Z_{2t}} \quad . \quad (87a)$$

Similarly,

$$I_2 = \frac{Y_{2t}Y_{3t}(E_{\mathrm{II}} - E_{\mathrm{III}})}{Y_{1t} + Y_{2t} + Y_{3t}} + \frac{Y_{2t}Y_{1t}(E_{\mathrm{II}} - E_{\mathrm{I}})}{Y_{1t} + Y_{2t} + Y_{3t}} \quad . \quad . \quad (88)$$
$$= \frac{E_{\mathrm{II}} - E_{\mathrm{III}}}{Z_{2t} + Z_{3t} + Z_{2t}Z_{3t}/Z_{1t}} + \frac{E_{\mathrm{II}} - E_{\mathrm{I}}}{Z_{1t} + Z_{2t} + Z_{1t}Z_{2t}/Z_{3t}} \quad . \quad (88a)$$

and

$$I_3 = \frac{Y_{3t}Y_{1t}(E_{III} - E_I)}{Y_{1t} + Y_{2t} + Y_{3t}} + \frac{Y_{3t}Y_{2t}(E_{III} - E_{II})}{Y_{1t} + Y_{2t} + Y_{3t}} \quad . \quad . \quad (89)$$

$$= \frac{E_{III} - E_I}{Z_{3t} + Z_{1t} + Z_{3t}Z_{1t}/Z_{2t}} + \frac{E_{III} - E_{II}}{Z_{2t} + Z_{3t} + Z_{2t}Z_{3t}/Z_{1t}} \quad . \quad (89a)$$

These are the general equations for the line currents.

With symmetrical systems, however, the equations can be expressed in simpler form. Thus, if the phase rotation is clockwise and the magnitude of the no-load E.M.F. of each phase is denoted by E, then

$$E_I = EJ^0, \quad E_{II} = EJ^{-120/90}, \quad E_{III} = EJ^{-240/90}$$

$$E_I - E_{II} = E(J^0 - J^{-120/90}) = E(1 - J^{-120/90})$$

$$E_I - E_{III} = E(J^0 - J^{-240/90}) = E(1 - J^{-240/90})$$

$$E_{II} - E_{III} = E(J^{-120/90} - J^{-240/90})$$

When these expressions are substituted in equation (87) we have, upon re-arrangement,

$$I_1 = \frac{Y_{1t}}{Y_{1t} + Y_{2t} + Y_{3t}} E\{Y_{2t}(1 - J^{-120/90}) + Y_{3t}(1 - J^{-240/90})\}$$

$$= \frac{Y_{1t}}{Y_{1t} + Y_{2t} + Y_{3t}} E(Y_{2t} + Y_{3t} - Y_{2t}J^{-120/90} - Y_{3t}J^{-240/90})$$

Introducing the quantity $Y_{1t}J^0$ into the right-hand side, and noting that $J^0 \equiv 1$, we have

$$I_1 = \frac{Y_{1t}}{Y_{1t} + Y_{2t} + Y_{3t}} E(Y_{1t} + Y_{2t} + Y_{3t} - (Y_{1t}J^0 + Y_{2t}J^{-120/90}) + Y_{3t}J^{-240/90})$$

$$= EY_{1t}\left(1 - \frac{Y_{1t}J^0 + Y_{2t}J^{-120/90} + Y_{3t}J^{-240/90}}{Y_{1t} + Y_{2t} + Y_{3t}}\right) \quad . \quad . \quad (87b)$$

Similarly when substitutions are made in equation (88) for E_I, E_{II}, and E_{III}, we have, upon re-arrangement,

$$I_2 = \frac{Y_{2t}}{Y_{1t} + Y_{2t} + Y_{3t}} E\{Y_{3t}(J^{-120/90} - J^{-240/90}) + Y_{1t}(J^{-120/90} - J^0)\}$$

Introducing the quantity $Y_{2t}J^{120/90}$,

$$I_2 = \frac{Y_{2t}}{Y_{1t}+Y_{2t}+Y_{3t}}E\left\{Y_{3t}(J^{120/90}-J^{240/90}) + Y_{2t}(J^{120/90}-J^{120/90}) + Y_{1t}(J^{120/90}-J^{0})\right\}$$

$$= \frac{Y_{2t}}{Y_{1t}+Y_{2t}+Y_{3t}}E\left\{1J^{120/90}(Y_{1t}+Y_{2t}+Y_{3t}) - (Y_{1t}J^{0}+Y_{2t}J^{120/90}+Y_{3t}J^{240/90})\right\}$$

$$= EY_{2t}\left(1J^{120/90} - \frac{Y_{1t}J^{0}+Y_{2t}J^{120/90}+Y_{3t}J^{240/90}}{Y_{1t}+Y_{2t}+Y_{3t}}\right) \qquad (88b)$$

Similarly, equation (89) finally reduces to

$$I_3 = EY_{3t}\left(1J^{240/90} - \frac{Y_{1t}J^{0}+Y_{2t}J^{120/90}+Y_{3t}J^{240/90}}{Y_{1t}+Y_{2t}+Y_{3t}}\right) \quad . \quad (89b)$$

These equations may be readily evaluated when Y_{1t}, Y_{2t}, Y_{3t} are known, since for clockwise phase rotation $J^{0} = 1 + j0 = 1$; $J^{120/90} = \cos - 120° - j \sin - 120° = -0.5 - j0.866$; $J^{240/90} = \cos - 240° - j \sin 240° = -0.5 + j0.866$

Whence

$$I_1 = EY_{1t}\left(1 - \frac{Y_{1t}+Y_{2t}(-0.5-j0.866)+Y_{3t}(-0.5+j0.866)}{Y_{1t}+Y_{2t}+Y_{3t}}\right) \quad . \quad . \quad (87c)$$

$$I_2 = EY_{2t}\left(-0.5-j0.866 - \frac{Y_{1t}+Y_{2t}(-0.5-j0.866)+Y_{3t}(-0.5+j0.866)}{Y_{1t}+Y_{2t}+Y_{3t}}\right) \quad . \quad (88c)$$

$$I_3 = EY_{3t}\left(-0.5+j0.866 - \frac{Y_{1t}+Y_{2t}(-0.5-j0.866)+Y_{3t}(-0.5+j0.866)}{Y_{1t}+Y_{2t}+Y_{3t}}\right) \quad . \quad (89c)$$

The difference of potential (V_0) between the neutral points of generator and load is given by

$$V_0 = E_{\mathrm{I}} - I_1Z_{1t} = E_{\mathrm{II}} - I_2Z_{2t} = E_{\mathrm{III}} - I_3Z_{3t}$$

Hence, with symmetrical systems and clockwise phase rotation,

$$V_0 = EJ^{0} - E\left(1J^{0} - \frac{Y_{1t}J^{0}+Y_{2t}J^{120/90}+Y_{3t}J^{240/90}}{Y_{1t}+Y_{2t}+Y_{3t}}\right)$$

$$= E\left(\frac{Y_{1t}J^{0}+Y_{2t}J^{120/90}+Y_{3t}J^{240/90}}{Y_{1t}+Y_{2t}+Y_{3t}}\right) \quad . \quad . \quad . \quad (90)$$

$$= E\left(\frac{Y_{1t}+Y_{2t}(-0.5-j0.866)+Y_{3t}(-0.5+j0.866)}{Y_{1t}+Y_{2t}+Y_{3t}}\right) \quad (90a)$$

With symmetrical systems and counter-clockwise phase rotation, we have

$$\dot{E}_{\mathrm{I}} = EJ^0, \qquad \dot{E}_{\mathrm{II}} = EJ^{120/90}, \qquad \dot{E}_{\mathrm{III}} = EJ^{240/90}$$
$$\dot{E}_{\mathrm{I}} - \dot{E}_{\mathrm{II}} = E(J^0 - J^{120/90}) = E(1 - J^{120/90})$$
$$\dot{E}_{\mathrm{I}} - \dot{E}_{\mathrm{III}} = E(J^0 - J^{240/90}) = E(1 - J^{240/90})$$
$$\dot{E}_{\mathrm{II}} - \dot{E}_{\mathrm{III}} = E(J^{120/90} - J^{240/90})$$

When these expressions are substituted in equations (87), (88), (89), we have, upon final re-arrangement and reduction,

$$\dot{I}_1 = E\dot{Y}_{1t}\left(1 - \frac{\dot{Y}_{1t}J^0 + \dot{Y}_{2t}J^{120/90} + \dot{Y}_{3t}J^{240/90}}{\dot{Y}_{1t} + \dot{Y}_{2t} + \dot{Y}_{3t}}\right) \quad . \quad . \quad (87d)$$

$$\dot{I}_2 = E\dot{Y}_{2t}\left(1J^{120/90} - \frac{\dot{Y}_{1t}J^0 + \dot{Y}_{2t}J^{120/90} + \dot{Y}_{3t}J^{240/90}}{\dot{Y}_{1t} + \dot{Y}_{2t} + \dot{Y}_{3t}}\right) \quad . \quad (88d)$$

$$\dot{I}_3 = E\dot{Y}_{3t}\left(1J^{240/90} - \frac{\dot{Y}_{1t}J^0 + \dot{Y}_{2t}J^{120/90} + \dot{Y}_{3t}J^{240/90}}{\dot{Y}_{1t} + \dot{Y}_{2t} + \dot{Y}_{3t}}\right) \quad . \quad (89d)$$

In the present case, $J^{120/90} = \cos 120° + j \sin 120° = -\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}$; $J^{240/90} = \cos 240° + j \sin 240° = -\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866}$. Whence

$$\dot{I}_1 = E\dot{Y}_{1t}\left(1 - \frac{\dot{Y}_{1t} + \dot{Y}_{2t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}) + \dot{Y}_{3t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})}{\dot{Y}_{1t} + \dot{Y}_{2t} + \dot{Y}_{3t}}\right) \quad . \quad . \quad (87e)$$

$$\dot{I}_2 = E\dot{Y}_{2t}\left(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866} - \frac{\dot{Y}_{1t} + \dot{Y}_{2t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}) + \dot{Y}_{3t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})}{\dot{Y}_{1t} + \dot{Y}_{2t} + \dot{Y}_{3t}}\right) \quad . \quad (88e)$$

$$\dot{I}_3 = E\dot{Y}_{3t}\left(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866} - \frac{\dot{Y}_{1t} + \dot{Y}_{2t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}) + \dot{Y}_{3t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})}{\dot{Y}_{1t} + \dot{Y}_{2t} + \dot{Y}_{3t}}\right) \quad . \quad (89e)$$

and the potential difference between the neutral points of generator and load is given by

$$\dot{V}_0 = E\left(\frac{\dot{Y}_{1t} + \dot{Y}_{2t}(-\mathbf{0{\cdot}5} + j\mathbf{0{\cdot}866}) + \dot{Y}_{3t}(-\mathbf{0{\cdot}5} - j\mathbf{0{\cdot}866})}{\dot{Y}_{1t} + \dot{Y}_{2t} + \dot{Y}_{3t}}\right) \quad (90b)$$

Case IIa. Calculation of currents in a star-connected load supplied from a large three-phase system. The system to which the load is connected is assumed to be so large that the line voltages are unaffected by any unbalancing of the loads.

The equations (87), (88), (89) deduced for the preceding case are therefore applicable to the present case if the line voltages are substituted for the differences between the phase E.M.Fs.

Thus

$$I_1 = \frac{Y_1}{Y_1 + Y_2 + Y_3} (Y_2 V_{1-2} - Y_3 V_{3-1}) \quad . \quad . \quad (91)$$

$$I_2 = \frac{Y_2}{Y_1 + Y_2 + Y_3} (Y_3 V_{2-3} - Y_1 V_{1-2}) \quad . \quad . \quad (92)$$

$$I_3 = \frac{Y_3}{Y_1 + Y_2 + Y_3} (Y_1 V_{3-1} \quad Y_2 V_{2-3}) \quad . \quad . \quad (93)$$

If we wish to calculate with impedances instead of admittances, then

$$I_1 = \frac{V_{1-2}}{Z_1 + Z_2 + Z_1 Z_2 / Z_3} - \frac{V_{3-1}}{Z_3 + Z_1 + Z_3 Z_1 / Z_2} \quad . \quad (91a)$$

$$I_2 = \frac{V_{2-3}}{Z_2 + Z_3 + Z_2 Z_3 / Z_1} - \frac{V_{1-2}}{Z_1 + Z_2 + Z_1 Z_2 / Z_3} \quad . \quad (92a)$$

$$I_3 = \frac{V_{3-1}}{Z_3 + Z_1 + Z_3 Z_1 / Z_2} - \frac{V_{2-3}}{Z_2 + Z_3 + Z_2 Z_3 / Z_1} \quad . \quad (93a)$$

With a symmetrical system and clockwise phase rotation

$$V_{1-2} = VJ^0; \qquad V_{2-3} = VJ^{-120/90}; \qquad V_{3-1} = VJ^{-240/90}$$

Under these conditions, we have

$$I_1 = V \left\{ \frac{Y_1}{Y_1 + Y_2 + Y_3} (Y_2 J^0 - Y_3 J^{-240/90}) \right\} \quad . \quad . \quad (91b)$$

$$= V \left(\frac{Y_1}{Y_1 + Y_2 + Y_3} \{ Y_2(1 + j0) - Y_3(-0{\cdot}5 + j0{\cdot}866) \} \right) \quad . \quad . \quad (91c)$$

$$I_2 = V \left\{ \frac{Y_2}{Y_1 + Y_2 + Y_3} (Y_3 J^{-120/90} - Y_1 J^0) \right\} \quad . \quad . \quad (92b)$$

$$= V \left(\frac{Y_2}{Y_1 + Y_2 + Y_3} \{ Y_3(-0{\cdot}5 - j0{\cdot}866) - Y_1(1 + j0) \} \right) \quad . \quad . \quad (92c)$$

$$I_3 = V \left\{ \frac{Y_3}{Y_1 + Y_2 + Y_3} (Y_1 J^{-240/90} - Y_2 J^{-120/90}) \right\} \quad . \quad (93b)$$

$$= V \left(\frac{Y_3}{Y_1 + Y_2 + Y_3} \{ Y_1(-0{\cdot}5 + j0{\cdot}866) - Y_2(-0{\cdot}5 - j0{\cdot}866) \} \right) \quad . \quad . \quad (93c)$$

or, alternatively,

$$I_1 = V\left(\frac{1 + j0}{Z_1 + Z_2 + Z_1Z_2/Z_3} - \frac{-0{\cdot}5 + j0{\cdot}866}{Z_3 + Z_1 + Z_3Z_1/Z_2}\right). \quad (91d)$$

$$I_2 = V\left(\frac{-0{\cdot}5 - j0{\cdot}866}{Z_2 + Z_3 + Z_2Z_3/Z_1} - \frac{1 + j0}{Z_1 + Z_2 + Z_1Z_2/Z_3}\right). \quad (92d)$$

$$I_3 = V\left(\frac{-0{\cdot}5 + j0{\cdot}866}{Z_3 + Z_1 + Z_3Z_1/Z_2} - \frac{-0{\cdot}5 - j0{\cdot}866}{Z_2 + Z_3 + Z_2Z_3/Z_1}\right). \quad (93d)$$

Example. The diagram (Fig. 174*a*) represents a star-connected unbalanced load which is connected to a three-phase system having sinusoidal E.M.Fs.,

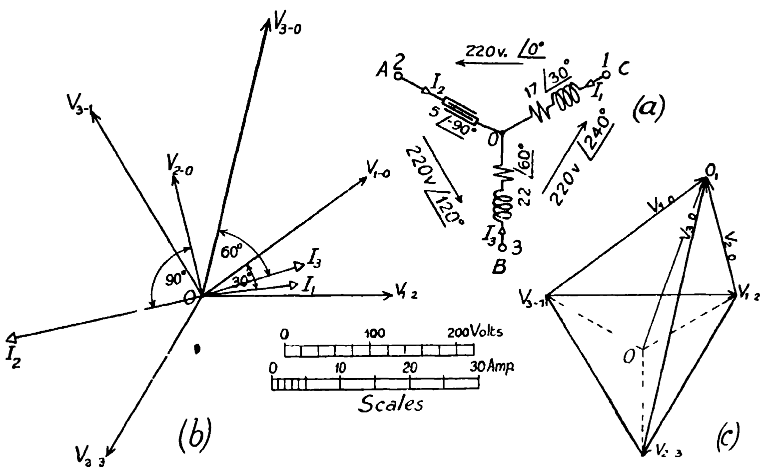


FIG. 174.—Vector Diagrams for Worked Example.

the phase relations between the E.M.Fs. being indicated in the diagram. Determine the current in the condenser branch OA. (L. U., 1921.)

[NOTE. The reactance of the condenser branch OA is $5/\underline{-90°}$ ohms, and the impedances of the other branches, OB, OC, are $22/\underline{60°}$ ohms, and $17/\underline{30°}$ ohms respectively.]

Denoting the terminals C, A, B, of the load by the numerals 1, 2, 3, respectively, we have, for the impedances of the branches,

$$Z_1 = 17/\underline{30°} = 17(\cos 30° + j\sin 30°) = \mathbf{14{\cdot}74 + j8{\cdot}5}$$

$$Z_2 = 5/\underline{-90°} = 5(\cos -90° + j\sin -90°) = \mathbf{0 - j5}$$

$$Z_3 = 22/\underline{60°} = 22(\cos 60° + j\sin 60°) = \mathbf{11 + j19{\cdot}06}$$

Whence $Z_1Z_2/Z_3 = (17 \times 5/22)J^{(30-90-60)/90} = 3{\cdot}86J^{-120/90}$

$$= 3{\cdot}86(-0{\cdot}5 - 0{\cdot}866)$$

$$= \mathbf{-1{\cdot}93 - j3{\cdot}35}$$

$$Z_2Z_3/Z_1 = (5 \times 22/17)J^{(-90+60-30)/90} = 6{\cdot}47J^{-60/90}$$
$$= 6{\cdot}47\,(0{\cdot}5 - j0{\cdot}866)$$
$$= \mathbf{3{\cdot}235 - j5{\cdot}6}$$
$$Z_1 + Z_2 + Z_1Z_2/Z_3 = \mathbf{12{\cdot}81 + j0{\cdot}15}$$
$$Z_2 + Z_3 + Z_2Z_3/Z_1 = \mathbf{14{\cdot}23 + j8{\cdot}46}$$

Hence, from equation (92d), we have

$$I_2 = V\left(\frac{-0{\cdot}5 - j0{\cdot}866}{Z_2 + Z_3 + Z_2Z_3/Z_1} - \frac{1 + j0}{Z_1 + Z_2 + Z_1Z_2/Z_3}\right)$$
$$= 220\left(\frac{(-0{\cdot}5 - j0{\cdot}866)\,(14{\cdot}23 - j8{\cdot}46)}{14{\cdot}23^2 + 8{\cdot}46^2} - \frac{12{\cdot}81 - j0{\cdot}15}{12{\cdot}81^2 + 0{\cdot}15^2}\right)$$
$$= \mathbf{-28{\cdot}78 - j6{\cdot}3}$$

Whence $I_2 = \sqrt{(28{\cdot}78^2 + 6{\cdot}3^2)} = 29{\cdot}46$ A.

Phase difference between I_2 and potential difference (V_{1-2}) between lines 1 and 2 $= \tan^{-1} - 6{\cdot}3/-28{\cdot}78 = -(180 - 12{\cdot}4)° = -167{\cdot}6°$.

As an extension of the problem, it will be of interest to calculate the currents in the other branches and the potential difference across each branch, and to draw a vector diagram for the circuit.

First, the quantity $Z_3 + Z_1 + Z_3Z_1/Z_2$ is evaluated, thus

$$Z_3Z_1/Z_2 = (17 \times 22/5)\,J^{(30+60+90)/90} = 74{\cdot}8\,J^{180/90}$$
$$= 74{\cdot}8(-\mathbf{1} + \mathbf{j0})$$
$$Z_3 + Z_1 + Z_3Z_1/Z_2 = \mathbf{-49 + j27{\cdot}55}$$

From equation (91d) we have, for the current in the inductive branch OC,

$$I_1 = V\left(\frac{1 + j0}{Z_1 + Z_2 + Z_1Z_2/Z_3} - \frac{-0{\cdot}5 + j0{\cdot}866}{Z_3 + Z_1 + Z_3Z_1/Z_2}\right)$$
$$= 220\left(\frac{12{\cdot}81 - j0{\cdot}15}{12{\cdot}81^2 + 0{\cdot}15^2} - \frac{(-0{\cdot}5 + j0{\cdot}866)\,(-49 + j27{\cdot}55)}{49^2 + 27{\cdot}55^2}\right)$$
$$= \mathbf{13{\cdot}85 + j1{\cdot}794}$$

Whence

$$I_1 = \sqrt{(13{\cdot}85^2 + 1{\cdot}794^2)} = 13{\cdot}96 \text{ A.}$$

Phase difference between I_1 and V_{1-2} is $\tan^{-1} 1{\cdot}794/13{\cdot}85 = 7{\cdot}4°$

Also from equation (93d) the current in the inductive branch OB is given by

$$I_3 = V\left(\frac{-0{\cdot}5 + j0{\cdot}866}{Z_3 + Z_1 + Z_3Z_1/Z_2} - \frac{-0{\cdot}5 - j0{\cdot}866}{Z_2 + Z_3 + Z_2Z_3/Z_1}\right)$$
$$= 220\left(\frac{(-0{\cdot}5 + j0{\cdot}866)\,(-49 - j27{\cdot}55)}{49^2 + 27{\cdot}55^2} - \frac{(-0{\cdot}5 - j0{\cdot}866)\,(14{\cdot}23 - j8{\cdot}46)}{14{\cdot}23^2 + 8{\cdot}46^2}\right)$$
$$= \mathbf{14{\cdot}94 + j4{\cdot}5}$$

Whence

$I_3 = \sqrt{(14{\cdot}94^2 + 4{\cdot}5^2)} = 15{\cdot}6$ A.

Phase difference between I_3 and $V_{1\text{-}2}$ is $\tan^{-1} 4{\cdot}5/14{\cdot}94 = 16{\cdot}8°$.

The potential difference across the branch OA is given by

$V_{2\text{-}0} = I_2 Z_2 = (-28{\cdot}78 - j6{\cdot}3)\ (0 - j5) = -31{\cdot}5 + j144$

Whence

$V_{2\text{-}0} = \sqrt{(31{\cdot}5^2 + 144)} = 147{\cdot}6$ V.

Phase difference between $V_{1\text{-}2}$ and $V_{2\text{-}0}$ is

$\tan^{-1} 144/-31{\cdot}5 = (180 - 77{\cdot}6)° = 102{\cdot}4°$

The potential difference across the branch OC is given by

$V_{1\text{-}0} = I_1 Z_1 = (13{\cdot}85 + j1{\cdot}794)\ (14{\cdot}74 + j8{\cdot}5) = 188{\cdot}8 + j144$

Whence

$V_{1\text{-}0} = \sqrt{(188{\cdot}8^2 + 144^2)} = 237$ V.

Phase difference between $V_{1\text{-}2}$ and $V_{1\text{-}0}$ is $\tan^{-1} 144/188{\cdot}8 = 37{\cdot}4°$.

The potential difference across the branch OB is given by

$V_{3\text{-}0} = I_3 Z_3 = (14{\cdot}94 + j4{\cdot}5)\ (11 + j19{\cdot}06) = 78{\cdot}7 + j334{\cdot}5$

Whence

$V_{3\text{-}0} = \sqrt{(78{\cdot}7^2 + 334{\cdot}5^2)} = 344$ V.

Phase difference between $V_{1\text{-}2}$ and $V_{3\text{-}0}$ is $\tan^{-1} 334{\cdot}5/78{\cdot}7 = 76{\cdot}8°$.

Check calculations for these potential differences are as follow—

$V_{2\text{-}0} = I_2 Z_2 = 29{\cdot}46 \times 5 = 147{\cdot}3$ V.

$V_{1\text{-}0} = I_1 Z_1 = 13{\cdot}96 \times 17 = 237$ V.

$V_{3\text{-}0} = I_3 Z_3 = 15{\cdot}6 \times 22 = 344$ V.

$$V_{1\text{-}0} - V_{2\text{-}0} (= V_{1\text{-}2}) = (188{\cdot}8 + j144) - (-31{\cdot}5 + j144) = 220{\cdot}3 + j0$$

$$V_{2\text{-}0} - V_{3\text{-}0} (= V_{2\text{-}3}) = (-31{\cdot}5 + j144) - (78{\cdot}7 + j334{\cdot}5) = -110{\cdot}2 - j190{\cdot}5$$

$$V_{3\text{-}0} - V_{1\text{-}0} (= V_{3\text{-}1}) = (78{\cdot}7 + 334{\cdot}5) - (188{\cdot}8 + j\,144) = -110{\cdot}1 + j190{\cdot}5$$

The vector diagram for the load circuit is given in Fig. 174*b*, in which the line voltages are represented by the vectors $OV_{1\text{-}2}$, $OV_{2\text{-}3}$, $OV_{3\text{-}1}$, the voltages across the branches of the load by $OV_{1\text{-}0}$, $OV_{2\text{-}0}$, $OV_{3\text{-}0}$, and the line currents by OI_1, OI_2, OI_3.

If the line-voltage vectors are drawn in the form of a triangle, as in Fig. 174*c*, and the vectors representing the voltages across

the branches of the load are so drawn from the corners of this triangle as to meet in a common point, O_1, then this point represents the potential of the neutral point of the load. Moreover, since the point, O (which is the centre of gravity of the line-voltage vector triangle) represents the potential of the neutral point of the supply system (which, according to the circuit diagram (Fig. 174*a*), is symmetrical), the potential difference between the two neutral points is given by

$$\dot{V}_0 = \dot{V}_{10} - \dot{V}'_{10} = (\mathbf{188{\cdot}8} + j\mathbf{144}) - (\mathbf{0{\cdot}866} - j\mathbf{0{\cdot}5})220/\sqrt{3}$$
$$= \mathbf{77{\cdot}8} + j\mathbf{207{\cdot}5}$$

where $\dot{V}'_{10}[= (\mathbf{0{\cdot}866} - j\mathbf{0{\cdot}5})220/\sqrt{3}\,]$ denotes the potential difference between line 2 and the neutral point of the supply system.

Whence

$$\dot{V}_0 = \sqrt{(77{\cdot}8^2 + 207{\cdot}5^2)} = 221{\cdot}5 \text{ V}.$$

Case III. Calculation of currents in three-phase, three-wire system with delta-connected generator and star-connected load. Let the no-load

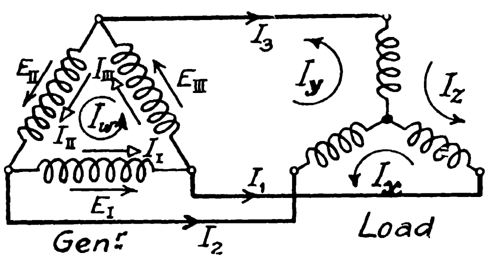


FIG. 175.—Circuit Diagram for Delta/Star Three-phase System

phase E.M.Fs. of the generator be denoted by $\dot{E}_I$, $\dot{E}_{II}$, $\dot{E}_{III}$; the phase currents by $\dot{I}_I$, $\dot{I}_{II}$, $\dot{I}_{III}$; the corresponding line currents by $\dot{I}_1$, $\dot{I}_2$, $\dot{I}_3$; and the impedances of the load, connecting line wires and generator, by $\dot{Z}_1$, $\dot{Z}_{1l}$, $\dot{Z}_{1g}$, etc. The delta-star system is then equivalent to four meshes (Fig. 175), of which one mesh is formed by the delta-connected generator, and three meshes by the interconnected phases of the load, the line wires, and the phases of the generator. Let the fictitious circulating currents in these meshes be denoted by $\dot{I}_w$, $\dot{I}_x$, $\dot{I}_y$, $\dot{I}_z$.

Then the E.M.F. equations for the meshes are

$$\dot{E}_I + \dot{E}_{II} + \dot{E}_{III} - (\dot{I}_w + \dot{I}_x)\dot{Z}_{1g} - (\dot{I}_w + \dot{I}_y)\dot{Z}_{2g} - (\dot{I}_w + \dot{I}_z)\dot{Z}_{3g} = 0 \quad . \quad . \quad (a)$$

$$\dot{E}_{\text{I}} - (\dot{I}_w + \dot{I}_x)\dot{Z}_{1g} - (\dot{I}_x - \dot{I}_z)(\dot{Z}_1 + \dot{Z}_{1l}) - (\dot{I}_x - \dot{I}_y)(\dot{Z}_2 + \dot{Z}_{2l}) = 0 \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

$$\dot{E}_{\text{II}} - (\dot{I}_w + \dot{I}_y)\dot{Z}_{2g} - (\dot{I}_y - \dot{I}_x)(\dot{Z}_2 + \dot{Z}_{2l}) - (\dot{I}_y - \dot{I}_z)(\dot{Z}_3 + \dot{Z}_{3l}) = 0 \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

$$\dot{E}_{\text{III}} - (\dot{I}_w + \dot{I}_z)\dot{Z}_{3g} - (\dot{I}_z - \dot{I}_y)(\dot{Z}_3 + \dot{Z}_{3l}) - (\dot{I}_z - \dot{I}_x)(\dot{Z}_1 + \dot{Z}_{1l}) = 0 \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

Now $\dot{I}_w + \dot{I}_x = \dot{I}_{\text{I}}$; $\dot{I}_w + \dot{I}_y = \dot{I}_{\text{II}}$; $\dot{I}_w + \dot{I}_z = \dot{I}_{\text{III}}$;
$\dot{I}_x - \dot{I}_z = \dot{I}_1 = \dot{I}_{\text{I}} - \dot{I}_{\text{III}}$; $\dot{I}_y - \dot{I}_x = \dot{I}_2 = \dot{I}_{\text{II}} - \dot{I}_{\text{I}}$;
$\dot{I}_z - \dot{I}_y = \dot{I}_3 = \dot{I}_{\text{III}} - \dot{I}_{\text{I}}$.

Substituting these values in the above equations, we obtain a set of equations containing the phase currents, I_{I}, I_{II}, I_{III}, as the unknown quantities. Thus

$$\dot{E}_{\text{I}} + \dot{E}_{\text{II}} + \dot{E}_{\text{III}} - \dot{I}_{\text{I}}\dot{Z}_{1g} - \dot{I}_{\text{II}}\dot{Z}_{2g} - \dot{I}_{\text{III}}\dot{Z}_{3g} = 0 \quad . \quad . \quad (\alpha')$$

$$\dot{E}_{\text{I}} - \dot{I}_{\text{I}}(\dot{Z}_{1g} + \dot{Z}_1 + \dot{Z}_{1l} + \dot{Z}_2 + \dot{Z}_{2l}) + \dot{I}_{\text{II}}(\dot{Z}_2 + \dot{Z}_{2l}) = 0 \quad . \quad (\beta')$$

$$\dot{E}_{\text{II}} + \dot{I}_{\text{I}}(\dot{Z}_2 + \dot{Z}_{2l}) - \dot{I}_{\text{II}}(\dot{Z}_{2g} + \dot{Z}_2 + \dot{Z}_{2l} + \dot{Z}_3 + \dot{Z}_{3l}) + \dot{I}_{\text{III}}(\dot{Z}_3 + \dot{Z}_{3l}) = 0. \quad . \quad (\gamma')$$

$$\dot{E}_{\text{III}} + \dot{I}_{\text{I}}(\dot{Z}_1 + \dot{Z}_{1l}) + \dot{I}_{\text{II}}(\dot{Z}_3 + \dot{Z}_{3l}) - \dot{I}_{\text{III}}(\dot{Z}_{3g} + \dot{Z}_3 + \dot{Z}_{3l} + \dot{Z}_1 + \dot{Z}_{1l}) = 0 \quad . \quad (\delta')$$

The solution to these equations is easily obtained by means of determinants.

Note to the solution of simultaneous equations by determinants. If we have three simultaneous equations, each of which contains three unknown quantities, x, y, z, and if the equations are expressed in the form

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

the solution is given by

$$x = -(D_1/D); \quad y = D_2/D; \quad z = -(D_3/D),$$

where D_1, D_2, D_3, D denote the following determinants of the third order—

$$D_1 = \begin{vmatrix} d_1b_1c_1 \\ d_2b_2c_2 \\ d_3b_3c_3 \end{vmatrix} \qquad D_2 = \begin{vmatrix} d_1a_1c_1 \\ d_2a_2c_2 \\ d_3a_3c_3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} d_1a_1b_1 \\ d_2a_2b_2 \\ d_3a_3b_3 \end{vmatrix} \qquad D = \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} \neq 0$$

Now
$$D_1 = d_1(b_2c_3 - b_3c_2) + b_1(c_2d_3 - c_3d_2) + c_1(d_2b_3 - d_3b_2)$$
$$D_2 = d_1(a_2c_3 - a_3c_2) + a_1(c_2d_3 - c_3d_2) + c_1(d_2a_3 - d_3a_2)$$
$$D_3 = d_1(a_2b_3 - a_3b_2) + a_1(b_2d_3 - b_3d_2) + b_1(d_2a_3 - d_3a_2)$$
$$D = a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$$

Alternatively, a solution may be obtained by ordinary algebra by substituting for, say, I_{I} in terms of I_{II} and I_{III}, in two of the equations and solving these, as ordinary simultaneous equations, for I_{II} and I_{III}. I_{I} is then obtained by substituting for I_{II} and I_{III} in one of the original equations. This process, however, usually involves more labour than the solution by determinants.

Substituting Z_{1t}', Z_{2t}', Z_{3t}', for the quantities $(Z_1 + Z_{1l})$, $(Z_2 + Z_{2l})$, $(Z_3 + Z_{3l})$, respectively, in the equations (β'), (γ'), (δ') for the delta-star circuit, and assuming the generator impedances to have equal values, Z_g, we obtain*

$$-I_{\mathrm{I}}(Z_g + Z_{1t}' + Z_{2t}') + I_{\mathrm{II}}Z_{2t}' + I_{\mathrm{III}}Z_{1t}' + E_{\mathrm{I}} = 0 \quad . \quad . \quad . \quad . \quad (\beta'')$$

$$+ I_{\mathrm{I}}Z_{2t}' - I_{\mathrm{II}}(Z_g + Z_{2t}' + Z_{3t}') + I_{\mathrm{III}}Z_{3t}' + E_{\mathrm{II}} = 0 \quad . \quad . \quad . \quad . \quad (\gamma'')$$

$$+ I_{\mathrm{I}}Z_{1t}' + I_{\mathrm{II}}Z_{3t}' - I_{\mathrm{III}}(Z_g + Z_{3t}' + Z_{1t}') + E_{\mathrm{III}} = 0 \quad . \quad . \quad . \quad . \quad (\delta'')$$

The determinants to these equations are—

$$D_1 = \begin{vmatrix} E_{\mathrm{I}} & + Z_{2t}' & + Z_{1t}' \\ E_{\mathrm{II}} & -(Z_g + Z_{2t}' + Z_{3t}') & + Z_{3t}' \\ E_{\mathrm{III}} & + Z_{3t}' & -(Z_g + Z_{3t}' + Z_{1t}') \end{vmatrix}$$

$$\begin{aligned} &= E_{\mathrm{I}}[(Z_g + Z_{2t}' + Z_{3t}')(Z_g + Z_{3t}' + Z_{1t}') - Z_{3t}'^2] \\ &\quad + Z_{2t}'[Z_{3t}'E_{\mathrm{III}} + E_{\mathrm{II}}(Z_g + Z_{3t}' + Z_{1t}')] \\ &\quad + Z_{1t}'[E_{\mathrm{II}}Z_{3t}' + E_{\mathrm{III}}(Z_g + Z_{2t}' + Z_{3t}')] \\ &= (E_{\mathrm{I}} + E_{\mathrm{II}} + E_{\mathrm{III}})(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}') \\ &\quad + Z_g[E_{\mathrm{I}}(Z_g + 2Z_{3t}') + Z_{1t}'(E_{\mathrm{I}} + E_{\mathrm{III}}) + Z_{2t}'(E_{\mathrm{I}} + E_{\mathrm{II}})] \end{aligned}$$

$$D_2 = \begin{vmatrix} E_{\mathrm{I}} & -(Z_g + Z_{1t}' + Z_{2t}') & + Z_{1t}' \\ E_{\mathrm{II}} & + Z_{2t}' & + Z_{3t}' \\ E_{\mathrm{III}} & + Z_{1t}' & -(Z_g + Z_{3t}' + Z_{1t}') \end{vmatrix}$$

$$\begin{aligned} &= E_{\mathrm{I}}[- Z_{2t}'(Z_g + Z_{3t}' + Z_{1t}') - Z_{1t}'Z_{3t}'] \\ &\quad - (Z_g + Z_{1t}' + Z_{2t}')[Z_{3t}'E_{\mathrm{III}} + (Z_g + Z_{3t}' + Z_{1t}')E_{\mathrm{II}}] \\ &\quad + Z_{1t}'[E_{\mathrm{II}}Z_{1t}' - E_{\mathrm{III}}Z_{2t}'] \\ &= - \{(E_{\mathrm{I}} + E_{\mathrm{II}} + E_{\mathrm{III}})(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}') \\ &\quad + Z_g[E_{\mathrm{II}}(Z_g + 2Z_{1t}') + Z_{2t}'(E_{\mathrm{II}} + E_{\mathrm{I}}) + Z_{3t}'(E_{\mathrm{II}} + E_{\mathrm{III}}) +]\} \end{aligned}$$

* Since equation (α'), p. 285, represents the sum of equations (β'), (γ'), (δ') its further consideration is unnecessary for the solution by determinants.

$$D_3 = \begin{vmatrix} E_{\text{I}} & -(Z_g + Z_{1t}' + Z_{2t}') & + Z_{2t}' \\ E_{\text{II}} & + Z_{2t}' & -(Z_g + Z_{2t}' + Z_{3t}') \\ E_{\text{III}} & + Z_{1t}' & + Z_{3t}' \end{vmatrix}$$

$$\begin{aligned}
&= E_{\text{I}}[Z_{2t}'Z_{3t}' + Z_{1t}'(Z_g + Z_{2t}' + Z_{3t}')] \\
&\quad - (Z_g + Z_{1t}' + Z_{2t}')[-(Z_g + Z_{2t}' + Z_{3t}')E_{\text{III}} - Z_{3t}'E_{\text{II}}] \\
&\quad + Z_{2t}'[E_{\text{II}}Z_{1t}' - E_{\text{III}}Z_{2t}'] \\
&= (E_{\text{I}} + E_{\text{II}} + E_{\text{III}})(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}') \\
&\quad + Z_g[E_{\text{III}}(Z_g + 2Z_{2t}') + Z_{3t}'(E_{\text{III}} + E_{\text{II}}) + Z_{1t}'(E_{\text{III}} + E_{\text{I}})]
\end{aligned}$$

$$D = \begin{vmatrix} -(Z_g + Z_{1t}' + Z_{2t}') & + Z_{2t}' & + Z_{1t}' \\ + Z_{2t}' & -(Z_g + Z_{2t}' + Z_{3t}') & + Z_{3t}' \\ + Z_{1t}' & + Z_{3t}' & -(Z_g + Z_{3t}' + Z_{1t}') \end{vmatrix}$$

$$\begin{aligned}
&= -(Z_g + Z_{1t}' + Z_{2t}')[(Z_g + Z_{2t}' + Z_{3t}')(Z_g + Z_{3t}' + Z_{1t}') - Z_{3t}'^2] \\
&\quad + Z_{2t}'[Z_{3t}'Z_{1t}' + (Z_g + Z_{3t}' + Z_{1t}')Z_{2t}'] \\
&\quad + Z_{1t}'[Z_{2t}'Z_{3t}' + Z_{1t}'(Z_g + Z_{2t}' + Z_{3t}')] \\
&= -\{Z_g[Z_g^2 + 2Z_g(Z_{1t}' + Z_{2t}' + Z_{3t}')] \\
&\quad + 3(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')\}
\end{aligned}$$

Hence $\quad I_{\text{I}} = -(D_1/D), \qquad I_{\text{II}} = (D_2/D), \qquad I_{\text{III}} = -(D_3/D).$

In the case of a *symmetrical system* the expressions for the determinants may be considerably simplified, since $E_{\text{I}} + E_{\text{II}} + E_{\text{III}} = 0$; $E_{\text{I}} + E_{\text{II}} = -E_{\text{III}}$; $E_{\text{II}} + E_{\text{III}} = -E_{\text{I}}$; $E_{\text{III}} + E_{\text{I}} = -E_{\text{II}}$. Thus

$$\begin{aligned}
D_1 &= Z_g[E_{\text{I}}(Z_g + 2Z_{3t}') - E_{\text{II}}Z_{1t}' - E_{\text{III}}Z_{2t}'] \\
D_2 &= -Z_g[E_{\text{II}}(Z_g + 2Z_{1t}') - E_{\text{III}}Z_{2t}' - E_{\text{I}}Z_{3t}'] \\
D_3 &= Z_g[E_{\text{III}}(Z_g + 2Z_{2t}') - E_{\text{I}}Z_{3t}' - E_{\text{II}}Z_{1t}']
\end{aligned}$$

Whence

$$I_{\text{I}} = \frac{E_{\text{I}}(Z_g + 2Z_{3t}') - E_{\text{II}}Z_{1t}' - E_{\text{III}}Z_{2t}'}{Z_g[Z_g + 2(Z_{1t}' + Z_{2t}' + Z_{3t}')] + 3(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')} \quad . \quad . \quad (94)$$

$$I_{\text{II}} = \frac{E_{\text{II}}(Z_g + 2Z_{1t}') - E_{\text{III}}Z_{2t}' - E_{\text{I}}Z_{3t}'}{Z_g[Z_g + 2(Z_{1t}' + Z_{2t}' + Z_{3t}')] + 3(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')} \quad . \quad . \quad (95)$$

$$I_{\text{III}} = \frac{E_{\text{III}}(Z_g + 2Z_{2t}') - E_{\text{I}}Z_{3t}' - E_{\text{II}}Z_{1t}'}{Z_g[Z_g + 2(Z_{1t}' + Z_{2t}' + Z_{3t}')] + 3(Z_{1t}'Z_{2t}' + Z_{2t}'Z_{3t}' + Z_{3t}'Z_{1t}')} \quad . \quad . \quad (96)$$

The line currents I_1, I_2, I_3 are readily obtained from the phase currents, since $I_1 = I_{\mathrm{I}} - I_{\mathrm{III}}$, $I_2 = I_{\mathrm{II}} - I_{\mathrm{I}}$, $I_3 = I_{\mathrm{III}} - I_{\mathrm{II}}$.

In the *special case* when the system is symmetrical and the impedances of the branches of the load and the connecting line wires have the same value, Z_t', equations (94), (95), (96) reduce to

$$I_{\mathrm{I}} = \frac{E_{\mathrm{I}}(Z_g + 3Z_t')}{Z_g(Z_g + 6Z_t') + 9Z_t'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (94a)$$

$$I_{\mathrm{II}} = \frac{E_{\mathrm{II}}(Z_g + 3Z_t')}{Z_g(Z_g + 6Z_t') + 9Z_t'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (95a)$$

$$I_{\mathrm{III}} = \frac{E_{\mathrm{III}}(Z_g + 3Z_t')}{Z_g(Z_g + 6Z_t') + 9Z_t'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (96a)$$

Again, if the impedance of the generator is ignored and the no-load phase E.M.Fs. are replaced by the potential differences $V_{1\text{-}2}$, $V_{2\text{-}3}$, $V_{3\text{-}1}$ between the terminals of the generator, we have

$$I_{\mathrm{I}} = \frac{3V_{1\text{-}2}Z_t'}{9Z_t'^2} = \frac{V_{1\text{-}2}}{3Z_t'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94b)$$

$$I_{\mathrm{II}} = \frac{3V_{2\text{-}3}Z_t'}{9Z_t'^2} = \frac{V_{2\text{-}3}}{3Z_t'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95b)$$

$$I_{\mathrm{III}} = \frac{3V_{3\text{-}1}Z_t'}{9Z_t'^2} = \frac{V_{3\text{-}1}}{3Z_t'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96b)$$

Numerically, if the terminal pressure of the generator is V, the current (per phase) in the generator is given by

$$I_{ph} = V/3Z_t',$$

and the line (or load) current is given by

$$I \quad = I_{ph}\sqrt{3} = (V/\sqrt{3})/Z_t'.$$

Case IV. Calculation of currents in three-phase, three-wire system in which both generator and load are delta-connected. This system may be considered as a network containing five meshes (Fig. 176), and accordingly five E.M.F. equations may be formed. Instead of assuming fictitious currents in each of the meshes, as hitherto, it will be more convenient in the present case to consider, as far as possible, the actual currents in the conductors forming the meshes, and obtain the E.M.F. equations for these conditions. By these means the number of unknown quantities in the equations are

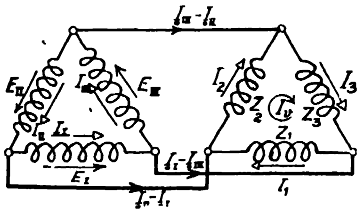


FIG. 176.—Circuit Diagram for Delta/Delta Three-phase System

reduced and expressions are obtained directly for the actual currents.

For example, if the generator currents are denoted by I_{I}, I_{II}, I_{III}, the currents in the line wires will be given by $I_{I} - I_{III}$, $I_{II} - I_{I}$, $I_{III} - I_{II}$, and the currents in the branches of the load by $(I_{I} - I_{III} + I_{v})$ $(I_{II} - I_{I} + I_{v})$, $(I_{III} - I_{II} + I_{v})$, where I_{v} is the assumed fictitious current circulating in the load mesh.

The five E.M.F. equations for the network are

$$E_{I} - I_{I} Z_{1g} - (I_{I} - I_{III})Z_{1l} - (I_{I} - I_{III} + I_{v})Z_{1} + (I_{II} - I_{I})Z_{2l} = 0 \quad . \qquad (\alpha)$$

$$E_{II} - I_{II} Z_{2g} - (I_{II} - I_{I})Z_{2l} - (I_{II} - I_{I} + I_{v})Z_{2} + (I_{III} - I_{II})Z_{3l} = 0 \quad . \qquad (\beta)$$

$$E_{III} - I_{III} Z_{3g} - (I_{III} - I_{II})Z_{3l} - (I_{III} - I_{II} + I_{v})Z_{3} + (I_{I} - I_{III})Z_{1l} = 0 \quad . \qquad (\gamma)$$

$$E_{I} + E_{II} + E_{III} - I_{I}Z_{1g} - I_{II}Z_{2g} - I_{III}Z_{3g} = 0 \quad . \quad . \qquad (\delta)$$

$$(I_{I} - I_{III} + I_{v})Z_{1} + (I_{II} - I_{I} + I_{v})Z_{2} + (I_{III} - I_{II} + I_{v})Z_{3} = 0 \quad . \qquad (\varepsilon)$$

which, when re-arranged, give

$$E_{I} - I_{I}(Z_{1g} + Z_{1l}' + Z_{2l}) + I_{II}Z_{2l} + I_{III}Z_{1l}' - I_{v}Z_{1} = 0 \qquad (\alpha')$$

$$E_{II} + I_{I}Z_{2l}' - I_{II}(Z_{2g} + Z_{2l}' + Z_{3l}) + I_{III}Z_{3l} - I_{v}Z_{2} = 0 \qquad (\beta')$$

$$E_{III} + I_{I}Z_{1l} + I_{II}Z_{3l}' - I_{III}(Z_{3g} + Z_{3l}' + Z_{1l}) - I_{v}Z_{3} = 0 \qquad (\gamma')$$

$$E_{I} + E_{II} + E_{III} - I_{I}Z_{1g} - I_{II}Z_{2g} - I_{III}Z_{3g} = 0 \qquad (\delta')$$

$$I_{I}(Z_{1} + Z_{2}) + I_{II}(Z_{2} + Z_{3}) + I_{III}(Z_{3} + Z_{1}) + I_{v}(Z_{1} + Z_{2} + Z_{3}) = 0 \qquad (\varepsilon')$$

From equation (ε') we obtain I_{v} in terms of I_{I}, I_{II}, I_{III}, Z_{1}, Z_{2}, Z_{3}, thus*

$$I_{v} = \frac{I_{I}(Z_{1} - Z_{2}) + I_{II}(Z_{2} - Z_{3}) + I_{III}(Z_{3} - Z_{1})}{Z_{1} + Z_{2} + Z_{3}} \quad . \qquad (\varepsilon'')$$

* Observe that $I_{v} = 0$ when $Z_{1} = Z_{2} = Z_{3}$

and if this value be substituted in equations (α'), (β'), (γ'), we have then only three unknown quantities $\dot{I}_{\mathrm{I}}$, $\dot{I}_{\mathrm{II}}$, $\dot{I}_{\mathrm{III}}$. Thus

$$-\dot{I}_{\mathrm{I}}\left(\dot{Z}_{1g} + \dot{Z}_{1t}' + \dot{Z}_{2t} + \frac{\dot{Z}_1(\dot{Z}_1 - \dot{Z}_2)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right)$$
$$+ \dot{I}_{\mathrm{II}}\left(\dot{Z}_{2t} + \frac{\dot{Z}_1(\dot{Z}_2 - \dot{Z}_3)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right)$$
$$+ \dot{I}_{\mathrm{III}}\left(\dot{Z}_{1t}' + \frac{\dot{Z}_1(\dot{Z}_3 - \dot{Z}_1)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right) + \dot{E}_{\mathrm{I}} = 0 \quad . \quad . \quad (\alpha'')$$

$$\dot{I}_{\mathrm{I}}\left(\dot{Z}_{2t}' + \frac{\dot{Z}_2(\dot{Z}_1 - \dot{Z}_2)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right)$$
$$- \dot{I}_{\mathrm{II}}\left(\dot{Z}_{2g} + \dot{Z}_{2t}' + \dot{Z}_{3t} + \frac{\dot{Z}_2(\dot{Z}_2 - \dot{Z}_3)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right)$$
$$+ \dot{I}_{\mathrm{III}}\left(\dot{Z}_{3t} + \frac{\dot{Z}_2(\dot{Z}_3 - \dot{Z}_1)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right) + \dot{E}_{\mathrm{II}} = 0 \quad . \quad (\beta'')$$

$$+ \dot{I}_{\mathrm{I}}\left(\dot{Z}_{1t} + \frac{\dot{Z}_3(\dot{Z}_1 - \dot{Z}_2)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right)$$
$$+ \dot{I}_{\mathrm{II}}\left(\dot{Z}_{3t}' + \frac{\dot{Z}_3(\dot{Z}_2 - \dot{Z}_3)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right)$$
$$- \dot{I}_{\mathrm{III}}\left(\dot{Z}_{3g} + \dot{Z}_{3t}' + \dot{Z}_{1t} + \frac{\dot{Z}_3(\dot{Z}_3 - \dot{Z}_1)}{\dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3}\right) + \dot{E}_{\mathrm{III}} = 0 \quad (\gamma'')$$

These equations may be solved in a similar manner to those of the preceding section, but in view of the number of terms in each of the coefficients it is desirable to evaluate these before the general solution is attempted. For this reason no general expressions are given here for the generator phase currents $\dot{I}_{\mathrm{I}}$, $\dot{I}_{\mathrm{II}}$, $\dot{I}_{\mathrm{III}}$.

When the generator phase currents have been determined, the fictitious current, $\dot{I}_v$, circulating in the load is calculated by means of equation (ε''), and the actual load currents are then readily determined, since

$$\dot{I}_1 = \dot{I}_{\mathrm{I}} - \dot{I}_{\mathrm{III}} + \dot{I}_v\,; \quad \dot{I}_2 = \dot{I}_{\mathrm{II}} - \dot{I}_{\mathrm{I}} + \dot{I}_v\,; \quad \dot{I}_3 = \dot{I}_{\mathrm{III}} - \dot{I}_{\mathrm{II}} + \dot{I}_v$$

Case V. Calculation of currents in three-phase, four-wire system (i.e. three-phase, star-connected system with neutral wire). This system may be considered as a network containing three meshes

(Fig. 177), and accordingly three E.M.F. equations may be formed. Adopting the same symbols as in the previous cases, and denoting the line currents by I_1, I_2, I_3, I_0, and the impedance of the neutral wire by Z_{0l}, we have

$$E_{\text{I}} - I_1(Z_{1g} + Z_{1l} + Z_1) - (I_1 + I_2 + I_3)Z_{0l} = 0 \quad . \quad (\alpha)$$

$$E_{\text{II}} - I_2(Z_{2g} + Z_{2l} + Z_2) - (I_1 + I_2 + I_3)Z_{0l} = 0 \quad . \quad (\beta)$$

$$E_{\text{III}} - I_3(Z_{3g} + Z_{3l} + Z_3) - (I_1 + I_2 + I_3)Z_{0l} = 0 \quad . \quad (\gamma)$$

Applying Kirchoff's first law to the neutral point of the generator, we have

$$I_1 + I_2 + I_3 + I_0 = 0$$

whence
$$I_0 = -(I_1 + I_2 + I_3)$$

Re-arranging terms and replacing the quantities $(Z_{1g} + Z_{1l} + Z_1)$,

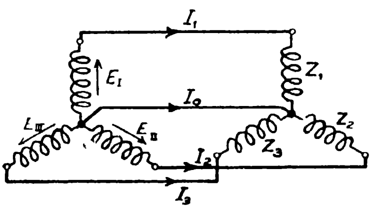


FIG. 177.—Circu Diagram for Three-phase Four-wire System

$(Z_{2g} + Z_{2l} + Z_2)$, $(Z_{3g} + Z_{3l} + Z_3)$ by the quantities Z_{1t}, Z_{2t}, Z_{3t}, respectively, we have

$$-I_1(Z_{1t} + Z_{0l}) - I_2Z_{0l} - I_3Z_{0l} + E_{\text{I}} = 0 \quad . \quad (\alpha')$$

$$-I_1Z_0 - I_2(Z_{2t} + Z_{0l}) - I_3Z_{0l} + E_{\text{II}} = 0 \quad . \quad (\beta')$$

$$-I_1Z_{0l} - I_2Z_{0l} - I_3(Z_{3t} + Z_{0l}) + E_{\text{III}} = 0 \quad . \quad (\gamma')$$

The solution to these equations is easily obtained by determinants. The expressions for the line currents are

$$I_1 = \frac{E_{\text{I}}Z_{2t}Z_{3t} + (E_{\text{I}} - E_{\text{III}})Z_{0l}Z_{2t} + (E_{\text{I}} - E_{\text{II}})Z_{0l}Z_{3t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})} \quad . \quad (97)$$

$$I_2 = \frac{E_{\text{II}}Z_{3t}Z_{1t} + (E_{\text{II}} - E_{\text{I}})Z_{0l}Z_{3t} + (E_{\text{II}} - E_{\text{III}})Z_{0l}Z_{1t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})} \quad . \quad (98)$$

$$I_3 = \frac{E_{\text{III}}Z_{1t}Z_{2t} + (E_{\text{III}} - E_{\text{II}})Z_{0l}Z_{1t} + (E_{\text{III}} - E_{\text{I}})Z_{0l}Z_{2t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})} \quad . \quad (99)$$

$$I_0 = -(I_1 + I_2 + I_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

Example. Calculate the line currents in a three-phase, four-wire system in which the branches of the load have resistances of 1·0, 1·3, and 1·85 ohms, the power-factor of all branches being unity. The load is supplied from a symmetrical three-phase star-connected generator through a four-core cable, the resistances of the principal conductors being 0·1 O., and the resistance of the neutral conductor being 0·2 O. The no-load E.M.F. of the generator per phase is 240 V.; the phase rotation is clockwise, and the impedance per phase is $0\cdot13/\underline{76^\circ}$ ohm.

[NOTE. A full solution to this problem, by a method which does not involve the application of Kirchoff's laws, has been given on pp. 270–272. The present solution will be obtained by the application of the equations deduced above.]

From the data of the example on p. 270 we have

$$Z_{1t} = 1\cdot1314 + j0\cdot1262, \quad Z_{2t} = 1\cdot4314 + j0\cdot1262,$$

$$Z_{3t} = 1\cdot9814 + j0\cdot1262, \quad Z_{0l} = 0\cdot2 + j0$$

$$E_{\mathrm{I}} = 240 + j0, \quad E_{\mathrm{II}} = -120 - j208, \quad E_{\mathrm{III}} = -120 + j208$$

Hence,

$$E_{\mathrm{I}} - E_{\mathrm{III}} = 360 - j208, \quad E_{\mathrm{I}} - E_{\mathrm{II}} = 360 + j208$$

$$E_{\mathrm{II}} - E_{\mathrm{I}} = -360 - j208, \quad E_{\mathrm{II}} - E_{\mathrm{III}} = 0 - j416$$

$$E_{\mathrm{III}} - E_{\mathrm{I}} = -360 + j208, \quad E_{\mathrm{III}} - E_{\mathrm{II}} = 0 + j416$$

$$Z_{1t}Z_{2t} = (1\cdot1314 + j0\cdot1262)(1\cdot4314 + j0\cdot1262) = 1\cdot606 + j0\cdot324$$

$$Z_{2t}Z_{3t} = (1\cdot4314 + j0\cdot1262)(1\cdot9814 + j0\cdot1262) = 2\cdot824 + j0\cdot4313$$

$$Z_{3t}Z_{1t} = (1\cdot9814 + j0\cdot1262)(1\cdot1314 + j0\cdot1262) = 2\cdot226 + j0\cdot3934$$

$$Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t} = 6\cdot656 + j1\cdot1487$$

$$Z_{0l}Z_{1t} = 0\cdot2(1\cdot1314 + j0\cdot1262) = 0\cdot2263 + j0\cdot02524$$

$$Z_{0l}Z_{2t} = 0\cdot2(1\cdot4314 + j0\cdot1262) = 0\cdot2863 + j0\cdot02524$$

$$Z_{0l}Z_{3t} = 0\cdot2(1\cdot9814 + j0\cdot1262) = 0\cdot3963 + j0\cdot02524$$

$$Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t}) = 0\cdot2(6\cdot656 + j1\cdot1149) = 1\cdot331 + j0\cdot2297$$

$$Z_{1t}Z_{2t}Z_{3t} = (1\cdot1314 + j0\cdot1262)(2\cdot824 + j0\cdot4313) = 3\cdot1456 + j0\cdot8495$$

$$I_1 = \frac{E_{\mathrm{I}}Z_{2t}Z_{3t} + (E_{\mathrm{I}} - E_{\mathrm{III}})Z_{0l}Z_{2t} + (E_{\mathrm{I}} - E_{\mathrm{II}})Z_{0l}Z_{3t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})}$$

$$= \frac{240(2\cdot824 + j0\cdot4313) + (360 - j208)(0\cdot2863 + j0\cdot02524) + (360 + j208)(0\cdot3963 + j0\cdot02524)}{3\cdot1456 + j0\cdot8495 + 1\cdot331 + j0\cdot2297}$$

$$= 202\cdot8 - j16\cdot55$$

$$I_1 = \sqrt{(202\cdot8^2 + 16\cdot55^2)} = 203\cdot3 \text{ A.}$$

$$I_2 = \frac{E_{\mathrm{II}}Z_{3t}Z_{1t} + (E_{\mathrm{II}} - E_{\mathrm{I}})Z_{0l}Z_{3t} + (E_{\mathrm{II}} - E_{\mathrm{III}})Z_{0l}Z_{1t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0l}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})}$$

$$= \frac{(-120 - j208)(2\cdot226 + j0\cdot3934) + (-360 - j208)(0\cdot3963 + j0\cdot02524) - j416(0\cdot2263 + j0\cdot0252)}{3\cdot1456 + j0\cdot8495 + 1\cdot331 + j0\cdot2297}$$

$$= -101\cdot6 - j131\cdot5$$

$$I_2 = \sqrt{(101{\cdot}6^2 + 131{\cdot}5^2)} = 166 \text{ A.}$$

$$I_3 = \frac{E_{III}Z_{1t}Z_{2t} + (E_{III} - E_{II})Z_{0t}Z_{1t} + (E_{III} - E_{I})Z_{0t}Z_{2t}}{Z_{1t}Z_{2t}Z_{3t} + Z_{0t}(Z_{1t}Z_{2t} + Z_{2t}Z_{3t} + Z_{3t}Z_{1t})}$$

$$= \frac{(-120 + j208)(1{\cdot}606 + j0{\cdot}324) + j416(0{\cdot}2263 + j0{\cdot}02524) + (-360 + j208)(0{\cdot}2863 + j0{\cdot}02524)}{3{\cdot}1456 + j0{\cdot}8495 + 1{\cdot}331 + j0{\cdot}2297}$$

$$= -57{\cdot}8 + j112$$

$$I_3 = \sqrt{(57{\cdot}8^2 + 112^2)} = 126 \text{ A}$$

These values agree with those obtained on p. 271

CHAPTER X

COMMERCIAL AND NON-SINUSOIDAL WAVE FORMS

In the preceding chapters sinusoidal currents and voltages have been considered almost exclusively to enable the fundamental principles of alternating-current circuits to be deduced in a simple manner and to allow graphical methods to be applied to the solution of problems. Although the sine wave is the ideal wave form and is closely approached in modern alternators operating at no-load, the load conditions in generators and commercial circuits frequently cause considerable deviations from the sine wave. It is necessary, therefore, to consider some of the causes of wave distortion and the manner in which the relationship between current and E.M.F. is affected by this distortion.

Equation to a complex wave. By the application of Fourier's theorem any single-valued* periodic function can be completely expressed by a series of simple harmonic functions (i.e. sine curves) having frequencies which are multiples of that of the complex function. These simple harmonic functions are called the *harmonics* of the complex function; the function which has the same frequency as the complex function is called the *first harmonic*, or the *fundamental*; that of double frequency, the *second harmonic*; that of triple frequency, the *third harmonic*, and so on. For example, in the case of a complex wave the fundamental (e_1) may be represented by

$$e_1 = E_{1m} \sin(\omega t + \alpha_1);$$

the second harmonic by

$$e_2 = E_{2m} \sin(2\omega t + \alpha_2);$$

the third harmonic by

$$e_3 = E_{3m} \sin(3\omega t + \alpha_3); \quad \text{etc.},$$

and the complex wave may be represented by the equation

$$e = E_{1m} \sin(\omega t + \alpha_1) + E_{2m} \sin(2\omega t + \alpha_2) + E_{3m} \sin(3\omega t + \alpha_3) + \ldots$$

where E_{1m}, E_{2m}, E_{3m}, . . . denote the maximum values, or amplitudes, of the first, second, and third harmonics respectively, and α_1, α_2, α_3 . . . denote the phase differences with respect to the

* A single-valued function is one in which the dependent variable has only one value for each value of the independent variable.

complex wave (i.e. the angles between the zero value of the complex wave and the corresponding zero values of the harmonic waves).

The number of terms in the series depends on the shape of the complex wave. Under certain conditions the number of terms may be indefinite, but under other conditions only a few terms may be involved. Again, the series may contain both even and odd harmonics, or only, alternatively, odd or even harmonics.

Shape of complex wave containing only even harmonics. A complex wave containing only even harmonics is unsymmetrical,

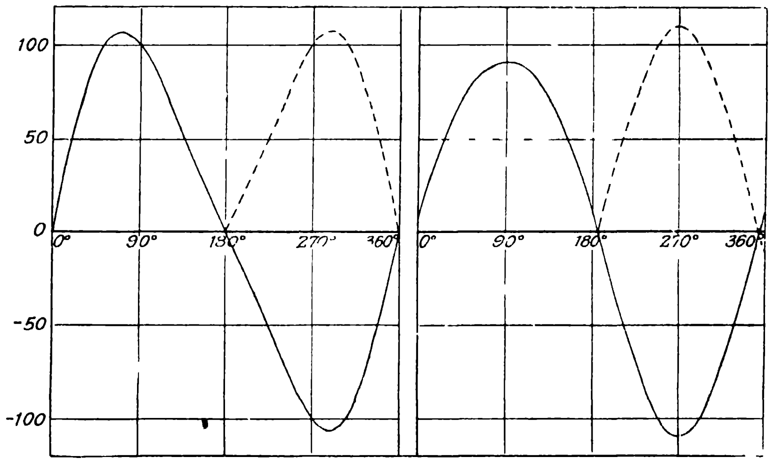


FIG. 178.—Wave Forms containing a Fundamental and a Second Harmonic

(a) Second harmonic in phase with fundamental; amplitude 20 per cent of that of fundamental (b) Second harmonic leading fundamental by 90°, amplitude 10 per cent of that of fundamental

i.e. the shape of the curve when rising positively from a zero value differs from that when rising negatively from another zero value. For example, in Fig 178, which shows complex wave-forms containing a fundamental and a second harmonic, if the negative half-cycle be reversed in sign and plotted, as shown dotted, above the horizontal axis, the dissimilarity in the shape of the two half-waves is emphasized. Observe that the two half-waves may have dissimilar shapes, as shown in the wave-form (b).

The analytical proof of the asymmetry of a complex wave containing only even harmonics is as follows—

Let the ordinate at any abscissa, ωt, in the positive half-wave be given by

$$e_1 = E_{1m} \sin(\omega t + a_1) + E_{2m} \sin(2\omega t + a_2) + E_{4m} \sin(4\omega t + a_4) + \ldots$$

The corresponding ordinate in the negative half-wave is obtained by substituting $(\omega t + \pi)$ for ωt in the preceding equation, and is therefore given by

$$\begin{aligned} e_2 &= E_{1m}\sin(\omega t + \pi + a_1) + E_{2m}\sin[2(\omega t + \pi) + a_2] \\ &\qquad + E_{4m}\sin[4(\omega t + \pi) + a_4] + \ldots \\ &= -E_{1m}\sin(\omega t + a_1) + E_{2m}\sin(2\omega t + a_2) \\ &\qquad + E_{4m}\sin(4\omega t + a_4) + \ldots \\ &= -[E_{1m}\sin(\omega t + a_1) - E_{2m}\sin(2\omega t + a_2) \\ &\qquad - E_{4m}\sin(4\omega t + a_4) - \ldots] \end{aligned}$$

Hence the ordinate at abscissa $(\omega t + \pi)$ is not equal to the ordinate at abscissa ωt.

Shape of complex wave containing only odd harmonics. A complex wave containing only odd harmonics is always symmetrical, the negative half-wave being an exact reproduction (with the reversed sign) of the positive half-wave. Examples are shown in Figs. 1, 3, . . .

The majority of the waves met with in alternating-current engineering are of this type, because of the symmetrical construction of the field magnets and the armature coils of alternating-current generators. Even harmonics, however, may also occur (in addition to the odd harmonics) under certain conditions of loading, and may also be produced when certain classes of apparatus (e.g. arc lamps, and electro-magnetic apparatus working with an unsymmetrical magnetization curve or loop) are connected to the circuit.

The analytical proof of the symmetry of a complex wave containing only odd harmonics is as follows—

Let the ordinate at abscissa ωt in the positive half-wave be given by

$$\begin{aligned} e_1 &= E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) \\ &\qquad + E_{5m}\sin(5\omega t + a_5) + \ldots \end{aligned}$$

Then the corresponding ordinate in the negative half-wave is given by

$$\begin{aligned} e_2 &= E_{1m}\sin(\omega t + \pi + a_1) + E_{3m}\sin[3(\omega t + \pi) + a_3] \\ &\qquad + E_{5m}\sin[5(\omega t + \pi) + a_5] + \ldots \\ &= -E_{1m}\sin(\omega t + a_1) - E_{3m}\sin(3\omega t + a_3) \\ &\qquad - E_{5m}\sin(5\omega t + a_5) - \ldots \\ &= -[E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) \\ &\qquad + E_{5m}\sin(5\omega t + a_5) + \ldots] \\ &= -e_1. \end{aligned}$$

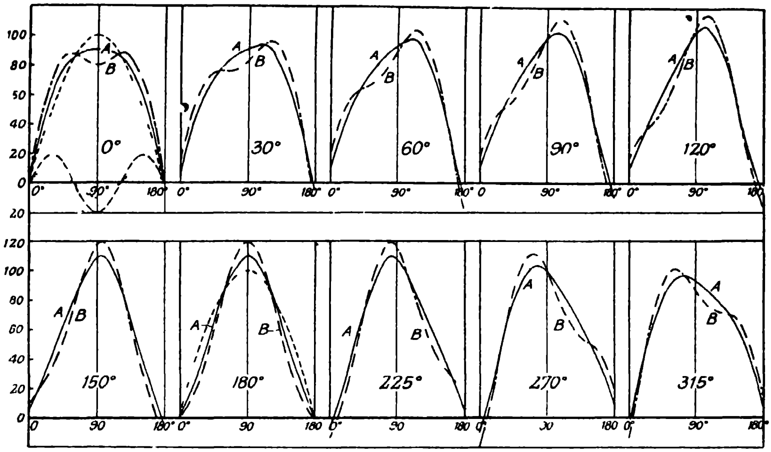


FIG. 179 Wave Forms containing a Fundamental and a Third Harmonic

A Amplitude of third harmonic 10 per cent of that of fundamental
B Amplitude of third harmonic 20 per cent of that of fundamental
Phase difference between fundamental and harmonic is indicated in diagrams
The additional dotted curves in the first diagram show the fundamental and third harmonics separately.

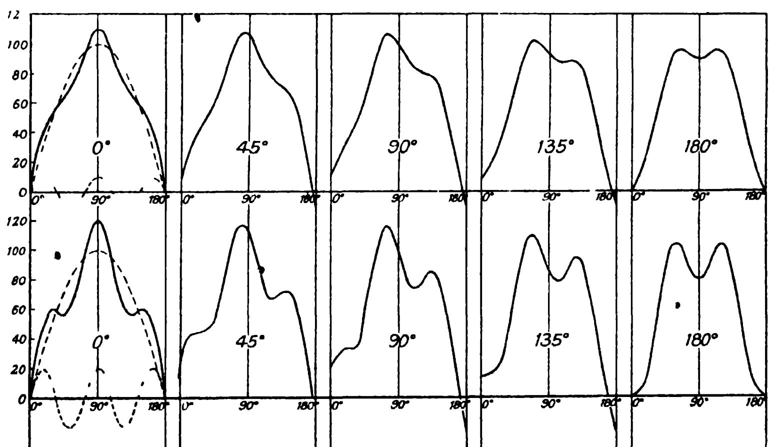


FIG. 180 —Wave Forms containing a Fundamental and a Fifth Harmonic

Amplitude of fifth harmonic is 10 per cent of that of fundamental for upper set of wave-forms and 20 per cent for lower set of wave-forms Phase difference between fundamental and harmonic is indicated in the diagrams.

Effect of phase positions of given harmonic on shape of complex wave. The deviation of a complex wave from a sine wave depends not only on the relative magnitude and order of the several harmonics, but also on their phase with respect to the complex wave. Fig. 179 illustrates the effect of superimposing a third harmonic of given amplitude, but of varying phase, on a given fundamental sine wave. Fig. 180 illustrates the effect produced

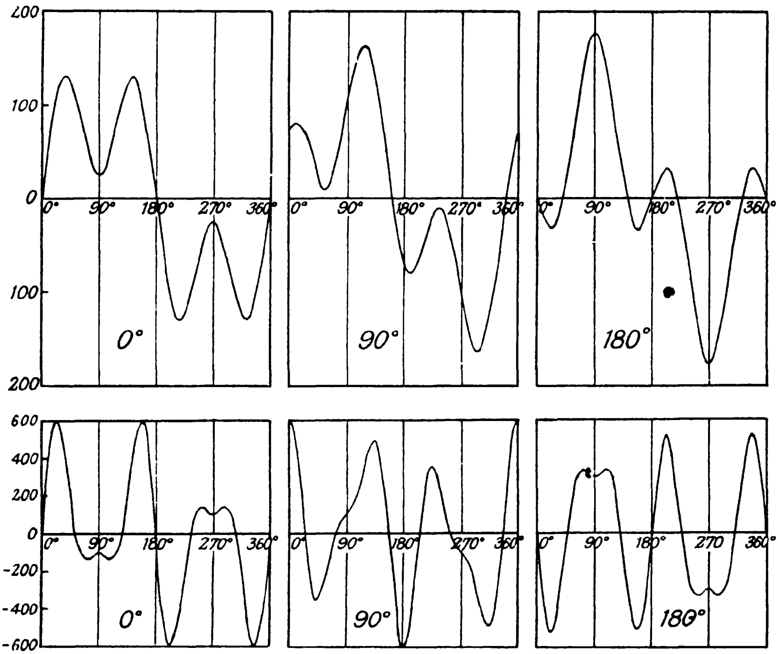


FIG. 181.—Wave Forms containing Fundamental and Pronounced Third Harmonics

Amplitude of third harmonic is 75 per cent of fundamental for upper set of wave forms and 600 per cent for lower set of wave forms. Phase difference between fundamental and harmonic is indicated in the diagrams.

under similar conditions by a fifth, and Fig. 181 illustrates the effect produced when the amplitude of the fundamental is small in comparison with the amplitudes of the harmonics.*

These illustrations show clearly that the combination of a fundamental with only one or two harmonics may produce a very great variety of wave forms. In many cases the order of the harmonic

* Complex curves similar to Fig. 181 occur in alternators under sustained short circuit.

can be ascertained by inspection, but in general a systematic analysis, as discussed later (p. 317), is required to determine the character of the several harmonics. For the present it will be desirable to investigate the manner in which the harmonics affect the relationship between current and E.M.F. in the simpler types of circuits.

CURRENT WAVE FORMS IN SINGLE-PHASE CIRCUITS SUPPLIED WITH NON-SINUSOIDAL E.M.F.

General. The current wave-form in a circuit, for which the constants (i.e. resistance, inductance, capacity) are invariable, is, in general, of different shape to that of the impressed E.M.F., and only in special cases are the two wave forms similar. We have shown in Chapters III and IV that for circuits containing constant resistance, inductance, or capacity, the current due to a sinusoidal impressed E.M.F. is of the same frequency as the latter. Hence if a number of sinusoidal E.M.Fs. of different frequencies be impressed upon the circuit each E.M.F. will produce a current component of its own frequency, quite independently of the others. The instantaneous value of the current in the circuit is therefore the algebraic sum of the instantaneous currents due to the several E.M.Fs. This principle of superposition enables us to determine readily the current in such circuits when the equation to the impressed E.M.F. is known.

Relation between impressed E.M.F. and current for circuits containing resistance. Let the impressed E.M.F. be represented by

$$e = E_{1m} \sin(\omega t + \alpha_1) + E_{3m} \sin(\omega t + \alpha_3) + E_{5m} \sin(\omega t + \alpha_5).$$

Then, if R is the resistance of the circuit, the current (i_1) due to the fundamental (e_1) is given by

$$i_1 = \frac{e_1}{R} = \frac{E_{1m}}{R} \sin(\omega t + \alpha_1);$$

that (i_3) due to the third harmonic (e_3) is given by

$$i_3 = \frac{e_3}{R} = \frac{E_{3m}}{R} \sin(3\omega t + \alpha_3);$$

that and (i_5) due to the fifth harmonic (e_5) is given by

$$i_5 = \frac{e_5}{R} = \frac{E_{5m}}{R} \sin(5\omega t + \alpha_5).$$

Hence the current (i) in the circuit is given by

$$i = i_1 + i_2 + i_3$$

$$= \frac{E_{1m}}{R}\sin(\omega t + a_1) + \frac{E_{3m}}{R}\sin(3\omega t + a_3)$$

$$+ \frac{E_{5m}}{R}\sin(5\omega t + a_5) \quad . \quad (101)$$

Thus the wave form of the current is similar to that of the impressed E.M.F.

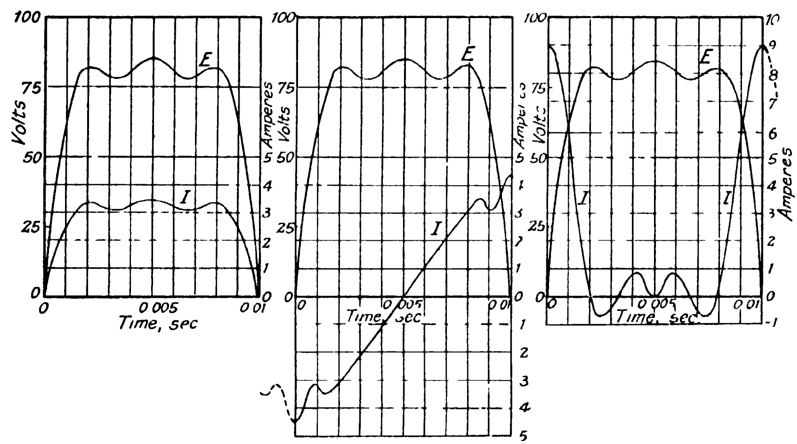


FIG. 182 FIG. 183 FIG. 184
Wave Forms of E.M.F. and Current for Simple Circuits containing Resistance (Fig. 182), Inductance (Fig. 183), and Capacity (Fig. 184)

Example. Let a non-inductive resistance of 25 ohms be connected to a circuit of which the E.M.F. follows the law—

$$e = 100 \sin 314t + 25 \sin 942t + 10 \sin 1570t.$$

The current in the circuit is given by

$$i = \frac{100}{25}\sin 314t + \frac{25}{25}\sin 942t + \frac{10}{25}\sin 1570t$$

$$4 \sin 314t + \sin 942t + 0{\cdot}4 \sin 1570t$$

The current curve is shown in Fig. 182. The impressed E.M.F. curve is also shown and a comparison of the two will show that the wave forms are of similar shape.

Relation between impressed E.M.F. and current for circuits containing inductance. Consider a purely inductive circuit of

inductance L and negligible resistance. Let the impressed E.M.F. be represented by the equation

$$e = E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) + E_{5m}\sin(5\omega t + a_5).$$

Then the current (i_1) due to the first harmonic (e_1) is

$$i_1 = \frac{e_1}{X_1} = \frac{E_{1m}}{\omega L}\sin\left(\omega t + a_1 - \tfrac{1}{2}\pi\right),$$

that (i_3) due to the third harmonic (e_3) is

$$i_3 = \frac{e_3}{X_3} = \frac{E_{3m}}{3\omega L}\sin\left(3\omega t + a_3 - \tfrac{1}{2}\pi\right).$$

and that (i_5) due to the fifth harmonic (e_5) is

$$i_5 = \frac{e_5}{X_5} = \frac{E_{5m}}{5\omega L}\sin\left(\omega t + a_5 - \tfrac{1}{2}\pi\right),$$

where $X_1(=\omega L)$, $X_3(=3\omega L)$, and $X_5(=5\omega L)$ are the reactances due to the first, third, and fifth harmonics respectively.

Hence the current (i) in the circuit is given by

$$\begin{aligned} i &= i_1 + i_3 + i_5 \\ &= \frac{E_{1m}}{\omega L}\sin\left(\omega t + a_1 - \tfrac{1}{2}\pi\right) + \frac{E_{3m}}{3\omega L}\sin\left(3\omega t + a_3 - \tfrac{1}{2}\pi\right) \\ &\qquad + \frac{E_{5m}}{5\omega L}\sin\left(5\omega t + a_5 - \tfrac{1}{2}\pi\right) \quad . \quad . \quad (102) \end{aligned}$$

Thus each component of the current has a phase difference of 90° (lagging) with respect to E.M.F. harmonic to which it is due, and, therefore, the wave form of the current differs from that of the impressed E.M.F. It will be observed, however, that the reactance due to a given harmonic is directly proportional to the order of that harmonic; hence the current components due to the higher harmonics will be very much smaller than those in the case of a circuit containing only resistance. Accordingly, *in an inductive circuit supplied with a non-sinusoidal E.M.F. the current wave form shows less distortion than the E.M.F. wave form, and the current in such a circuit more nearly approaches a sine curve than does the current in a circuit containing resistance.*

Example. Let an inductive coil, of inductance 0·08 henry and negligible resistance, be connected to a circuit of which the E.M.F. follows the law—

$$e = 100 \sin 314t + 25 \sin 942t + 10 \sin 1570t.$$

Then the current (i) in the circuit is given by

$$i = \frac{100}{314 \times 0{\cdot}08} \sin(314t - \tfrac{1}{2}\pi) + \frac{25}{942 \times 0{\cdot}08} \sin(942t - \tfrac{1}{2}\pi) + \frac{10}{1570 \times 0{\cdot}08} \sin(1570t - \tfrac{1}{2}\pi)$$

$$= 4 \sin(314t - \tfrac{1}{2}\pi) + 0{\cdot}33 \sin(942t - \tfrac{1}{2}\pi) + 0{\cdot}08 \sin(1570t - \tfrac{1}{2}\pi)$$

The current curve is shown in Fig. 183. On comparing this with the current curve in Fig. 182 a marked difference in wave shape will be noted, the curve of Fig. 183 showing considerably less distortion than the current curve in Fig. 182.

Relation between impressed E.M.F. and current for series circuits containing resistance and inductance. Let the impressed E.M.F. be represented by the equation

$$e = E_{1m} \sin(\omega t + a_1) + E_{3m} \sin(3\omega t + a_3) + E_{5m} \sin(5\omega t + a_5)$$

Then the current (i) in the circuit is given by

$$i = \frac{E_{1m}}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t + a_1 - \varphi_1) + \frac{E_{3m}}{\sqrt{(R^2 + 9\omega^2 L^2)}} \sin(3\omega t + a_3 - \varphi_3) + \frac{E_{5m}}{\sqrt{(R^2 + 25\omega^2 L^2)}} \sin(5\omega t + a_5 - \varphi_5) \quad . \quad (103)$$

where $\varphi_1 (= \tan^{-1} \omega L/R)$, $\varphi_3 (= \tan^{-1} 3\omega L/R)$, $\varphi_5 (= \tan^{-1} 5\omega L/R)$ are the phase differences between the E.M.Fs. and currents due to the respective harmonics.

Observe that these phase differences have different magnitudes.

Relation between impressed E.M.F. and current for circuits containing capacity. If an E.M.F. represented by the equation

$$e = E_{1m} \sin(\omega t + a_1) + E_{3m} \sin(3\omega t + a_3) + E_{5m} \sin(5\omega t + a_5)$$

be applied to a condenser of capacity C farads, the charging current will be given by the equation

$$i = \omega C E_{1m} \sin(\omega t + a_1 + \tfrac{1}{2}\pi) + 3\omega C E_{3m} \sin(3\omega t + a_3 + \tfrac{1}{2}\pi) + 5\omega C E_{5m} \sin(5\omega t + a_5 + \tfrac{1}{2}\pi) \quad . \quad . \quad . \quad . \quad (104)$$

Therefore, in this case, the amplitudes of the currents due to the higher harmonics are increased and the current wave will show more distortion than the E.M.F. wave. Thus the effect of capacity on wave distortion is exactly the reverse to that of inductance.

Example. Let a condenser having a capacity of 127·5 microfarads be connected to a circuit of which the E.M.F. follows the law—

$$e = 100 \sin 314t + 25 \sin 942t + 10 \sin 1570t.$$

The current (i) in the condenser is given by

$$i = 314 \times 127{\cdot}5 \times 10^{-6} \times 100 \sin(314t + \tfrac{1}{2}\pi) + 942 \times 127{\cdot}5 \times 10^{-6} \times 25 \sin(942t + \tfrac{1}{2}\pi) + 1570 \times 127{\cdot}5 \times 10^{-6} \times 10 \sin(1570t + \tfrac{1}{2}\pi)$$

$$= 4 \sin(314t + \tfrac{1}{2}\pi) + 3 \sin(942t + \tfrac{1}{2}\pi) + 2 \sin(1570t + \tfrac{1}{2}\pi)$$

The current curve is shown in Fig. 184. This curve should be compared with the current curve in Fig. 183, as the capacitive reactance and the inductive reactance for these examples have been chosen so as to give the same maximum value of the fundamental in each case.

Relation between impressed E.M.F. and current for series circuits containing resistance, inductance, and capacity. In the general case of a series-connected circuit containing resistance, inductance, and capacity, the current resulting from an impressed E.M.F. of complex wave form (which is represented by the equation $e = E_{1m}\sin(\omega t + \alpha_1) + \ldots$) is given by

$$i = \frac{E_{1m}}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}\sin(\omega t + \alpha_1 - \varphi_1)$$
$$+ \frac{E_{3m}}{\sqrt{[R^2 + (3\omega L - 1/3\omega C)^2]}}\sin(3\omega t + \alpha_3 - \varphi_3)$$
$$+ \frac{E_{5m}}{\sqrt{[R^2 + (5\omega L - 1/5\omega C)^2]}}\sin(5\omega t + \alpha_5 - \varphi_5) \quad . \qquad (104)$$

where

$$\varphi_1 = \tan^{-1}\left(\frac{\omega L}{R} - \frac{1}{\omega CR}\right), \quad \varphi_3 = \tan^{-1}\left(\frac{3\omega L}{R} - \frac{1}{3\omega CR}\right),$$
$$\varphi_5 = \tan^{-1}\left(\frac{5\omega L}{R} - \frac{1}{5\omega CR}\right)$$

are the phase differences between the E.M.Fs. and currents for the respective harmonics.

The amplitude (I_{nm}) of any harmonic, say the nth, is equal to

$$I_{nm} = \frac{E_{nm}}{\sqrt{[R^2 + (n\omega L - 1/n\omega C)^2]}}$$

and its phase difference (φ_n), with respect to the E.M.F. producing it, is

$$\varphi_n = \tan^{-1}\left(\frac{n\omega L}{R} - \frac{1}{n\omega CR}\right)$$

Resonance due to harmonics. When the angle (φ_n) is zero (i.e. when $n\omega L = 1/n\omega C$) resonance occurs with respect to this particular harmonic. Under resonance conditions considerable voltages may be produced at the terminals of the condenser and the inductive resistance, although the amplitude of the E.M.F. due to this harmonic may be relatively small. For example, if the E.M.F. wave of a 50-cycle alternator contains a 13th harmonic which has an amplitude equal to 1 per cent of the fundamental, and this alternator is connected to a series circuit containing an

inductive resistance ($R = 5$ O., $L = 0{\cdot}12$ H.) and a condenser ($C = 0{\cdot}5\mu$ F.), resonance occurs with the 13th harmonic (since $n\omega L = 13 \times 2\pi \times 50 \times 0{\cdot}12 = 490$, and $1/n\omega C = 10^6/(13 \times 2\pi \times 50 \times 0{\cdot}5) = 490$). The maximum value ($I_{13m}$) of the current due to this harmonic is given by

$$I_{13m} = \frac{E_{13m}}{R} = \frac{E_{1m}}{100R} = \frac{E_{1m}}{100 \times 5} = 0{\cdot}002E_{1m},$$

and the voltage across the terminals of the condenser due to this current is equal to

$$0{\cdot}002E_{1m} \times 10^{12}/(13 \times 2\pi \times 50 \times 0{\cdot}5)^2 = 0{\cdot}98E_{1m}.$$

Similarly the voltage at the terminals of the inductive resistance is equal to

$$0{\cdot}002E_{1m}\sqrt{[5^2 + (13 \times 2\pi \times 50 \times 0{\cdot}12)^2]} = 0{\cdot}98E_{1m}.$$

The actual voltages may be much higher and will depend on the relative amplitudes of the fundamental and other harmonics.

Pressure-rises due to these causes may occur in practice under certain conditions. For instance, when an alternator is connected to unloaded cables the capacity of the latter is in series with the inductance of the former, and the conditions may be favourable for obtaining resonance with a particular harmonic.

Experimental method of ascertaining presence of any particular harmonic in E.M.F. wave. The property of resonance may be utilized to ascertain the order of the harmonics in a complex E.M.F. wave. For this purpose an oscillograph (p. 444), a variable inductance, and a condenser of variable capacity are required. The inductance, condenser, a variable non-inductive resistance and a fixed non-inductive resistance, or shunt, for the oscillograph are connected in series and the combination is connected to the source of E.M.F. to be tested ; the oscillograph being connected to show the wave form of the voltage across the fixed non-inductive resistance. The inductance and capacity are adjusted successively to values which will give resonance conditions for the 1st, 3rd, 5th, 7th, etc., harmonics, and a record of the wave form is obtained by the oscillograph. From the shape of the latter the presence, or absence, of a given harmonic can be detected by inspection.

For example, if a particular harmonic is present, the wave form will show this harmonic with a relatively large amplitude, as under resonance conditions any other harmonics will have only a very small amplitude. Thus if the 5th harmonic is present, the current wave form under resonance conditions will show a wave of quintuple frequency superimposed upon a wave of fundamental frequency.

Fig. 185 shows a typical set of wave forms obtained by this method. An examination of these shows that the E.M.F. wave form contains 5th, 7th, 11th, 13th, 15th, and 17th harmonics.

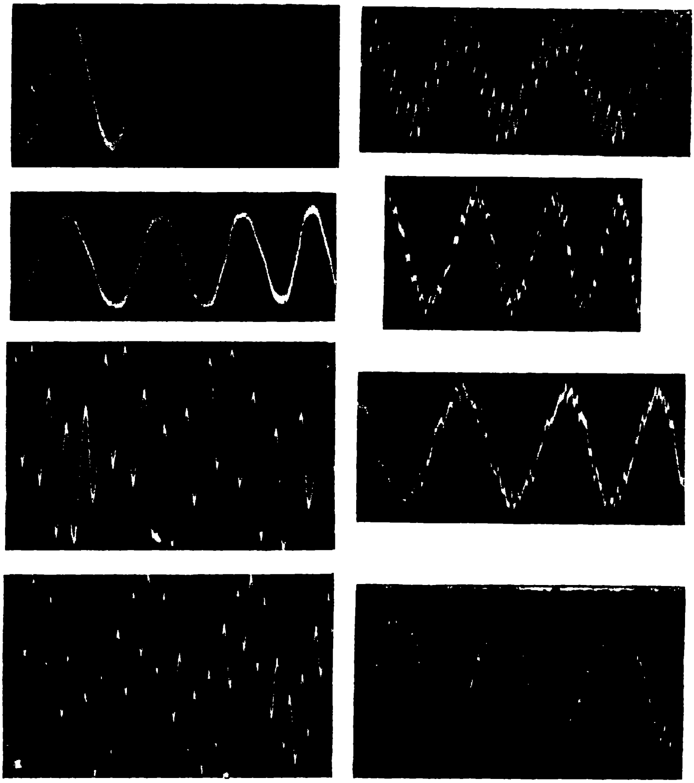


FIG. 185.—Oscillograms showing (*a*) the E.M.F. Wave Form of a Given Alternator, and the Wave Forms of Current (*b* to *h*) in a Special Resonant Circuit supplied by this Alternator, the Constants of the Circuit being Adjusted to give, successively, Resonance Conditions for the 3rd (wave form *b*), 5th (*c*), 7th (*d*), 11th (*e*), 13th (*f*), 15th (*g*), and 17th (*h*) Harmonics

NOTE. The following data give the values of inductance (*L*) and capacity (*C*) in the resonant circuit for the several current wave forms; also the values of the non-inductive shunt (*R*) to which the oscillograph vibrator was connected.

Wave form	*b*	*c*	*d*	*e*	*f*	*g*	*h*
L(henry)	0·059	0·059	0·059	0·059	0·059	0·059	0·059
C(μ F.)	18·9	6·9	3·52	1·4	1·0	0·76	0·59
R (ohms)	0·5	0·833	2·22	5	15	20	35

R.M.S. value of a complex wave. Let the wave be represented by $e = E_{1m} \sin(\omega t + \alpha_1) + E_{3m} \sin(3\omega t + \alpha_3) + E_{5m} \sin(\omega t + \alpha_5) + \ldots$. Then if the R.M.S. value of this wave be denoted by E, we have

$$E^2 = \frac{1}{\pi}\int_0^{\pi} \{E_{1m} \sin(\omega t + \alpha_1) + E_{3m} \sin(3\omega t + \alpha_3) + E_{5m} \sin(5\omega t + \alpha_5) + \ldots\}^2 d\omega t$$

$$= \frac{1}{\pi}\int_0^{\pi} \{E_{1m}^2 \sin^2 (\omega t + \alpha_1) + E_{3m}^2 \sin^2(3\omega t + \alpha_3) + E_{5m}^2 \sin^2(5\omega t + \alpha_5) + \ldots + 2 E_{1m}E_{3m} \sin(\omega t + \alpha_1) \sin(3\omega t + \alpha_3) + 2 E_{1m}E_{5m} \sin(\omega t + \alpha_1) \sin(5\omega t + \alpha_5) + \ldots\} d\omega t.$$

Now, $\int_0^{\pi} \sin^2(n\omega t + \alpha_n) d\omega t = \frac{1}{2}\pi$;

and $\int_0^{\pi} \sin(\omega t + \alpha_1) \sin(n\omega t + \alpha_n) d\omega t = 0$;

provided that n is an integer greater than unity.

Hence when the above integral is evaluated, all terms involving the product of quantities having different frequencies become zero, so that

$$E^2 = \frac{1}{\pi}\{(E_{1m}^2 \times \tfrac{1}{2}\pi) + (E_{3m}^2 \times \tfrac{1}{2}\pi) + (E_{5m}^2 \times \tfrac{1}{2}\pi) + \ldots\}$$

$$= \tfrac{1}{2}(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)$$

and $E = 0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)}$ . . (105)

Examples. The R.M.S. values of the E.M.F. and current waves of Figs. 182–185 can now be obtained.

(1) *E.M.F. wave*, Fig. 182, p. 300.

$e = 100 \sin 314t + 25 \sin 942t + 10 \sin 1570t$

which may be written—

$e = 100 \sin \omega t + 25 \sin 3\omega t + 10 \sin 5\omega t$,

where $\omega = 314$.

Hence,

$E_{1m} = 100, \quad E_{3m} = 25, \quad E_{5m} = 10$

Therefore,

$E = 0{\cdot}707\sqrt{(100^2 + 25^2 + 10^2)} = 73{\cdot}2$.

R.M.S. value of fundamental $(E_1) = 70{\cdot}7$.

(2) *Current wave*, Fig. 182.

$i = 4 \sin 314\omega t + \sin 925t + 0{\cdot}4 \sin 1570t$

Therefore,

$I = 0{\cdot}707\sqrt{(4^2 + 1^2 + 0{\cdot}4^2)} = 2{\cdot}93$.

R.M.S. value of fundamental $= 0{\cdot}707\sqrt{4^2} = 2{\cdot}83$

(3) *Current wave*, Fig. 183, p. 300.

$$i = 4\sin(314t - \tfrac{1}{2}\pi) + 0{\cdot}33\sin(942t - \tfrac{1}{2}\pi) + 0{\cdot}08\sin(1570t - \tfrac{1}{2}\pi)$$

$$I = 0{\cdot}707\sqrt{(4^2 + 0{\cdot}33^2 + 0{\cdot}08^2)} = 2{\cdot}84$$

R.M.S. value of fundamental $= 0{\cdot}707\sqrt{4^2} = 2{\cdot}83$.

(4) *Current wave*, Fig. 184, p. 300.

$$i = 4\sin(314t + \tfrac{1}{2}\pi) + 3\sin(942t + \tfrac{1}{2}\pi) + 2\sin(1570t + \tfrac{1}{2}\pi)$$

$$I = 0{\cdot}707\sqrt{(4^2 + 3^2 + 2^2)} = 3{\cdot}8$$

R.M.S. value of fundamental $= 0{\cdot}707\sqrt{4^2} = 2{\cdot}83$.

(5) An electromotive force.

$$e = 2000\sin\omega t + 400\sin 3\omega t + 100\sin 5\omega t,$$

is connected to a circuit consisting of a resistance of 10 ohms, a variable inductance, and a condenser of 30μ F. capacity, arranged in series with a hot-wire ammeter. Find the value of the inductance which will give resonance with the triple frequency component of the pressure, and estimate the readings on the ammeter and on a hot-wire voltmeter connected across the supply when resonance exists. $\omega = 300$. (*L.U.* 1922.)

For resonance to occur at triple frequency we must have

$$3\omega L = 1/3\omega C.$$

Substituting numerical values for ω and C, we obtain

$$L = 10^6/(9 \times 300^2 \times 30) = 0{\cdot}0411 \text{ H}.$$

The R.M.S. value of the current is given by

$$I = 0{\cdot}707\sqrt{\left[\left(\frac{E_{1m}}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}\right)^2 + \left(\frac{E_{3m}}{\sqrt{[R^2 + (3\omega L - 1/3\omega C)^2]}}\right)^2 + \left(\frac{E_{5m}}{\sqrt{[R^2 + (5\omega L - 1/5\omega C)^2]}}\right)^2\right]}$$

where E_{1m}, E_{3m}, E_{5m} denote the maximum values of the fundamental, triple, and quintuple frequency components of the pressure. Substituting numerical values for these and the other quantities, and noting that, for the given conditions, $3\omega L - 1/3\omega C = 0$, we obtain

$$I = 0{\cdot}707\sqrt{\left[\left(\frac{2000}{\sqrt{[10^2 + (300 \times 0{\cdot}0411 - 10^6/(300 \times 30))^2]}}\right)^2 + \left(\frac{400}{10}\right)^2 + \left(\frac{100}{\sqrt{[10^2 + (5 \times 300 \times 0{\cdot}0411 - 10^6/(5 \times 300 \times 30))^2]}}\right)^2\right]}$$

$$= 0{\cdot}707\sqrt{(20{\cdot}15^2 + 40^2 + 2{\cdot}45^2)}$$

$$= 31{\cdot}75 \text{ A}.$$

The R.M.S. value of the supply pressure is given by

$$E = 0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2)} = 0{\cdot}707\sqrt{(2000^2 + 400^2 + 100^2)}$$

$$= 1445 \text{ V}.$$

Effect of wave distortion on measurements of inductance and capacity. Since alternating-current ammeters and voltmeters are calibrated to read R.M.S. values of current and pressure, respectively, the expression (105) deduced on p. 306 for the R.M.S. value of a complex wave enables the readings of the instruments connected in a circuit to be calculated when the "constants" of the circuit and the wave forms of current and pressure are known. Conversely,

the "constants" of the circuit may be calculated from the instrument readings, but corrections have, in general, to be applied to take into account the shapes of the current and pressure waves. In the special case of circuits containing pure resistance, however, the wave forms of current and impressed E.M.F. are similar, and therefore the ratio of the R.M.S. values of these quantities will be constant whether they have a sinusoidal or non-sinusoidal wave form. Hence the resistance of a non-inductive alternating-current circuit may be determined by means of the ammeter and voltmeter method without a knowledge of the wave form.* In all other cases a knowledge of the E.M.F. wave form is necessary to obtain correct results. Thus, consider a purely inductive circuit having an inductance of L henries, and let E and I denote, respectively, the R.M.S. values of impressed E.M.F. and current as read on instruments connected in the circuit. Then

$$E = 0{\cdot}707 \sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)}$$

$$I = 0{\cdot}707 \sqrt{\left[\left(\frac{E_{1m}}{\omega L}\right)^2 + \left(\frac{E_{3m}}{3\omega L}\right)^2 + \left(\frac{E_{5m}}{5\omega L}\right)^2 + \ldots\right]}$$

$$= \frac{0{\cdot}707}{\omega L} \sqrt{(E_{1m}^2 + \tfrac{1}{9} E_{3m}^2 + \tfrac{1}{25} E_{5m}^2 + \ldots)}$$

Whence

$$L = \frac{0{\cdot}707}{\omega I} \sqrt{(E_{1m}^2 + \tfrac{1}{9} E_{3m}^2 + \tfrac{1}{25} E_{5m}^2 + \ldots)}$$

This expression involves a knowledge of the absolute values of the amplitudes of the several harmonics in the E.M.F. wave. In practice it is more convenient to deal with the relative values of these amplitudes, and accordingly the expression for L is modified as follows—

Multiply and divide the right-hand side by E, writing the denominator in the form $[0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)}]$, thus

$$L = \left(\frac{0{\cdot}707}{\omega I} \sqrt{(E_{1m}^2 + \tfrac{1}{9} E_{3m}^2 + \tfrac{1}{25} E_{5m}^2 + \ldots)}\right) \left(\frac{E}{0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots)}}\right)$$

$$= \frac{E}{\omega I} \sqrt{\left(\frac{E_{1m}^2 + \tfrac{1}{9} E_{3m}^2 + \tfrac{1}{25} E_{5m}^2 + \ldots}{E_{1m}^2 + E_{3m}^2 + E_{5m}^2 + \ldots}\right)}$$

* The value of the resistance obtained by this method may differ slightly from that obtained by a test using direct current, owing to possible eddy-currents in the conductors and the non-uniform distribution of current over their cross-section when the testing current is alternating.

$$= \frac{E}{\omega I}\sqrt{\left(\frac{1+\frac{1}{9}(E_{3m}/E_{1m})^2+\frac{1}{25}(E_{5m}/E_{1m})^2+\dots}{1+(E_{3m}/E_{1m})^2+(E_{5m}/E_{1m})^2+\dots}\right)} \quad (106)$$

Now $E/\omega I$ is the apparent value of the inductance obtained from the uncorrected values of the instrument readings. Hence the quantity under the radical is the correction factor by which the apparent inductance must be multiplied to contain the true inductance.

Similarly, in the case of a circuit containing pure capacity ($= C$ farads), let E and I denote the R.M.S. values of impressed E.M.F. and current as read on instruments connected in the circuit. Then

$$E = 0{\cdot}707\sqrt{(E_{1m}^2+E_{3m}^2+E_{5m}^2+\dots)}$$

$$I = 0{\cdot}707\sqrt{[(\omega CE_{1m})^2+(3\omega CE_{3m})^2+(5\omega CE_{5m})^2+\dots]}$$

$$= 0{\cdot}707\omega C\sqrt{(E_{1m}^2+9E_{3m}^2+25E_{5m}^2+\dots)}$$

Whence

$$C = \frac{I}{0{\cdot}707\omega\sqrt{(E_{1m}^2+9E_{3m}^2+25E_{5m}^2+\dots)}}$$

$$= \left(\frac{I}{0{\cdot}707\omega E\sqrt{(E_{1m}^2+9E_{3m}^2+25E_{5m}^2+\dots)}}\right)\left(0{\cdot}707\sqrt{(E_{1m}^2+E_{3m}^2+E_{5m}^2+\dots)}\right)$$

$$= \frac{I}{\omega E}\sqrt{\left(\frac{E_{1m}^2+E_{3m}^2+E_{5m}^2+\dots}{E_{1m}^2+9E_{3m}^2+25E_{5m}^2+\dots}\right)}$$

$$= \frac{I}{\omega E}\sqrt{\left(\frac{1+(E_{3m}/E_{1m})^2+(E_{5m}/E_{1m})^2+\dots}{1+9(E_{3m}/E_{1m})^2+25(E_{5m}/E_{1m})^2+\dots}\right)} \quad (107)$$

Observe that the expression for the correction factor differs from that for the preceding case.

For example, if the E.M.F. wave contains 3rd and 5th harmonics, having amplitudes of 15 per cent and 10 per cent, respectively, of the fundamental, the correction factor for measurements of inductance is equal to

$$\sqrt{\left(\frac{1+\frac{1}{9}(E_{3m}/E_{1m})^2+\frac{1}{25}(E_{5m}/E_{1m})^2}{1+(E_{3m}/E_{1m})^2+(E_{5m}/E_{1m})^2}\right)}$$

$$= \sqrt{\left(\frac{1+\frac{1}{9}(\frac{15}{100})^2+\frac{1}{25}(\frac{10}{100})^2}{1+(\frac{15}{100})^2+(\frac{10}{100})^2}\right)}$$

$$= 0{\cdot}985$$

Hence the apparent value of the inductance, as obtained from the uncorrected readings of the instruments, is about $1\frac{1}{2}$ per cent

higher than the true value. Thus the error introduced by not taking into account the wave form is relatively small.

Similarly the correction factor for measurements of capacity with this E.M.F. wave form is equal to

$$\sqrt{\left(\frac{1 + (E_{3m}/E_{1m})^2 + (E_{5m}/E_{1m})^2}{1 + 9\,(E_{3m}/E_{1m})^2 + 25(E_{5m}/E_{1m})^2}\right)}$$

$$= \sqrt{\left(\frac{1 + (\frac{15}{100})^2 + (\frac{10}{100})^2}{1 + 9\,(\frac{15}{100})^2 + 25(\frac{10}{100})^2}\right)}$$

$$= 0{\cdot}844$$

Hence the apparent value of the capacity, as obtained from the uncorrected values of the instrument readings is 15·6 per cent higher than the true value of the capacity. In this case the error introduced by not taking into account wave distortion is very large.

Example. A current of 50 frequency containing first, third, and fifth harmonics of crest values 100, 15, 12 amperes, respectively, is sent through an ammeter and an inductive coil of negligibly small losses. A voltmeter connected to the terminals shows 75 volts. What will be the current indicated on the ammeter, and what is the exact value of the inductance of the coil in henries ? (C. and G., 1918.)

The R.M.S. current (I) is given by

$$\begin{aligned} I &= 0{\cdot}707\sqrt{(I_{1m}^2 + I_{3m}^2 + I_{5m}^2)} \\ &= 0{\cdot}707\sqrt{(100^2 + 15^2 + 12^2)} \\ &= 0{\cdot}707 \times 101{\cdot}9 = 72 \text{ A.} \end{aligned}$$

Let L = inductance of coil in henries and let E_{1m}, E_{3m}, E_{5m}, represent the crest values of the first, third, and fifth harmonics of the E.M.F. wave. Then the R.M.S. value (75 volts) indicated by the voltmeter is given by

$$E = 75 = 0{\cdot}707\sqrt{(E_{1m}^2 + E_{3m}^2 + E_{5m}^2)}$$

But $I_{1m} = E_{1m}/\omega L$; $I_{3m} = E_{3m}/3\omega L$; $I_{5m} = E_{5m}/5\omega L$.

Hence

$$\begin{aligned} E &= 0{\cdot}707\sqrt{[(I_{1m}\omega L)^2 + (3I_{3m}\omega L)^2 + (5I_{5m}\omega L)^2]} \\ &= 0{\cdot}707\omega L\sqrt{(I_{1m}^2 + 9I_{3m}^2 + 25I_{5m}^2)} \\ &= 0{\cdot}707\omega L I_{1m}\sqrt{[1 + 9(I_{3m}/I_{1m})^2 + 25(I_{5m}/I_{1m})^2]} \end{aligned}$$

Substituting the numerical values given above, we have

$$75 = 0{\cdot}707 \times 2\pi \times 50 \times L \times 100\sqrt{[1 \times 9(0{\cdot}15)^2 + 25(0{\cdot}12)^2]}$$

$$\therefore \quad L = 0{\cdot}0027 \text{ H.}$$

[Note. Apparent value of inductance = 75/(314 × 72) = 0·00331 H.]

POWER IN SINGLE-PHASE CIRCUITS IN WHICH E.M.F. AND CURRENT ARE NON-SINUSOIDAL

Let the E.M.F. and current waves be represented by the equations

$$\begin{aligned} e &= E_{1m}\sin(\omega t + \alpha_1) + E_{3m}\sin(3\omega t + \alpha_3) \\ &\qquad + E_{5m}\sin(5\omega t + \alpha_5) + \dots \\ i &= I_{1m}\sin(\omega t + \alpha_1 - \varphi_1) + I_{3m}\sin(3\omega t + \alpha_3 - \varphi_3) \\ &\qquad + I_{5m}\sin(5\omega t + \alpha_5 - \varphi_5) + \dots \end{aligned}$$

Then the instantaneous power (p) is given by

$$p = ei = E_{1m}I_{1m}\sin(\omega t + a_1)\sin(\omega t + a_1 - \varphi_1)$$
$$+ E_{3m}I_{3m}\sin(3\omega t + a_3)\sin(3\omega t + a_3 - \varphi_3) +$$
$$+ E_{1m}I_{3m}\sin(\omega t + a_1)\sin(3\omega t + a_3 - \varphi_3) + \dots$$
$$+ E_{3m}I_{1m}\sin(3\omega t + a_3)\sin(\omega t + a_1 - \varphi_1) + \dots$$

and the mean power (P) is given by

$$P = \frac{1}{\pi}\int_0^\pi ei\,d\omega t = \frac{1}{\pi}\int_0^\pi \{E_{1m}I_{1m}\sin(\omega t + a_1)\sin(\omega t + a_1 - \varphi_1)$$
$$+ E_{3m}I_{3m}\sin(3\omega t + a_3)\sin(3\omega t + a_3 - \varphi_3) + \dots$$
$$+ E_{1m}I_{3m}\sin(\omega t + a_1)\sin(3\omega t + a_3 - \varphi_3) + \dots$$
$$+ E_{3m}I_{1m}\sin(3\omega t + a_3)\sin(\omega t + a_1 - \varphi_1) + \dots\}d\omega t$$

Now the integral, taken between 0 and π, of all products of different frequencies is zero, and the integral of terms such as

$$E_mI_m\sin(\omega t + a)\sin(\omega t + a - \varphi) = \tfrac{1}{2}\pi E_mI_m\cos\varphi.*$$

Hence the expression for the mean power reduces to

$$P = \frac{1}{\pi}(\tfrac{1}{2}\pi E_{1m}I_{1m}\cos\varphi_1 + \tfrac{1}{2}\pi E_{3m}I_{3m}\cos\varphi_3$$
$$+ \tfrac{1}{2}\pi E_{5m}I_{5m}\cos\varphi_5 + \dots)$$
$$= \tfrac{1}{2}(E_{1m}I_{1m}\cos\varphi_1 + E_{3m}I_{3m}\cos\varphi_3 + E_{5m}I_{5m}\cos\varphi_5 + \dots)$$
$$= E_1I_1\cos\varphi_1 + E_3I_3\cos\varphi_3 + E_5I_5\cos\varphi_5 + \dots \quad . \quad (108)$$

Thus the mean power due to distorted current and E.M.F. waves is equal to the sum of the mean powers due to the several harmonic components.

Power factor of circuits in which current and E.M.F. are non-sinusoidal. In Chapter IV power factor was defined in two ways : (1) the ratio of the mean power to the apparent power, or volt-amperes ; (2) the cosine of the angle of phase difference between impressed E.M.F. and current. The power factor determined according to the first definition is a definite quantity for any particular circuit, whether the supply E.M.F. is sinusoidal or non-sinusoidal. The second definition of power factor, however, requires further consideration when the E.M.F. and current waves are non-sinusoidal. Thus, with sinusoidal current and E.M.F.

* Thus, $\int_0^\pi \sin(\omega t + a)\sin(\omega t + a - \varphi)d\omega t = \int_0^\pi [\sin(\omega t + a)\sin(\omega t + a)\cos\varphi - \cos(\omega t + a)\sin\varphi]d\omega t = \int_0^\pi \sin^2(\omega t + a)\cos\varphi\, d\omega t - \int_0^\pi (\sin(\omega t + a)\cos(\omega t + a)\sin\varphi)d\omega t = \tfrac{1}{2}\pi\cos\varphi$

the phase difference, when the "constants" of the circuit are invariable, is constant at every instant, but with distorted waves the phase difference between the maximum values of E.M.F. and current is not necessarily the same as that between the zero values of these quantities, since the current and E.M.F. waves may be of different shapes. In these circumstances ambiguity may be avoided by employing the term "phase difference" (or angle of lag, or lead) only in connection with the *equivalent* sine waves of current and E.M.F. (i.e. the sine waves having the same frequency and R.M.S. values as the distorted waves*). Hence, if φ_e is the phase difference between these (equivalent sine) waves, then $\cos \varphi_e$ will represent the equivalent power factor of the circuit; i.e.

$$\cos \varphi_e = P/EI,$$

where P is the mean power (as measured by a wattmeter or calculated from equation (108) and E, I are the R.M.S. values of E.M.F. and current, respectively.

DISTORTED E.M.F. AND CURRENT WAVES IN POLYPHASE CIRCUITS

The method of treatment is in general similar to that given above for single phase circuits, viz., each harmonic is treated separately, and the current or voltage harmonics for the several "phases" of the circuit are compounded geometrically in the same manner as these quantities were compounded in the case of simple sine waves.

Expressions for "phase" E.M.Fs. Considering the symmetrical three-phase system, let the "phase" E.M.Fs. be represented by the equations

$$\begin{aligned} e_{\text{I}} &= E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) \\ &\qquad + E_{5m}\sin(5\omega t + a_5) + \ldots \\ e_{\text{II}} &= E_{1m}\sin(\omega t + a_1 - \tfrac{2}{3}\pi) + E_{3m}\sin(3\omega t + a_3 - 3 \times \tfrac{2}{3}\pi) \\ &\qquad + E_{5m}\sin(5\omega t + a_5 - 5 \times \tfrac{2}{3}\pi) + \ldots \\ e_{\text{III}} &= E_{1m}\sin(\omega t + a_1 - \tfrac{4}{3}\pi) + E_{3m}\sin(3\omega t + a_3 - 3 \times \tfrac{4}{3}\pi) \\ &\qquad + E_{5m}\sin(5\omega t + a_5 - 5 \times \tfrac{4}{3}\pi) + \ldots \end{aligned}$$

These, upon simplification, become

$$\begin{aligned} e_{\text{I}} &= E_{1m}\sin(\omega t + a_1) + E_{3m}\sin(3\omega t + a_3) \\ &\qquad + E_{5m}\sin(5\omega t + a_5) + \ldots \quad . \quad (109) \end{aligned}$$

* See p. 16 for method of determining the equivalent sine wave.

$$e_{\mathrm{II}} = E_{1m}\sin(\omega t + \alpha_1 - \tfrac{2}{3}\pi) + E_{3m}\sin(3\omega t + \alpha_3) + E_{5m}\sin(5\omega t + \alpha_5 + \tfrac{2}{3}\pi) + \ldots \quad (110)$$

$$e_{\mathrm{III}} = E_{1m}\sin(\omega t + \alpha_1 - \tfrac{4}{3}\pi) + E_{3m}\sin(3\omega t + \alpha_3) + E_{5m}\sin(5\omega t + \alpha_5 + \tfrac{4}{3}\pi) + \ldots \quad (111)$$

$$= E_{1m}\sin(wt + \alpha_1 + \tfrac{2}{3}\pi) + E_{3m}\sin(3\omega t + \alpha_3) + E_{5m}\sin(5\omega t + \alpha_5 - \tfrac{2}{3}\pi) + \ldots \quad (111a)$$

Thus all harmonics of triple frequency as well as their multiples (i.e. the 9th, 15th, 21st, etc.) are equal in all the phases of the circuit. Further, at any instant these E.M.Fs. have the same direction (i.e. in a star-connected system they are all directed either away from, or towards, the neutral point, and in a Δ-connected system they all act in the *same* direction around the circuit). Moreover, all harmonics which are not a multiple of three are displaced 120° from one another, and can be dealt with in the usual manner. It is to be observed, however, that these harmonics do not have all the same phase rotation as the fundamental; the phase rotation for the 5th, 11th, 17th, 23rd, 29th, etc., harmonics being opposite to that of the fundamental, and that for the 7th, 13th, 19th, 25th, 31st, etc., harmonics being the same as that of the fundamental.

Expressions for the "line" E.M.Fs. in a star-connected three-phase system. Having obtained the equations to the "phase" E.M.Fs. the equations to the "line" E.M.Fs. readily follow, thus

$$\begin{aligned}
v_{\mathrm{I\,II}} = e_{\mathrm{I}} - e_{\mathrm{II}} &= E_{1m}\{\sin(\omega t + \alpha_1) - \sin(\omega t + \alpha_1 - \tfrac{2}{3}\pi)\} \\
&\quad + E_{3m}\{\sin(3\omega t + \alpha_3) - \sin(3\omega t + \alpha_3)\} \\
&\quad + E_{5m}\{\sin(5\omega t + \alpha_5) - \sin(5\omega t + \alpha_5 - \tfrac{4}{3}\pi)\} + \ldots \\
&= 2E_{1m}\{\cos\tfrac{1}{2}[2(\omega t + \alpha_1) - \tfrac{2}{3}\pi]\sin(\tfrac{1}{2} \times \tfrac{2}{3}\pi)\} \\
&\quad + 2E_{5m}\{\cos\tfrac{1}{2}[2(5\omega t + \alpha_5) - \tfrac{4}{3}\pi]\sin(\tfrac{1}{2} \times \tfrac{4}{3}\pi)\} + \ldots \\
&= 2E_{1m}\{\tfrac{1}{2}\sqrt{3}\cos(\omega t + \alpha_1 - \tfrac{1}{3}\pi)\} \\
&\quad + 2E_{5m}\{\tfrac{1}{2}\sqrt{3}\cos(5\omega t + \alpha_5 - \tfrac{2}{3}\pi)\} + \ldots \\
&= \sqrt{3}\{E_{1m}\sin(\omega t + \alpha_1 - \tfrac{1}{3}\pi + \tfrac{1}{2}\pi) \\
&\quad + E_{5m}\sin(5\omega t + \alpha_5 - \tfrac{2}{3}\pi + \tfrac{1}{2}\pi) + \ldots\} \\
&= \sqrt{3}\{E_{1m}\sin(\omega t + \alpha_1 + \tfrac{1}{6}\pi) + E_{5m}\sin(5\omega t \\
&\quad + \alpha_5 - \tfrac{1}{6}\pi) + \ldots\} \quad (112)
\end{aligned}$$

$$v_{\mathrm{II\text{-}III}} = e_{\mathrm{II}} - e_{\mathrm{III}} = \sqrt{3}\{E_{1m}\sin(\omega t + \alpha_1 - \tfrac{1}{2}\pi) + E_{5m}\sin(5\omega t + \alpha_5 + \tfrac{1}{2}\pi) + \ldots\} \quad (113)$$

$$v_{\mathrm{III\,I}} = e_{\mathrm{III}} - e_{\mathrm{I}} = \sqrt{3}\{E_{1m}\sin(\omega t + \alpha_1 + \tfrac{5}{6}\pi) + E_{5m}\sin(5\omega t + \alpha_5 - \tfrac{5}{6}\pi) + \ldots\} \quad . \quad (114)$$

Thus in a star-connected symmetrical three-phase system no E.M.F. of triple frequency, or a multiple thereof, appears in the line pressures, notwithstanding that these E.M.Fs. may be present in the phase pressures.

If, in the above equations of line E.M.Fs. we replace $(\omega t + \frac{1}{6}\pi)$ by $\omega t'$, which means that time is now reckoned from an instant 30° in advance of the previous zero, we have

$$v_{\mathrm{I\,II}} = \sqrt{3}\{E_{1m}\sin(\omega t' - \alpha_1) - E_{5m}\sin(5\omega t' - \alpha_5) + \ldots\} \quad (112a)$$

$$v_{\mathrm{II\,III}} = \sqrt{3}\{E_{1m}\sin(\omega t' - \alpha_1 - \tfrac{2}{3}\pi) - E_{5m}\sin(5\omega t' - \alpha_5 - \tfrac{4}{3}\pi) + \ldots\} \quad . \quad (113a)$$

$$v_{\mathrm{III\,I}} = \sqrt{3}\{E_{1m}\sin(\omega t' - \alpha_1 - \tfrac{4}{3}\pi) - E_{5m}\sin(5\omega t' - \alpha_5 - \tfrac{2}{3}\pi) + \ldots\} \quad . \quad (114a)$$

These equations are now similar to those for the phase E.M.Fs. except that (1) there are no third harmonic terms; (2) the signs of the fifth harmonics are changed*; and (3) the factor $\sqrt{3}$ is introduced.

R.M.S. values of "phase" and "line" E.M.Fs. in a three-phase system. In a symmetrical three-phase system the amplitude of any particular harmonic component of the "phase" E.M.F. is the same for each phase of the circuit, so that when considering R.M.S. values we may denote the R.M.S. value of the fundamental of the phase E.M.F. for each circuit by E_1, and the R M.S. values of the several harmonics by E_3, E_5, E_7, etc.

Similarly, the R M.S. value of the fundamental of the line E.M.F. may be denoted by V_1, and the R.M.S. values of the several harmonics by V_5, V_7, etc.

Then, from equations (112), (113), (114),

$$V_1 = \sqrt{3}\,E_1, \quad V_5 = \sqrt{3}\,E_5, \quad V_7 = \sqrt{3}\,E_7, \quad V_{11} = \sqrt{3}\,E_{11}, \quad V_{13} = \sqrt{3}\,E_{13}$$

The R.M.S. value of the line voltage (V) of the system is therefore given by

$$V = \sqrt{(V_1^2 + V_5^2 + V_7^2 + \ldots)}$$

Similarly the R.M.S. value of the "phase" E.M.F. is given by

$$E = \sqrt{(E_1^2 + E_3^2 + E_5^2 + E_7^2 + \ldots)}$$

* The seventh harmonics also have the minus sign.

Hence

$$\begin{aligned}\frac{V}{E} &= \sqrt{\left(\frac{V_1^2 + V_5^2 + V_7^2 + \ldots}{E_1^2 + E_3^2 + E_7^2 + \ldots}\right)} \\ &= \sqrt{3}\sqrt{\left(\frac{E_1^2 + E_5^2 + E_7^2 + \ldots}{E_1^2 + E_3^2 + E_5^2 + E_7^2 + \ldots}\right)} \\ &= \sqrt{3}\sqrt{\left(\frac{1 + (E_5/E_1)^2 + (E_7/E_1)^2 + \ldots}{1 + (E_3/E_1)^2 + (E_5/E_1)^2 + (E_7/E_1)^2 + \ldots}\right)} \qquad (155)\end{aligned}$$

Thus in the case of distorted E.M.F. wave forms the value of the ratio of the " line " and " phase " E.M.Fs. is not the same as that ($\sqrt{3}$) when the E.M.Fs. are sinusoidal. An exception occurs, however, when the third harmonic and its multiples are not present in the phase E.M.Fs.

Example. If the phase E.M.F. of a star-connected, three-phase alternator contains first, third, fifth, seventh, and ninth harmonics of amplitudes 100, 13, 5, 1·5, 1, respectively, the ratio (line E.M.F./phase E.M.F.) is

$$\sqrt{3}\sqrt{\left(\frac{1 + (5/100)^2 + (1\cdot5/100)^2 + \ldots}{1 + (13/100)^3 + (5/100)^2 + (1\cdot5/100)^2 + (1/100)^2}\right)} = 1\cdot732 \times 0\cdot9915$$
$$= 1\cdot717.$$

If, however, the third and fifth harmonics are 30 per cent and 10 per cent, respectively, of the fundamental, the ratio (line E.M.F./phase E.M.F.) is

$$\sqrt{3}\sqrt{\left(\frac{1 + (10/100)^2 + (1\cdot5/100)^2 + \ldots}{1 + (30/100)^2 + (10/100)^2 + (1\cdot5/100)^2 + (1/100)^2}\right)}$$
$$= 1\cdot732 \times 0\cdot9583$$
$$= 1\cdot66.$$

In the former case, which is taken from actual practice, the ratio of the " line " and " phase " E.M.Fs. is 0·87 per cent less than the value for sinusoidal waves, while in the latter case, which represents excessive wave distortion, the ratio is 4·15 per cent less than the value for sinusoidal waves.

Circulating current in delta-connected alternator in which E.M.F. wave is non-sinusoidal. Consider the case of an alternator in which the phase E.M.Fs. are symmetrical and represented by the equations

$$\begin{aligned} e_{\mathrm{I}} &= E_{1m}\sin(\omega t + \alpha_1) + E_{3m}\sin(3\omega t + \alpha_3) \\ &\qquad + E_{5m}\sin(5\omega t + \alpha_5) + \ldots \\ e_{\mathrm{II}} &= E_{1m}\sin(\omega t + \alpha_1 - \tfrac{2}{3}\pi) + E_{3m}\sin(3\omega t + \alpha_3) \\ &\qquad + E_{5m}\sin(5\omega t + \alpha_5 + \tfrac{2}{3}\pi) + \ldots \\ e_{\mathrm{III}} &= E_{1m}\sin(\omega t + \alpha_1 - \tfrac{4}{3}\pi) + E_{3m}\sin(3\omega t + \alpha_3) \\ &\qquad + E_{5m}\sin(5\omega t + \alpha_5 + \tfrac{4}{3}\pi) + \ldots \end{aligned}$$

Then the resultant E.M.F. acting in the delta-connected armature winding is

$$\begin{aligned} e_{\mathrm{I}} + e_{\mathrm{II}} + e_{\mathrm{III}} &= 3E_{3m}\sin(3\omega t + \alpha_3) + 3E_{9m}\sin(9\omega t + \alpha_9) \\ &\qquad + 3E_{15m}\sin(15\omega t + \alpha_{15}) + \ldots \end{aligned}$$

Hence if R is the resistance per phase and L the inductance per phase of the armature winding, the circulating current (i_c) due to the resultant E.M.F. is

$$i_c = \frac{3E_{3m}\sin(3\omega t + a_3)}{3\sqrt{(R^2 + 9\omega^2L^2)}} + \frac{3E_{9m}\sin(9\omega t + a_9)}{3\sqrt{(R^2 + 81\omega^2L^2)}} + \frac{3E_{15m}\sin(15\omega t + a_{15})}{3\sqrt{(R^2 + 225\omega^2L^2)}} +$$

$$= \frac{E_{3m}\sin(3\omega t + a_3)}{\sqrt{(R^2 + 9\omega^2L^2)}} + \frac{E_{9m}\sin(9\omega t + a_9)}{\sqrt{(R^2 + 81\omega^2L^2)}} + \frac{E_{15m}\sin(15\omega t + a_{15})}{\sqrt{(R^2 + 225\omega^2L^2)}} + \ldots$$

and its R.M.S. value is

$$I_c = 0{\cdot}707\sqrt{[E_{3m}{}^2/(R^2 + 9\omega^2L^2) + E_{9m}{}^2/(R^2 + 81\omega^2L^2) + E_{15m}{}^2/(R^2 + 225\omega^2L^2) + \ldots]}$$

Example. A three-phase 50-cycle alternator has a delta-connected armature winding for which the resistance and inductance per phase are 0·025 O. and 0·4 mH. respectively. The no-load E.M.F. wave form contains third, ninth, and fifteenth harmonics (together with others) which have amplitudes of 4 per cent, 2 per cent, and 1·5 per cent of that of the fundamental. Calculate the circulating current at no-load when the excitation is such that the amplitude of the fundamental of the E.M.F. is 850 volts.

The reactance per phase (ωL) at 50 frequency

$$314 \times 0{\cdot}4 \times 10^{-3} = 0{\cdot}126 \text{ O}.$$

Hence, for the 3rd, 9th, and 15th harmonics we have

$$9\omega^2L^2 = 0{\cdot}142 \text{ O}; \quad 81\omega^2L^2 = 1{\cdot}28 \text{ O}.; \quad 225\omega^2L^2 = 3{\cdot}55 \text{ O},$$

whence the values of the corresponding (impedance per phase)^2 are

$$R^2 + 9\omega^2L^2 = 0{\cdot}1426 \text{ O}; \quad R^2 + 81\omega^2L^2 = 1{\cdot}2806 \text{ O};$$

$$R^2 + 225\omega^2L^2 = 3{\cdot}55 \text{ O}.$$

The maximum values, or amplitudes, of the E.M.Fs. due to the 3rd, 9th, and 15th harmonics are

$$E_{3m} = 0{\cdot}04 \times 850 = 34 \text{ V.}; \quad E_{9m} = 0{\cdot}02 \times 850 = 17 \text{ V.};$$

$$E_{15m} = 0{\cdot}015 \times 850 = 12{\cdot}7 \text{ V.}$$

Hence $I_c = 0{\cdot}707\sqrt{[(34^2/0{\cdot}1426) + (17^2/1{\cdot}28) + (12{\cdot}7^2/3{\cdot}55)]} = 64{\cdot}8$ A.

Power in three-phase circuits in which the E.M.Fs. and currents are non-sinusoidal. The total power in a polyphase system in which the E.M.Fs. and currents are non-sinusoidal is equal to the algebraic sum of the power supplied by the several harmonic components of the current and E.M.F. Hence in the case of a three-phase, three-wire system with balanced loads we have

$$P = \sqrt{3}.V_1I_1\cos\varphi_1 + \sqrt{3}.V_5I_5\cos\varphi_5 + \sqrt{3}.V_7I_7\cos\varphi_7 + \ldots \quad (156)$$

where V_1, V_5, V_7, . . . I_1, I_5, I_7, . . . denote the line voltages and currents, respectively, due to the several harmonics.

In the case of a four-wire system, the neutral wire provides a path for the circulation of currents of triple frequency and multiples thereof. The power is therefore given by

$$P = \sqrt{3}.V_1I_1\cos\varphi_1 + \sqrt{3}.V_3I_3\cos\varphi_3 + \sqrt{3}.V_5I_5\cos\varphi_5 + \sqrt{3}.V_7I_7\cos\varphi_7 + \dots \quad (157)$$

which is greater than that in the corresponding three-wire system by the amount of the power due to the third harmonic and its multiples.

HARMONIC ANALYSIS

General. The process of determining the magnitude, order, and phase of the several harmonics in a complex periodic curve is called *harmonic analysis*. The analysis may be effected by analytical, graphical, or mechanical methods, the mechanical method involving the use of a special instrument called a "harmonic analyser."

All methods of analysis are based upon Fourier's theorem, which states that any single-valued complex periodic function can be resolved into a series of simple harmonic curves, the first of which has a frequency equal to that of the complex function. Thus, if y_θ denotes the magnitude of any ordinate at abscissa θ of the complex curve, then, generally,

$$y_\theta = F_0 + F_1\sin(\theta + \alpha_1) + F_2\sin(2\theta + \alpha_2) + F_3\sin(3\theta + \alpha_3) + \dots$$

where F_0, F_1, F_2, F_3, . . . are constants, and α_1, α_2, α_3 . . . are the phase differences of the harmonic components with respect to the complex wave. The constant F_0 is required only in cases where the mean value of the ordinates for a cycle of the complex wave is not zero. Hence if the reference axis is so drawn that the mean value of the ordinates for a cycle is zero, the equation to the curve is given by

$$y_\theta = F_1\sin(\theta + \alpha_1) + F_2\sin(2\theta + \alpha_2) + F_3\sin(3\theta + \alpha_3) + \dots \quad (158)$$

and, when only odd harmonics are present, we have

$$y_\theta = F_1\sin(\theta + \alpha_1) + F_3\sin(3\theta + \alpha_3) + F_5\sin(5\theta + \alpha_5) + \dots \quad (158a)$$

This equation is representative of the complex E.M.F. and current waves which are met with in alternating-current work.

The number of terms which may be necessary to express the function with absolute exactness depends upon the extent to which the complex curve deviates from a sine curve

The number of terms to be included in the analysis depends on the degree of accuracy desired. For commercial purposes analysis as far as the fifth harmonic is generally sufficient, as the principal harmonics which occur in alternating-current systems are the third and the fifth. In special cases it may be desirable to carry out the analysis so as to include harmonics of the 17th, or higher, orders. These higher harmonics occur principally in generators and are due to magnetic pulsations; their amplitude is usually very small in comparison with that of the fundamental.

Analytical methods of analysis. With these methods the constants F_1, F_3, . . . , and the phase angles a_1, a_3, . . . , are obtained indirectly. Thus, since $\sin(\theta + a) = \sin\theta\cos a + \cos\theta\sin a$, each term of the series

$$y_\theta = F_1\sin(\theta + a_1) + F_3\sin(3\theta + a_3) + F_5\sin(5\theta + a_5) + \ldots$$

can be expressed as the sum of a sine and cosine term.

Effecting this transformation, we have—

$$\begin{aligned} y_\theta = {} & F_1\sin\theta\cos a_1 + F_3\sin 3\theta\cos a_3 + F_5\sin 5\theta\cos a_5 + \ldots \\ & + F_1\cos\theta\sin a_1 + F_3\cos 3\theta\sin a_3 + F_5\cos 5\theta\sin a_5 + \ldots \end{aligned} \quad (158b)$$

Substituting A_1 for $F_1\cos a_1$, A_3 for $F_3\cos a_3$, etc.; B_1 for $F_1\sin a_1$, B_3 for $F_3\sin a_3$, etc., we have

$$\begin{aligned} y_\theta = {} & A_1\sin\theta + A_3\sin 3\theta + A_5\sin 5\theta + \ldots \\ & + B_1\cos\theta + B_3\cos 3\theta + B_5\cos 5\theta + \ldots \end{aligned} \quad (159)$$

Hence $A_1^2 + B_1^2 = F_1^2\cos^2 a_1 + F_1^2\sin^2 a_1 = F_1^2$,

Whence $F_1 = \sqrt{(A_1^2 + B_1^2)}$

Similarly $F_3 = \sqrt{(A_3^2 + B_3^2)}$; $F_5 = \sqrt{(A_5^2 + B_5^2)}$; etc.

Also $A_1/B_1 = F_1\cos a_1/F_1\sin a_1 = \cot a_1$

Whence $a_1 = \tan^{-1} B_1/A_1$

Similarly $a_3 = \tan^{-1} B_3/A_3$;

$a_5 = \tan^{-1} B_5/A_5$; etc.

The coefficients A_1, A_3, A_5, . . . , B_1, B_3, B_5, . . . , are best evaluated by a summation, or integration, process, and the labour may be shortened considerably by suitably selecting the ordinates

and grouping the terms as shown later. The summation process is based upon the following theorems of the integral calculus—

$$\int_0^{\pi} \sin^2 \theta . d\theta = \tfrac{1}{2}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

$$\int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = \begin{Bmatrix} 0 \text{ when } m \neq n \\ 0 \text{ when } m = 0, n = 0 \end{Bmatrix} . \quad . \quad (161)$$

$$\int_0^{\pi} \cos^2 \theta \, d\theta = \tfrac{1}{2}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (162)$$

$$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = \begin{Bmatrix} 0 \text{ when } m \neq n \\ \pi \text{ when } m = 0, n = 0 \end{Bmatrix} \quad . \quad (163)$$

$$\int^{\pi} \sin m\theta \cos n\theta \, d\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

where m and n are any positive integers.

For instance, to evaluate A_1 multiply equation (159) throughout by $\sin \theta$ and integrate between the limits 0 and π,* thus

$$\begin{aligned} \int^{\pi} (y_\theta \sin \theta) d\theta = \int^{\pi} \{ & A_1 \sin^2 \theta + A_3 \sin 3\theta \sin \theta \\ & + A_5 \sin 5\theta \sin \theta + \ldots \\ & + B_1 \cos \theta \sin \theta + B_3 \cos 3\theta \sin \theta \\ & + B_5 \cos 5\theta \sin \theta + \ldots \} d\theta \\ = \; & \tfrac{1}{2}\pi A_1 \end{aligned}$$

Hence

$$\begin{aligned} A_1 &= 2 \times \frac{1}{\pi} \int_0^{\pi} (y_\theta \sin \theta) \, d\theta \\ &= 2 \times \text{mean value of } y_\theta \sin \theta \text{ for the half period.} \end{aligned}$$

Similarly,

$$A_3 = 2 \times \text{mean value of } y_\theta \sin 3\theta \text{ for the half period.}$$

. . . . . .

$$B_1 = 2 \times \text{mean value of } y_\theta \cos \theta \text{ for the half period.}$$

$$B_3 = 2 \times \text{mean value of } y_\theta \cos 3\theta \text{ for the half period.}$$

. . . . . .

Method of summation with ungrouped terms. The mean value of $y_\theta \sin \theta$ may be obtained simply by dividing the half period into a number of equal parts (the number of parts depending upon the accuracy desired), by means of ordinates erected at equidistant intervals, the first ordinate being erected at abscissa zero;

* With wave forms in which the two half-waves are symmetrical, only one half of the wave need be considered in analysis.

multiplying each ordinate by the sine of the angle corresponding to the abscissa; adding the products and finally dividing the sum by the number of intervals.

The mean value of the other products—$y \sin 3\theta$, $y \sin 5\theta$, etc.—is obtained in a similar manner, except that the ordinates are multiplied by $\sin 3\theta$, $\sin 5\theta$, etc., instead of $\sin \theta$.

For example, if the half period is divided into six equal parts by ordinates erected at abscissæ 0°, 30°, 60°, 90°, 120°, 150°, 180°, and the values of these ordinates are denoted by y_0, y_{30}, y_{60}, y_{90}, y_{120}, y_{150}, y_{180}, respectively, then

$$A_1 = 2\left\{\tfrac{1}{6}(y_{30}\sin 30° + y_{60}\sin 60° + y_{90}\sin 90° + y_{120}\sin 120° + y_{150}\sin 150° + y_{180}\sin 180°)\right\}$$

Similarly

$$A_3 = 2\left\{\tfrac{1}{6}(y_{30}\sin 90° + y_{60}\sin 180° + y_{90}\sin 270° + y_{120}\sin 360° + y_{150}\sin 450° + y_{180}\sin 540°)\right\}$$

. . . . . .

$$B_1 = 2\left\{\tfrac{1}{6}(y_{30}\cos 30° + y_{60}\cos 60° + y_{90}\cos 90° + y_{120}\cos 120° + y_{150}\cos 150° + y_{180}\cos 180°)\right\}$$

$$B_3 = 2\left\{\tfrac{1}{6}(y_{30}\cos 90° + y_{60}\cos 180° + y_{90}\cos 270 + y_{120}\cos 360° + y_{150}\cos 450° + y_{180}\cos 180°)\right\}$$

. . . . . .

On the assumption that all harmonics present in the wave form have been included in the analysis, a check upon the accuracy of the calculations may be obtained by the application of the following rules*—

(1) Ordinate at 90° $= y_{90} = A_1 - A_3 + A_5 - A_7 + \ldots$

(2) Ordinate at 0° $= y_0 = B_1 + B_3 + B_5 + B_7 + \ldots$

Example. Analysis of E.M.F. wave (Fig. 186) for odd harmonics up to, and including, the fifth.

The half-period is divided into twelve equal parts by ordinates erected at abscissæ 0°, 15°, 30°, etc., the values of the ordinates being as follows –

Abscissa	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Ordinate	0	24·5	51	84·2	126	171	191·5	180	152·3	119	82	41	0

The calculations are best carried out in tabular form and are given in Table IX.

* $y_{90} = A_1 \sin 90° + A_3 \sin(3 \times 90)° + A_5 \sin(5 \times 90)° \ldots + B_1 \cos 90° + B_3 \cos(3 \times 90)° + B_5 \cos(5 \times 90)° + \ldots$
$= A_1 - A_3 + A_5 - \ldots$

$y_0 = A_1 \sin 0° + A_3 \sin 0° + A_5 \sin 0° + \ldots + B_1 \cos 0° + B_3 \cos 0° + B_5 \cos 0° + \ldots$
$= B_1 + B_3 + B_5 + \ldots$

TABLE IX

Analysis of E.M.F. curve up to fifth harmonic

		Sine Terms						Cosine Terms					
θ	y_θ	$\sin\theta$	$y\sin\theta$	$\sin 3\theta$	$y\sin 3\theta$	$\sin 5\theta$	$y\sin 5\theta$	$\cos\theta$	$y\cos\theta$	$\cos 3\theta$	$y\cos 3\theta$	$\cos 5\theta$	$y\cos 5\theta$
15°	24·5	0·259	6·34	0·707	17·33	0·966	23·7	0·966	23·7	0·707	17·33	0·259	6·34
30°	51	0·5	25·5	1·0	51	0·5	25·5	0·866	44·1	0	0	−0·866	−44·1
45°	84·2	0·707	59·5	0·707	59·5	−0·707	−59·5	0·707	59·5	−0·707	−59·5	−0·707	−59·5
60°	126	0·866	109	0	0	−0·866	−109	0·5	63	−1·0	−126	0·5	63
75°	171	0·966	165·2	−0·707	−121	0·259	44·3	0·259	44·3	−0·707	−121	0·966	165·2
90°	191·5	1·0	191·5	−1·0	−191·5	1·0	191·5	0	0	0	0	0	0
105°	180	0·966	173·9	−0·707	−127·2	0·259	46·6	−0·259	−46·6	0·707	127·2	−0·966	−173·9
120°	152·3	0·866	132	0	0	−0·866	−132	−0·5	−76·1	1·0	152·3	−0·5	−76·1
135°	119	0·707	84·1	0·707	84·1	−0·707	−84·1	−0·707	−84·1	0·707	84·1	0·707	84·1
150°	82	0·5	41	1·0	82	0·5	41	−0·866	−71	0	0	0·866	71
165°	41	0·259	10·62	0·707	29	0·966	39·6	−0·966	−39·6	−0·707	−29	−0·259	−10·62
180°	0	0	0	0	0	0	0	−1·0	0	−1·0	0	−1·0	0
Total . .			998·7		−116·8		27·6		−82·8		45·4		25·4
$\frac{1}{6}$ × total .			(A_1) 166·4		(A_3) −19·5		(A_5) 4·6		(B_1) −13·8		(B_3) 7·6		(B_5) 4·2
Check .		$A_1 - A_3 + A_5 = y_{90}$; i.e. 166·4 + 19·5 + 4·6 = 190·5*						$B_1 + B_3 + B_5 = 0$; i.e. −13·8 + 7·6 + 4·2 = −2*					

$F_1 = \sqrt{(166{\cdot}4^2 + 13{\cdot}8^2)} = 167.$ $\quad F_3 = \sqrt{(19{\cdot}5^2 + 7{\cdot}6^2)} = 20{\cdot}9.$ $\quad F_5 = \sqrt{(4{\cdot}6^2 + 4{\cdot}2^2)} = 6{\cdot}23$

$a_1 = \tan^{-1} - 13{\cdot}8/166{\cdot}4 = -4{\cdot}8^\circ$ $\quad a_3 = \tan^{-1} 7{\cdot}6/-19{\cdot}5 = 158{\cdot}7^\circ$ $\quad a_5 = \tan^{-1} 4{\cdot}2/4{\cdot}6 = 42{\cdot}4^\circ$

Equation to curve—$e = 167 \sin(\theta - 4{\cdot}8^\circ) + 20{\cdot}9 \sin(3\theta + 158{\cdot}7^\circ) + 6{\cdot}2 \sin(5\theta + 42{\cdot}4^\circ)$

* These discrepancies indicate that other harmonics are present in the wave form, and that the present analysis only gives an approximation to the equation of the wave.

Summation methods employing grouped terms. With these methods the A and B coefficients are obtained by so choosing the number of ordinates that certain terms may be grouped together to enable the multiplication operations to be effected upon an assemblage of terms instead of upon individual terms. By these means the labour involved in the summation is shortened considerably, but in certain cases chiefly those in which the analysis includes only harmonics of low orders, and the number of ordinates is relatively small- the accuracy is not equal to that of the longer methods in which a relatively large number of ordinates are employed. The accuracy of the shorter methods, however, is usually sufficient for commercial purposes.

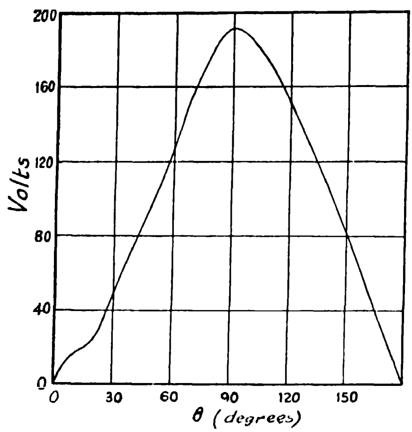


FIG. 186 Wave Form of E.M.F. analysed in Tables IX and XII.

In the method due to *Runge and Thompson* the number of intermediate ordinates in the half-period of the wave to be analysed is chosen equal to the order of the highest harmonic required, and these ordinates are treated in supplemental pairs, the sum and difference of each pair being determined. The sums contain only A, or sine, coefficients and the differences contain only B, or cosine coefficients.* The terms are then grouped so as to avoid repetition of the multiplication operations, special groupings being possible

* Thus,

$$\begin{aligned} y_\theta + y_{(\pi-\theta)} = {} & [A_1 \sin \theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \ldots \\ & + B_1 \cos \theta + B_3 \cos 3\theta + B_5 \cos 5\theta + \ldots] \\ & + [A_1 \sin (\pi - \theta) \\ & + A_3 \sin 3(\pi - \theta) + A_5 \sin 5(\pi - \theta) + \ldots \\ & + B_1 \cos(\pi - \theta) + B_3 \cos 3(\pi - \theta) \\ & + B_5 \cos 5(\pi - \theta) + \ldots] \\ = {} & 2A_1 \sin \theta + 2A_3 \sin 3\theta + 2A_5 \sin 5\theta + \ldots \\ y_\theta - y_{(\pi-\theta)} = {} & [A_1 \sin \theta + A_3 \sin 3\theta + \ldots + B_1 \cos \theta + B_3 \cos 3\theta + \ldots] \\ & - [A_1 \sin(\pi - \theta) + A_3 \sin 3(\pi - \theta) + \ldots \\ & + B_1 \cos(\pi - \theta) + B_3 \cos 3(\pi - \theta) + \ldots] \\ = {} & 2B_1 \cos \theta + 2B_3 \cos 3\theta + \ldots \end{aligned}$$

for the third harmonic and its multiples. The manner in which the grouping is effected is best shown by considering a few typical cases.

I. Analysis of a periodic curve for odd harmonics up to, and including, the fifth. Ordinates are erected at abscissæ 30°, 60°, 90°, 120°, 150°. The sums and differences of the ordinates of supplemental angles are determined, the sums being denoted by a_1, a_2, a_3, . . . and the differences by d_1, d_2, d_3, ... This portion of the analysis is best carried out by arranging the quantities in horizontal rows, thus

Ordinates	y_{30} y_{150}	y_{60} y_{120}	y_{90}
Sums	$(y_{30}+y_{150})=a_1$	$(y_{60}+y_{120})=a_2$	$y_{90}=a_3$
Differences	$(y_{30}-y_{150})=d_1$	$(y_{60}-y_{120})=d_2$	—

The method of obtaining the grouping of the terms for the multiplication processes is best shown by carrying out the summation according to the previous method. Thus, from p. 320, we have

$$A_1=2\left\{\tfrac{1}{6}(y_{30}\sin 30^\circ+y_{60}\sin 60^\circ+y_{90}\sin 90^\circ+y_{120}\sin 120^\circ+y_{150}\sin 150^\circ)\right\}$$
$$=\tfrac{1}{3}\left\{(y_{30}+y_{150})\sin 30^\circ+(y_{60}+y_{120})\sin 120^\circ+y_{90}\sin 90^\circ\right\}$$
$$=\tfrac{1}{3}(a_1\sin 30^\circ+a_2\sin 60^\circ+a_3\sin 90^\circ)$$

$$A_3=2\left\{\tfrac{1}{6}[y_{30}\sin(3\times 30)^\circ+y_{60}\sin(3\times 60)^\circ+y_{90}(3\times 90)^\circ+y_{120}\sin(3\times 120)^\circ+y_{150}\sin(3\times 150)^\circ]\right\}$$
$$=\tfrac{1}{3}\left\{(y_{30}+y_{150})\sin 90^\circ+(y_{60}+y_{120})\sin 180^\circ+y_{90}\sin 270^\circ\right\}$$
$$=\tfrac{1}{3}(a_1\sin 90^\circ+a_2\sin 180^\circ+a_3\sin 270^\circ)$$
$$=\tfrac{1}{3}(a_1-a_3)\sin 90^\circ$$
$$=\tfrac{1}{3}c_1,$$

where $c_1=a_1-a_3$.

$$A_5=2\left\{\tfrac{1}{6}[y_{30}\sin(5\times 30)^\circ+y_{60}\sin(5\times 60)^\circ+y_{90}\sin(5\times 90)^\circ+y_{120}\sin(5\times 120)^\circ+y_{150}\sin(5\times 150)^\circ]\right\}$$
$$=\tfrac{1}{3}\left\{(y_{30}+y_{150})\sin 150^\circ+(y_{60}+y_{120})\sin 300^\circ+y_{90}\sin 450^\circ\right\}$$
$$=\tfrac{1}{3}(a_1\sin 30^\circ-a_2\sin 60^\circ+a_3\sin 90^\circ)$$

$$B_1 = 2\left\{\tfrac{1}{6}(y_{30}\cos 30° + y_{60}\cos 60° + y_{90}\cos 90° + y_{120}\cos 120° + y_{150}\cos 150°)\right\}$$

$$= \tfrac{1}{3}\left\{(y_{30} - y_{150})\cos 30° + (y_{60} - y_{120})\cos 60° + y_{90}\cos 90°\right\}$$

$$= \tfrac{1}{3}(d_1 \cos 30° + d_2 \cos 60°)$$

$$B_3 = 2\left\{\tfrac{1}{6}(y_{30}\cos(3\times 30)° + y_{60}\cos(3\times 60)° + y_{90}\cos(3\times 90)° + y_{120}\cos(3\times 120)° + y_{150}\cos(3\times 150)°\right\}$$

$$= \tfrac{1}{3}\left\{(y_{30} - y_{150})\cos 90° + (y_{60} - y_{120})\cos 180°\right\}$$

$$= \tfrac{1}{3} d_2 \cos 180°$$

$$= -\tfrac{1}{3} d_2$$

$$B_5 = 2\left\{\tfrac{1}{6}(y_{30}\cos(5\times 30°) + y_{60}\cos(5\times 60°) + y_{90}\cos(5\times 90°) + y_{120}\cos(5\times 120°) + y_{150}\cos(5\times 150°)\right\}$$

$$= \tfrac{1}{3}\left\{(y_{30} - y_{150})\cos 150° + (y_{60} - y_{120})\cos 300°\right\}$$

$$= \tfrac{1}{3}(-d_1 \cos 30° + d_2 \cos 60°)$$

Observe that the same products are involved in the coefficients of both the first and fifth harmonics. Hence, if the products are arranged alternately in two vertical columns (see Table X), one-third of the sum of the totals of the columns gives the coefficient for the first harmonic, or fundamental, and one-third of the difference of the totals of the columns gives the coefficient for the fifth harmonic. Observe also that terms a_1, a_3, may be grouped together and treated as a single term (c_1) in obtaining the products for the third harmonics.

II. Analysis of a periodic curve for odd harmonics up to, and including, the eleventh. In this case we can obtain special groupings for both third and ninth harmonics.

Thus, denoting the eleven equidistant intermediate ordinates (erected at abscissæ 15°, 30°, 45°, etc.) by y_{15}, $y_{30°}$, etc., and taking the sums and differences of ordinates at supplemental angles, as above, we have

	$y_{15°}$	$y_{30°}$	$y_{45°}$	$y_{60°}$	$y_{75°}$	y
	$y_{165°}$	$y_{150°}$	$y_{135°}$	$y_{120°}$	$y_{105°}$	
Sum	a_1	a_2	a_3	a_4	a_5	a_6
Difference	d_1	d_2	d_3	d_4	d_5	—

TABLE X

Schedule for analysis of periodic curve up to fifth harmonic

Order of Harmonic.			1st.	5th.	3rd.	1st.	5th	3rd.
θ	$90^\circ - \theta$	$\sin\theta = \cos(90^\circ - \theta)$	Sine Terms.			Cosine Terms.		
30° 60° 90°	60° 30° 0°	0·5 0·866 1·0	$a_1 \sin 30^\circ$ $a_3 \sin 90^\circ$	$a_2 \sin 60^\circ$	$c_1 \sin 90^\circ$	$d_2 \cos 60^\circ$	$d_1 \cos 30^\circ$	$-d_2 \cos 0^\circ$
Total 1st column . . . . Total 2nd column								
$\frac{1}{3} \times$ Sum of totals . . . . $\frac{1}{3} \times$ Difference of totals . . .				A_1 A_5	A_3		B_1 B_5	B_3
Check			$A_1 - A_3 - A_5 = a_3$			$B_1 - B_3 + B_5 = 0$		

$F_1 = \sqrt{(A_1^2 - B_1^2)}$. $\Gamma_3 = \sqrt{(A_3^2 - B_3^2)}$. $\Gamma_5 = \sqrt{(A_5^2 - B_5^2)}$.

$a_1 = \tan^{-1} B_1/A_1$. $a_3 = \tan^{-1} B_3/A_3$. $a_5 = \tan^{-1} B_5/A_5$

Equation to curve—$y = F_1 \sin(\theta - a_1) + \Gamma_3 \sin(3\theta + a_3) + \Gamma_5 \sin(5\theta + a_5)$.

Now the coefficients for the third harmonics are given by

$$A_3 = \tfrac{1}{6}(a_1 \sin 45^\circ + a_2 \sin 90^\circ + a_3 \sin 135^\circ + a_4 \sin 180^\circ + a_5 \sin 225^\circ + a_6 \sin 270^\circ),$$

$$= \tfrac{1}{6}(a_1 \sin 45^\circ + a_2 \sin 90^\circ + a_3 \sin 45^\circ + 0 - a_5 \sin 45^\circ - a_6 \sin 90^\circ)$$

$$= \tfrac{1}{6}\{(a_1 + a_3 - a_5) \sin 45^\circ + (a_2 - a_6) \sin 90^\circ\}$$

$$= \tfrac{1}{6}(c_1 \sin 45^\circ + c_2 \sin 90^\circ)$$

where $c_1 = (a_1 + a_3 - a_5)$, and $c_2 = (a_2 - a_6)$

$$B_3 = \tfrac{1}{6}(d_1 \cos 45^\circ + d_2 \cos 90^\circ + d_3 \cos 135^\circ + d_4 \cos 180^\circ + d_5 \cos 225^\circ + d_6 \cos 270^\circ)$$

$$= \tfrac{1}{6}(d_1 \cos 45^\circ + 0 - d_3 \cos 45^\circ - d_4 - d_5 \cos 45^\circ + 0)$$

$$= \tfrac{1}{6}\{(d_1 - d_3 - d_5) \cos 45^\circ - d_4\}$$

$$= \tfrac{1}{6}(g_1 \cos 45^\circ - d_4)$$

where $g_1 = (d_1 - d_3 - d_5)$

The coefficients for the ninth harmonic reduce to

$$A_9 = \tfrac{1}{6}\{(a_1 + a_3 - a_5) \sin 45^\circ - (a_2 - a_6) \sin 90^\circ\}$$

$$= \tfrac{1}{6}(c_1 \sin 45^\circ - c_2 \sin 90^\circ)$$

$$B_9 = \tfrac{1}{6}\{(-d_1 + d_3 + d_5) \cos 45^\circ - d_4\}$$

$$= \tfrac{1}{3}(-g_1 \cos 45^\circ - d_4)$$

Hence the coefficients for the third and ninth harmonics can be evaluated together.

Similarly, the coefficients for the first and eleventh harmonics can be evaluated together, as well as those for the fifth and seventh harmonics. Thus

$$A_1 = \tfrac{1}{6}(a_1 \sin 15^\circ + a_2 \sin 30^\circ + a_3 \sin 45^\circ + a_4 \sin 60^\circ + a_5 \sin 75^\circ + a_6 \sin 90^\circ)$$

$$A_{11} = \tfrac{1}{6}(a_1 \sin 15^\circ - a_2 \sin 30^\circ + a_3 \sin 45^\circ - a_4 \sin 60^\circ + a_5 \sin 75^\circ - a_6 \sin 90^\circ)$$

$$B_1 = \tfrac{1}{6}(d_1 \cos 15^\circ + d_2 \cos 30^\circ + d_3 \cos 45^\circ + d_4 \cos 60^\circ + d_5 \cos 75^\circ)$$

$$B_{11} = \tfrac{1}{6}(-d_1 \cos 15^\circ + d_2 \cos 30^\circ - d_3 \cos 45^\circ + d_4 \cos 60^\circ - d_5 \cos 75^\circ)$$

$$A_5 = \tfrac{1}{6}(a_1 \sin 75^\circ + a_2 \sin 30^\circ - a_3 \sin 45^\circ - a_4 \sin 60^\circ$$
$$+ a_5 \sin 15^\circ + a_6 \sin 90^\circ)$$

$$A_7 = \tfrac{1}{6}(a_1 \sin 75^\circ - a_2 \sin 30^\circ - a_3 \sin 45^\circ + a_4 \sin 60^\circ$$
$$+ a_5 \sin 15^\circ - a_6 \sin 90^\circ)$$

$$B_5 = \tfrac{1}{6}(d_1 \cos 75^\circ - d_2 \cos 30^\circ - d_3 \cos 45^\circ + d_4 \cos 60^\circ$$
$$- d_5 \cos 15^\circ)$$

$$B_7 = \tfrac{1}{6}(- d_1 \cos 75^\circ - d_2 \cos 30^\circ + d_3 \cos 45^\circ + d_4 \cos 60^\circ$$
$$+ d_5 \cos 15^\circ)$$

The schedule is therefore arranged according to Table XI.

Example. Analysis of the E.M.F. curve given in Fig. 186 for odd harmonics up to, and including, the eleventh.

The values of the ordinates at abscissæ 15°, 30°, . . . are as follow—

Abscissa	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°
Ordinate	24·5	51	84·2	126	171	191·5	180	152·3	119	82	41

Taking sums and differences of ordinates of supplemental angles, we have

Ordinates	24·5	51	84·2	126	171	191·5
	41	82	119	152·3	180	
Sums	65·5 (a_1)	133 (a_2)	203·2 (a_3)	278·3 (a_4)	351 (a_5)	191·5 (a_6)
Differences	−16·5 (d_1)	31 (d_2)	34·8 ($-d_3$)	−26·3 (d_4)	−9 (d_5)	

Grouping for third harmonic

$c_1 \quad a_1 + a_3 - a_5 \quad 65{\cdot}5 + 203{\cdot}2 - 351 \quad -82{\cdot}3$

$c_2 \quad a_2 - a_6 \quad 133 - 191{\cdot}5 \quad -58{\cdot}5$

$q_1 \quad d_1 - d_3 \quad d_5 \quad -16{\cdot}5 + 34{\cdot}8 + 9 \quad 17{\cdot}3$

The remaining calculations are given in Table XII.

III. Analysis of a periodic curve for odd harmonics up to, and including, the seventeenth. In this case special groupings are obtained for the third, ninth, and fifteenth harmonics.

Seventeen equidistant intermediate ordinates are erected at abscissæ 10°, 20°, etc.

The sums and differences of ordinates at supplemental angles are obtained and tabulated thus—

	y_{10°	y_{20°	y_{30°	y_{40}	y_{50}	y_{60°	y_{70}	y_{80°	y_{90°
	y_{170°	y_{160°	y_{150°	y_{140°	y_{130}	y_{120°	y_{110°	y_{100°	y_{90°
Sums	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
Differences	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	

TABLE XI

Schedule for analysis of periodic curve up to eleventh harmonic

Order of Harmonic.			1st.	11th.	3rd.	9th.	5th.	7th.	1st.	11th.	3rd.	9th.	5th.	7th.
θ	$90°-\theta$	$\sin\theta = \cos(90°-\theta)$	Sine Terms.						Cosine Terms.					
15°	75°	0·259	$a_1 \sin 15°$				$a_5 \sin 15°$			$d_5 \cos 75°$				$d_1 \cos 75°$
30°	60°	0·5		$a_2 \sin 30°$				$a_2 \sin 30°$	$d_4 \cos 60°$				$d_4 \cos 60°$	
45°	45°	0·707	$a_3 \sin 45°$		$c_1 \sin 45°$		$-a_3 \sin 45°$			$d_3 \cos 45°$		$g_1 \cos 45°$		$-d_3 \cos 45°$
60°	30°	0·866		$a_4 \sin 60°$				$-a_4 \sin 60°$	$d_2 \cos 30°$				$-d_2 \cos 30°$	
75°	15°	0·966	$a_5 \sin 75°$				$a_1 \sin 75°$			$d_1 \cos 15°$				$+d_5 \cos 15°$
90°	0	1·0		$a_6 \sin 90°$		$c_2 \sin 90°$		$a_6 \sin 90°$			$-d_4 \cos 0°$			
Total 1st column . Total 2nd column .														
Sum Difference . .														
$\frac{1}{2}$ × Sum . . . $\frac{1}{2}$ × Difference . .			A_1 A_{11}		A_3 A_9		A_5 A_7		B_1 B_{11}		B_3 B_9		B_5 B_7	
Check . . .			$A_1 - A_3 + A_5 - A_7 + A_9 - A_{11} = a_6$						$B_1 - B_3 + B_5 + B_7 + B_9 + B_{11} = 0$					

$F_1 = \sqrt{(A_1^2 + B_1^2)}$. $F_3 = \sqrt{(A_3^2 + B_3^2)}$. $F_5 = \sqrt{(A_5^2 + B_5^2)}$. $F_7 = \sqrt{(A_7^2 + B_7^2)}$. $F_9 = \sqrt{(A_9^2 + B_9^2)}$. $F_{11} = \sqrt{(A_{11}^2 + B_{11}^2)}$.

$a_1 = \tan^{-1} B_1/A_1$. $a_3 = \tan^{-1} B_3/A_3$. $a_5 = \tan^{-1} B_5/A_5$. $a_7 = \tan^{-1} B_7/A_7$. $a_9 = \tan^{-1} B_9/A_9$. $a_{11} = \tan^{-1} B_{11}/A_{11}$.

Equation to curve—$y = F_1 \sin(\theta + a_1) + F_3 \sin(3\theta + a_3) + F_5 \sin(5\theta + a_5) + F_7 \sin(7\theta + a_7) + F_9 \sin(9\theta + a_9)$
$+ F_{11} \sin(11\theta + a_{11})$.

TABLE XII

Analysis of E.M.F. curve up to eleventh harmonic

Order of Harmonic.			1st.	11th.	3rd.	9th	5th.	7th.	1st.	11th.	3rd.	9th.	5th.	7th.
θ	$90° - \theta$	$\sin\theta = \cos(90° - \theta)$	Sine Terms.						Cosine Terms.					
15°	75°	0·259	17				91			−2·33				−4·27
30°	60°	0·5		66·5				66·5	−13·15				−13·15	
45°	45°	0·707	143·6		−58·2		−143·6			−24·6		12·23		24·6
60°	30°	0·866		241				−241	−26·9				26·9	
75°	15°	0·966	339				63·3			−15·94				8·7
90°	0°	1·0		191·5		−58·5		191·5			26·3			
Total 1st column			499·6		−58·2		10·7		−40		26·3		13·75	
Total 2nd column			499		−58·5		17		−42·9		12·23		29·03	
Sum			998·6		−116·7		27·7		−82·9		38·5		42·78	
Difference			0·6		0·3		−6·3		2·9		14·1		−15·28	
$\frac{1}{6}$ × Sum			166·5 (= A_1)		−19·45 (= A_3)		4·6 (= A_5)		−13·82 (= B_1)		6·4 (= B_3)		7·13 (= B_5)	
$\frac{1}{6}$ × Difference			0·1 (= A_{11})		0·05 (= A_9)		−1·05 (= A_7)		0·48 (= B_{11})		2·35 (= B_9)		−2·55 (= B_7)	
Check			$A_1 - A_3 + A_5 - A_7 + A_9 - A_{11} = a_6$ i.e. $166·5 + 19·45 + 4·6 - 1·05 - 0·05 - 0·1 = 191·55$						$B_1 + B_3 + B_5 + B_7 + B_9 + B_{11} = 0$ i.e. $13·82 + 6·4 + 7·13 - 2·55 + 2·35 + 0·48 = 0·01$					

$F_1 = \sqrt{[166·5^2 + (-13·82)^2]} = 167.$ $\quad F_3 = \sqrt{[(-19·45)^2 + 6·4^2]} = 20·5.$ $\quad F_5 = \sqrt{(4·6^2 + 7·13)} = 8·9.$

$F_7 = \sqrt{[(-1·05)^2 + (-2·55)^2]} = 2·76.$ $\quad F_9 = \sqrt{(0·05^2 + 2·35)^2} = 2·35.$ $\quad F_{11} = \sqrt{(0·1^2 + 0·49)} = 0·49.$

$\alpha_1 = \tan^{-1} - 13·82/166·5 = -4·8°.$ $\quad \alpha_3 = \tan^{-1} 6·4/-19·45 = 161·8°.$ $\quad \alpha_5 = \tan^{-1} 7·13/4·6 = 57·2°.$

$\alpha_7 = \tan^{-1} - 2·55/-1·05 = 247·6°.$ $\quad \alpha_9 = \tan^{-1} 2·35/0·05 = 88·8°.$ $\quad \alpha_{11} = \tan^{-1} 0·48/0·1 = 78·2°.$

Equation to curve—$e = 167 \sin(\theta - 4·8°) + 20·5 \sin(3\theta + 161·8°) + 8·9 \sin(5\theta + 57·2°) - 2·76 \sin(7\theta + 67·6°) + 2·35 \sin(9\theta + 88·8°) + 0·49 \sin(11\theta + 78·2°).$

The coefficients for the third, ninth, and fifteenth harmonics reduce to the following expressions—

$$A_3 = \tfrac{1}{6}\{(a_1 + a_5 - a_7)\sin 30^\circ + (a_2 + a_4 - a_8)\sin 60^\circ + (a_3 - a_9)\sin 90^\circ\}$$

$$= \tfrac{1}{6}(c_1 \sin 30^\circ + c_2 \sin 60^\circ + c_3 \sin 90^\circ)$$

where $c_1 = (a_1 + a_5 - a_7)$, $c_2 = (a_2 + a_4 - a_8)$ and $c_3 = (a_3 - a_9)$

$$A_9 = \tfrac{1}{6}\{(a_1 - a_3 + a_5 - a_7 + a_9)\sin 90^\circ\} = \tfrac{1}{6} c_4 \sin 90^\circ$$

where $c_4 = (a_1 - a_3 + a_5 - a_7 + a_9)$

$$A_{15} = \tfrac{1}{6}\{(a_1 + a_5 - a_7)\sin 30^\circ - (a_2 + a_4 - a_8)\sin 60^\circ + (a_3 - a_9)\sin 90^\circ\}$$

$$= \tfrac{1}{6}(c_1 \sin 30^\circ - c_2 \sin 60^\circ + c_3 \sin 90^\circ)$$

$$B_3 = \tfrac{1}{6}\{(d_1 - d_5 - d_7)\cos 30^\circ + (d_2 - d_4 - d_8)\cos 60^\circ - d_6\}$$

$$= \tfrac{1}{6}(g_1 \cos 30^\circ + g_2 \cos 60^\circ - d_6)$$

where $g_1 = (d_1 - d_5 - d_7)$, and $g_2 = (d_2 - d_4 - d_8)$

$$B_9 = \tfrac{1}{6}(-d_2 + d_4 - d_6 + d_8) = \tfrac{1}{6} g_3$$

$$B_{15} = \tfrac{1}{6}\{(-d_1 + d_5 + d_7)\cos 30^\circ + (d_2 - d_4 - d_8)\cos 60^\circ - d_6\}$$

$$= \tfrac{1}{6}(g_2 \cos 60^\circ - d_6 - g_1 \cos 30^\circ)$$

Hence the groupings for these harmonics are

	For Sine terms.	For Cosine terms.
Third and fifteenth harmonics	$a_1 + a_5 - a_7 = c_1$ $a_2 + a_4 - a_8 = c_2$ $a_3 - a_9 = c_3$	$d_1 - d_5 - d_7 = g_1$ $d_2 - d_4 - d_8 = g_2$
Ninth harmonic	$a_1 - a_3 + a_5 - a_7 + a_9 = c_4$	$-d_2 + d_4 - d_6 + d_8 = g_3$

The schedule is arranged according to Table XIII.

IV. Analyses for harmonics higher than the seventeenth. If an analysis is to include the 23rd harmonic, special groupings can be obtained for the 3rd, 9th, 15th, and 21st harmonics; while, if the 29th harmonic is to be included, special groupings can be obtained for the 3rd, 5th, 9th, 15th, 21st, 25th, and 27th harmonics.

In general, as the analysis is extended to include still higher harmonics, the number of harmonics for which special groupings of the terms may be obtained, increases.

The determination of the "grouping coefficients" (c, g) and the preparation of schedules for these cases are effected by methods similar to those already given.

CHAPTER XI

MAGNETIZATION OF IRON BY ALTERNATING CURRENTS

Effects due to alternating magnetization. The magnetization of iron by alternating current produces phenomena which are absent when the iron is magnetized by direct current. In the latter case, assuming a closed magnetic circuit, a given steady current in the magnetizing coil results in a definite flux in the iron, and hysteresis manifests itself only when changes occur in the magnetizing current, the resulting flux due to a given current being then dependent upon whether this current is greater or less than the previous current.

With a steady magnetizing current (i), the impressed E.M.F. (v) at the terminals of the magnetizing coil is given by $v = iR$, where R is the resistance of the coil. But, with a varying current, the impressed E.M.F. contains a component which balances the E.M.F. induced in the coil by the variations of the flux. Moreover, the changes in the flux also induce E.M.Fs in the iron core and set up *eddy currents* therein. The eddy currents cause a loss of energy—which is expended in heating the core—and produce a magnetic reaction tending to damp out the variations of the flux. An additional loss of energy, due to hysteresis, occurs in the iron core during each cycle of magnetization, and is expended in heating the core. Further, both hysteresis and eddy currents cause a phase displacement between the magnetizing current and the flux. Hence with alternating magnetization the current (maximum value) required to produce a given flux (maximum value) is larger than that required when the flux is non-alternating.

Therefore when iron is magnetized by alternating current, both the current in the coil and the potential difference at its terminals will have higher maximum values, for a given maximum value of the flux, than those when the magnetization is produced by steady current. Also, due to the varying permeability of the iron, and hysteresis, the wave-form of the current is distorted and is no longer of similar shape to that of the impressed E.M.F.

[NOTE. The distortion becomes greater as the magnetic saturation increases; it is affected similarly by the hysteresis loss.]

The wave-form of the current may be easily obtained when the hysteresis loop of the iron is available and the wave-form of the

flux is known; the method is a graphical one and is described later (p. 333).

Flux wave-form. The wave-form of the flux can be obtained when that of the impressed E.M.F. is known, provided that the voltage drop due to the resistance of the magnetizing coil is negligible. Thus, if v denotes the impressed E.M.F., i the current, R the resistance of the magnetizing coil, and e the E.M.F. induced in the latter due to the variations of the flux, we have

$$v = -(e + iR)$$

and, when iR is negligible, $v = -e$.

Now $e = -(N \times 10^{-8})d\Phi/dt$, where N denotes the number of turns in the magnetizing coil, and Φ denotes the value of the flux at the instant t. Hence

$$\Phi = -\frac{10^8}{N}\int e \, . \, dt = \frac{10^8}{N}\int v \, . \, dt \quad . \quad . \quad . \quad . \quad (165)$$

Thus the curve obtained by the integration of the wave-form of the impressed E.M.F. represents the wave-form of the flux.

If the impressed E.M.F. is sinusoidal, i.e. $v = V_m \sin \omega t$,

$$\Phi = \frac{V_m \times 10^8}{N}\int \sin \omega t \, . \, dt = -\frac{V_m \times 10^8}{N\omega}\cos \omega t$$

$$= \frac{V_m \times 10^8}{N\omega}\sin(\omega t - \tfrac{1}{2}\pi) \quad (166)$$

Thus the flux is sinusoidal and lags 90° with respect to the impressed E.M.F.

The maximum value (Φ_m) of the flux is given by

$$\Phi_m = V_m \times 10^8/N\omega = \sqrt{2} \, . \, V \times 10^8/2\pi f N$$
$$= V \times 10^8/4{\cdot}44 f N \quad . \quad . \quad . \quad . \quad (167)$$

where V denotes the R.M.S. value of the impressed E.M.F., and f the frequency.

If the impressed E.M.F. is non-sinusoidal, and only odd harmonics are present, let V_{av} denote the mean value of the E.M.F. during the half-period ($= \frac{1}{2}T$ seconds) between two successive zero values. Then

$$V_{av} = \int_0^{\frac{1}{2}T} \frac{v \, . \, dt}{\frac{1}{2}T}$$

and

$$\int_0^{\frac{1}{2}T} v \, . \, dt = \tfrac{1}{2}T V_{av}$$

Now $\int_0^{\frac{1}{2}T} v \, . \, dt$ represents, under the conditions stated, the maximum value of the indefinite integral $\int v \, . \, dt$, and therefore if the value $(\frac{1}{2}TV_{av})$ of the definite integral is substituted in equation (165), the resulting expression $[(\frac{1}{2}TV_{av})10^8/N]$ represents the maximum change $(\Delta\Phi)_m$ in the flux during a half-period, i.e. $(\Delta\Phi)_m = (\frac{1}{2}TV_{av})10^8/N$. Since the E.M.F. wave-form is assumed to contain only odd harmonics, then only odd harmonics are present in the flux wave-form, and the positive and negative half-waves are identical in shape. Hence, if Φ_m denotes the maximum value, or amplitude, of the flux wave, then the maximum change in the flux during a half-period is equal to twice the amplitude and is given by $(\Delta\Phi)_m = 2\Phi_m$.

Substituting the above value for $(\Delta\Phi)_m$, we have

$$2\Phi_m = (\tfrac{1}{2}TV_{av})10^8/N$$

Whence
$$\Phi_m = TV_{av} \times 10^8/4N$$
$$= 10^8 V_{av}/4fN \quad . \quad . \quad . \quad . \quad (167a)$$

since $T = 1/f$.

Hence for a given frequency (f) and number of turns (N), the maximum value of the flux depends only upon the *mean value* of the impressed E.M.F. Conversely, the mean E.M.F. induced in a coil by an alternating flux depends only upon the maximum value of the flux – the frequency and number of turns being constant – and is unaffected by the shape of the wave-form of the flux.

Denoting the form factor—i.e. the ratio (R.M.S. value/mean value)—of the E.M.F. wave by k_f, we have

$$V_{av} = V/k_f$$

where V is the R.M.S. value of the impressed E.M.F.

Hence $\quad \Phi_m = V \times 10^8/4k_f fN \quad . \quad . \quad . \quad . \quad (167b)$

If the E.M.F. in sinusoidal, $k_f = 1{\cdot}11$, and the expression (167b) reduces to (167), i.e.

$$\Phi_m = V \times 10^8/4{\cdot}44fN$$

Wave-form of the magnetizing current. The wave-form of the magnetizing current is deduced as follows—

Assuming the hysteresis loop and the wave-form of the flux, or flux density, to have been plotted in rectangular co-ordinates, select a convenient number of points, say twelve, on the flux curve and determine the magnetizing current for each point from the corresponding points on the hysteresis loop (Fig. 187). Plot

the currents so obtained to the same abscissæ as the flux curve, and draw a curve through the points. Thus, in Fig. 187, the flux-density curve, B, has been plotted to the same ordinate scale as the hysteresis loop (which has been plotted in terms of flux density, B, and magnetizing current, I) By arranging these curves side by side, the selected points, 1–12, on the flux curve are projected across to the hysteresis loop, and the corresponding values of currents are marked off as points 1′–12′ on the respective ordinates of the flux curve.

A similar method can be used to determine the wave form of

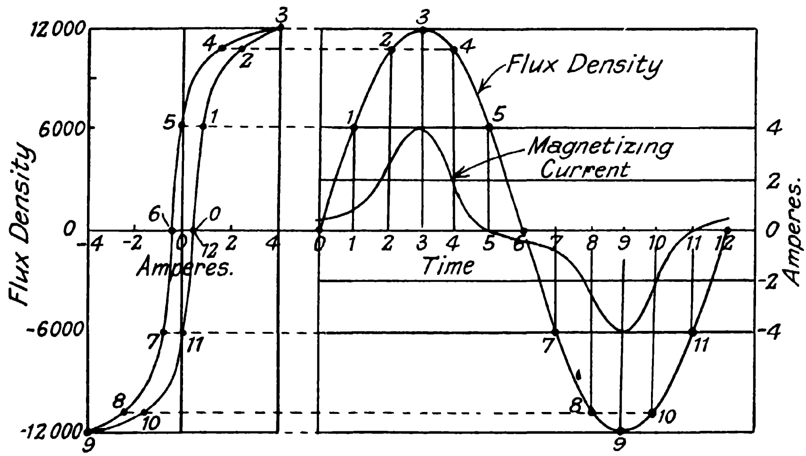


FIG. 187 Method of Determining Wave Form of Magnetizing Current

the flux corresponding to a given wave-form of current Fig. 188 gives the wave-form of the flux when the magnetizing current follows a sine law and the hysteresis loop is the same as that given in Fig. 187.

Wave-form of induced E.M.F. When the shape of the flux curve is known, the wave-form of the induced E M.F. may be readily obtained since each point on the E.M.F. wave is proportional to the differential coefficient of the corresponding point on the flux wave.

[NOTE. The differential coefficient of any point on the flux wave may be determined either graphically, by drawing the tangent at that point and calculating its value, or analytically, by calculating the quantity $\Delta\Phi/\Delta t$.]

Referring to Fig. 187 we observe that when the flux is sinusoidal

the magnetizing current wave-form is distorted. Moreover, although the maximum ordinate of the current wave coincides with the maximum value of the flux, the zero values of current occur earlier than those of the flux. This non-coincidence of the zero values of current and flux is due to hysteresis.*

We also observe that the half-waves of current are similar and symmetrical. Therefore only odd harmonics are present in the current wave.

Analysis of current wave-form. If the current wave is analysed into its harmonics the results will show the presence of a fundamental, and third, fifth, and higher harmonics; the harmonics

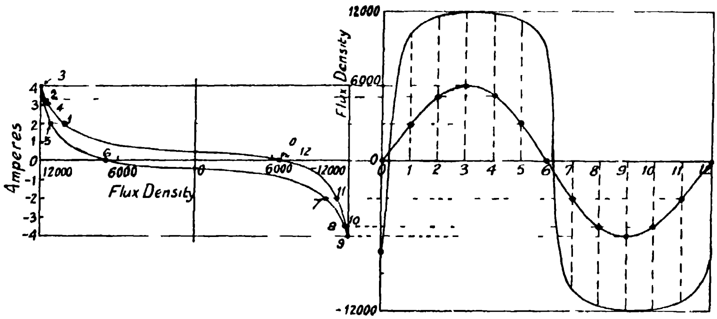


FIG 188.—Method of Determining Wave Form of Flux

higher than the fifth being small in comparison with the third and fifth harmonics and the fundamental. For example, the analysis of the current wave in Fig. 187 gives the following result—

$$i = 2{\cdot}707 \sin(\theta + 9{\cdot}3°) - 1{\cdot}11 \sin(3\theta + 10{\cdot}3°) + 0{\cdot}255 \sin (5\theta - 48{\cdot}2) - 0{\cdot}1 \sin(7\theta - 59{\cdot}3°) + 0{\cdot}085 \sin(9\theta - 83{\cdot}2°) - 0{\cdot}054 \sin(11\theta + 82{\cdot}6)°$$

In Fig. 189 the fundamental of the current wave of Fig. 187 is denoted by i_1 and the higher harmonics are grouped together in the curve marked i_2.

Equivalent exciting current. The power necessary to carry the iron through a cycle of magnetization is given by the product of the impressed E.M.F. and the fundamental of the current curve (since, with sinusoidal E.M.F. and non-sinusoidal current, only the fundamental of the current wave has a frequency equal to that of the E.M.F.), and this power is equal to the hysteresis loss. The

* If hysteresis were absent, or negligible, the current wave-form would be in phase with the flux wave-form, but the former would still show distortion due to the variation of permeability of the iron with variation of magnetizing current.

fundamental of the current wave may therefore be resolved into power and wattless components with respect to the impressed E.M.F. wave. These components are shown in Fig. 189 by the curves i_{1p} and i_{1w}.

The wattless component, i_{1w}, of the fundamental of the current wave, when compounded with the curve, i_2, representing the third, fifth, and higher harmonics (which are wattless with respect to a sinusoidal E.M.F.) gives the curve marked i_w, which represents the total wattless component of the current wave and is the true

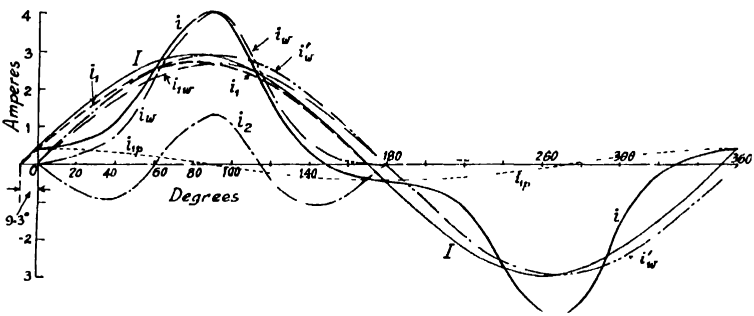


FIG. 189.— Components of Magnetizing and Exciting Currents

Explanation and data of curves –

i, non-sinusoidal magnetizing current, Fig. 187
i_1, fundamental of (i) equation $i_1 = 2{\cdot}7 \sin(\theta + 9{\cdot}3°)$
i_2, higher harmonics grouped together
i_1p, power component of i_1 – equation $i_1p = 0{\cdot}437 \cos\theta$
i_1w, wattless component of i_1 equation $i_1w = 2{\cdot}672 \sin\theta$
iw, total wattless component of current wave
iw', equivalent sine curve for iw – equation $iw' = 2{\cdot}9 \sin\theta$
I, equivalent exciting current—equation $I_{inst} = 2{\cdot}935 \sin(\theta + 9{\cdot}3°)$

magnetizing current. This curve, i_w, may be replaced by an equivalent sine wave, i_w', having an R.M.S. value equal to I_w. Compounding the sine curve i_w' with the power component i_{1p} of the fundamental of the current wave, we obtain the sine curve I, which represents the equivalent sinusoidal current necessary to carry the iron through a magnetic cycle. This current, since it is compounded from the true magnetizing current and the power component of the current supplying the hysteresis loss, is called the *exciting current*, and the angle by which it leads the flux is called the *hysteretic angle of advance, α*.

The vector diagram representing these conditions is given in Fig. 190, in which the vector $O\Phi_m$ represents the flux, OI_o the exciting current, OI_w the magnetizing current, and OI_p the power component of the exciting current supplying the losses. Hence

$$I_o = \sqrt{(I_p^2 + I_w^2)}.$$

Now the R.M.S. value of the exciting current may be obtained from the reading of an ammeter connected in the circuit of the magnetizing winding, and if the hysteresis loss is denoted by P_h watts, the R.M.S. value of the power component of the fundamental of the current wave is equal to P_h/E, where E denotes the R.M.S. value of the E.M.F. induced in the magnetizing winding.

Whence $$I_w = \sqrt{[I_o^2 - (P_h/E)^2]}$$

Thus the R.M.S. value of the magnetizing current may be obtained when the exciting current, the hysteresis loss, and the E.M.F. induced in the magnetizing winding are known. If the pressure drop in the winding is small in comparison with the impressed E.M.F., then the induced E.M.F. may be taken as equal to the impressed E.M.F.

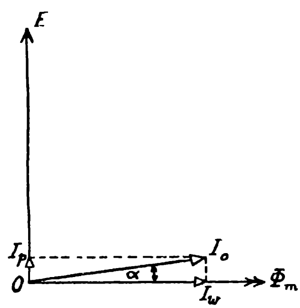


FIG. 190. Vector Diagram Showing Effect of Hysteresis

Effect of hysteresis and magnetic saturation on the "constants" of an iron-cored choking coil. The general effect of hysteresis and magnetic saturation is to cause the power-factor of any iron-cored coil to be higher than that calculated from the resistance and inductance of the magnetizing winding. Thus, with sinusoidal impressed E.M.F. and non-sinusoidal exciting current, the power component of the fundamental of the exciting current supplies the losses, which, in the present case, include the I^2R loss in the magnetizing winding and the hysteresis loss (with which should be included the eddy-current loss, if any) in the iron core. If P_i denotes the iron losses, the total loss, P, is given by

$$P = P_i + I^2R,$$

where I is the exciting current and R the resistance of the magnetizing winding. Dividing throughout by I^2, we have

$$P/I^2 = P_i/I^2 + R$$

The quantity P/I^2 represents the effective resistance R_{eff} of the choking coil, and includes the true resistance (R) of the winding and the apparent resistance (P_i/I^2) due to the losses in the iron core. The effective resistance also includes the apparent resistance due to any eddy currents in the conductors.

If E denotes the impressed E.M.F., the effective impedance (Z_{eff}) of the choking coil is given by

$$Z_{eff} = E/I$$

Hence the effective reactance is given by

$$X_{eff} = \sqrt{[(E/I)^2 - (P/I^2)^2]}$$

Now the non-sinusoidal exciting current (I) is larger than the sinusoidal magnetizing current which would be required to produce the same flux, and therefore both the effective reactance and the effective impedance of the choking coil are smaller than the calculated values obtained from the resistance and inductance of the magnetizing winding.

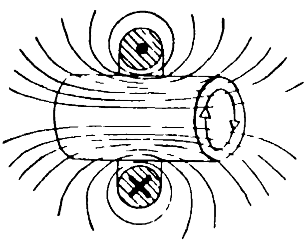


FIG. 191. Production of Eddy Currents in Solid Core

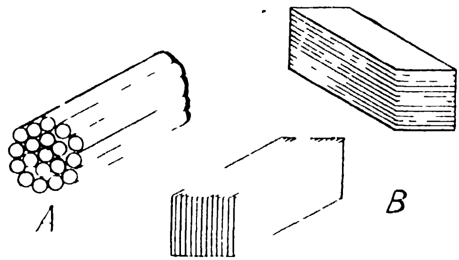


FIG. 192 Lamination of Cores Subjected to Alternating Flux

Hence, since the power factor is given by (effective resistance/effective impedance), it follows that, due to iron losses and magnetic saturation the actual power-factor of the choking coil will be higher than that calculated from the resistance and inductance of the magnetizing winding.

Eddy currents. When a solid iron or metal core is placed longitudinally in an alternating magnetic field, the lines of force cut the core radially, the motion of the flux being towards the centre of the core when the magnetizing force is decreasing, and *vice versâ* when the magnetizing force is increasing. Hence, E.M.Fs. are induced in the core, and these E.M.Fs. give rise to currents which circulate in the core in a direction parallel and opposite to that of the current in the magnetizing solenoid (*see* Fig. 191). These circulating currents are called *eddy currents*, and cause heating of the core.

The energy, which is dissipated in the form of heat, is obtained by electromagnetic induction from the circuit supplying current to

the magnetizing solenoid. Moreover, the circulating currents produce a magneto-motive force which acts in opposition to that producing magnetization, and therefore the flux due to a given current in the solenoid is smaller than that which would be obtained if the core were removed or replaced by one of non-magnetic material.

Hence when large masses of iron such as the cores of transformers and alternating-current machines, or heavy conductors — are subjected to alternating magnetic fields, means must be adopted

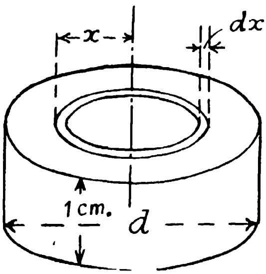


FIG. 193

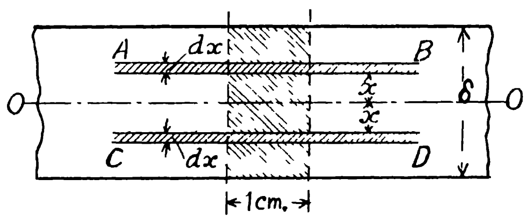


FIG. 194

Pertaining to the Calculation of Eddy Currents in Wires and Plates

to reduce the eddy currents to such values that the resulting heating and loss of energy is not detrimental to the efficiency and operation.

The most effective means of reducing the eddy current loss is to *laminate* the material subjected to the alternating flux, the laminations being parallel to the direction of the magnetic lines of force. For example, the core of Fig. 191 would be constructed either of iron wires of small diameter, or of thin iron sheets, as indicated at *A* and *B*, Fig. 192, respectively. In each case the individual wires or laminations must be lightly insulated from one another by a thin coating of insulating varnish, as, if the laminations are in electrical contact, the eddy current loss will be practically the same as with a solid core.

Calculation of eddy-current loss. The loss due to eddy-currents in wires and laminations may be calculated easily when the flux is uniformly distributed over the cross section of the material.

I. Eddy-current loss in round wires. Consider an elementary coaxial annulus of radius x cm. and width dx cm., in the cross section of the wire (Fig. 193). Then if d cm. denotes the external diameter of the wire, B_m, the maximum value of the flux density (assumed to be uniform throughout the cross section of the wire),

the flux linked with the annulus is $B_m \pi x^2$, and the R.M.S. value (E_x) of the induced E.M.F. is given by

$$E_x = 4k_f f B_m \pi x^2 \times 10^{-8} \text{ volts},$$

where k_f is the form-factor or the flux wave and f is the frequency.

The direction of the induced E.M.F. is along the mean perimeter of the annulus, and for unit length (1 cm.) of wire the resistance in the path of current is

$$r = 2\pi x \rho / dx,$$

where ρ denotes the specific resistance of the material in ohm-cm. units.

Hence the loss, in watts, in this elementary ring due to the current circulating in it is

$$\begin{aligned} P_e &= E_x^2/r = E_x^2\, dx/2\pi x\rho \\ &= (4k_f f B_m \pi x^2 \times 10^{-8})^2\, dx/2\pi x \rho \\ &= (8\pi k_f^2 f^2 B_m^2 x^3 \times 10^{-16}) dx/\rho \end{aligned}$$

Whence the loss in watts per cm. length of wire is

$$\begin{aligned} P_{c1} &= \int_0^{\frac{1}{2}d} P_e \, . \, dx = \left(\frac{8\pi k_f^2 f^2 B_m^2}{\rho \times 10^{16}}\right) \int_0^{\frac{1}{2}d} x^3 \, . \, dx \\ &= \frac{\pi k_f^2 f^2 B_m^2 d^4}{\rho \times 8 \times 10^{16}} \end{aligned}$$

Therefore the loss in watts per cubic cm. of the wire is

$$\begin{aligned} P &= P_{c1}/(\tfrac{1}{4}\pi d^2) \\ &= \tfrac{1}{2} k_f^2 f^2 B_m^2 d^2 \times 10^{-16}/\rho \qquad . \quad . \quad . \quad . \quad . \quad (168) \\ &= \tfrac{1}{2} \cdot \frac{1}{\rho \times 10^6}\left(d \frac{f}{100} \frac{k_f B_m}{1000}\right)^2 \end{aligned}$$

and the loss per kilogram of core is

$$\begin{aligned} P_{kg} &= P/\sigma \times 10^3 \\ &= \tfrac{1}{2}\left(\frac{1}{\sigma\rho \times 10^3}\right)\left(d \frac{f}{100} \frac{k_f B_m}{1000}\right)^2 \qquad . \quad . \quad . \quad (169) \end{aligned}$$

where σ is the density of the wire in grammes per cubic cm.

II. Eddy-current loss in thin plates. In this case, the thickness, δ, of the plate is assumed to be very small in comparison with the width, i.e. the dimension parallel to the direction in which the eddy currents circulate. The eddy currents may therefore be considered to flow parallel to the surfaces of the plate and will have opposite directions on either side of the centre line as indicated in Fig. 194.

Consider two parallel elementary layers, AB, CD, in a plate at distances x cm. on either side of the centre line OO, Fig. 194. Let the thickness of each element be dx cm. and its length perpendicular to the paper be 1 cm. Then the flux enclosed per cm. length of the central portion of the plate (shown shaded in Fig. 194) bounded by the elements is

$$B_m(2x \times 1) = 2x \,.\, B_m$$

Hence the E.M.F. (E_x) induced in the circuit formed by the elements (which is equivalent to an electric circuit of one turn) is

$$E_x = 4k_f f B_m \,.\, 2x \times 10^{-8} \text{ volts.}$$

Now if the thickness of the plate is very small in comparison with the width, the length of the elementary circuit $ABCD$ may be considered as equal to twice the distance between the transverse boundary planes, and the resistance of this circuit per cm. length and depth of the plate is

$$r = 2\rho/(dx \times 1) \text{ ohms.}$$

Hence the loss, in watts, due to the circulating current in this circuit is

$$P_e = E_x^2/r = E_x^2 dx/2\rho$$
$$= (4k_f f B_m \,.\, 2x \times 10^{-8})^2 dx/2\rho$$

and the total loss, in watts, per cm. length and depth of plate (i.e for a volume δ cubic cm.) is

$$P_{e1} = \frac{1}{2\rho}\int_0^{\frac{1}{2}\delta} E_x^2\, dx = \frac{4}{3\rho}\, k_f^2 f^2 B_m^2 \delta^3 \times 10^{-16}$$

Therefore the eddy current loss, in watts, per cubic cm. of the plate is

$$P = \frac{P_{e1}}{\delta} = \frac{4}{3\rho}\, k_f^2 f^2 B_m^2 \delta^2 \times 10^{-16} \quad . \quad . \quad . \quad (170)$$

and the loss per kilogram is

$$P_{kg} = \tfrac{4}{3}\left(\frac{1}{\sigma\rho \times 10^3}\right)\left(\delta \,\frac{f}{100}\, \frac{k_f B_m}{1000}\right)^2 \quad . \quad . \quad . \quad (171)$$

Example. Calculate the eddy-current loss in watts per kg. of alloyed iron plate for which the thickness is 0·5 mm., the specific resistance is 50×10^{-6} ohms per cm. cube, the density is 7·5 gm. per cubic cm., and the maximum value of the sinusoidal flux density is 13,000 lines per square cm. at a frequency of 50 cycles per second.

From the data we have

$\sigma = 7{\cdot}5$	$f = 50$
$\rho = 50 \times 10^{-6}$	$k_f = 1{\cdot}11$
$\delta = 0{\cdot}05$	$B_m = 13{,}000$

Hence, substituting in equation (171),

$$P_{kg} = \frac{4 \times 10^6}{3 \times 7.5 \quad 50 \times 10^3} \left(0.05 \times \frac{50}{100} \times \frac{1.11 \times 13000}{1000}\right)^2$$
$$= 0.162 \text{ W.}$$

NOTE. In practice the eddy-current loss is always in excess of that calculated by the above method, the discrepancy being due to a number of causes, such as imperfect insulation between the plates, non uniform distribution of the flux throughout the cross section of the core due to differences in the length of the magnetic path for different portions of the core, non-uniform distribution of the flux throughout the cross section of individual plates due to the effect of eddy currents (see p. 346).

Effect of eddy currents on flux distribution in iron cores. The E.M.F (E_t) induced in an element of the cores of Figs. 193, 194, by the alternations of the (sinusoidal) flux may be represented by a vector, OE_t, Fig. 195, lagging 90° with respect to the vector, OI_u, which represents the equivalent (sinusoidal) magnetizing current. The circulating, or eddy, current due to the E.M.F., E_x, may be represented by a vector, OI_x, lagging $\varphi°$ with respect to OE_x owing to the inductance of the path of the eddy current.

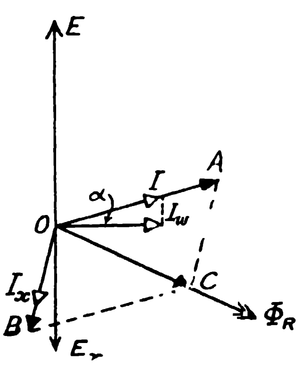


FIG. 195. Vector Diagram Showing Effect of Eddy Currents

Now when the core is concentric with the magnetizing winding, the path of this eddy current is concentric with that of the current in the magnetizing winding. Therefore the resultant magneto-motive force, to which the flux in the core is due, is equal to the vector sum of the M.M.Fs. due to the current in the magnetizing winding and the eddy current in the core. The component M.M.Fs. are represented by OA, OB in Fig. 195, the resultant M.M.F. is represented by OC, and the flux in the core (which is in phase with the resultant M.M.F.) is represented by $O\Phi_R$. In general, the resultant M.M.F. will be smaller than that due to the magnetizing current, and therefore the resultant flux will not be in phase with the magnetizing current, and will be smaller than the flux which would be obtained if there were no eddy currents, other conditions remaining unaltered.

When the effects of all the eddy currents in the core are considered—which requires analytical treatment and is given later—the results show that, except in the cases of very thin wires and plates, the flux distribution over the cross section of the core is not

uniform; the flux tending to concentrate towards the surface, or outer layers, of the plates, or wires, forming the core. The eddy currents, therefore, produce a magnetic screening effect on the central portions of the plate. Hence, with cores constructed of thick plates, a large percentage of the cross-section would be rendered magnetically ineffective by eddy currents, and therefore the average value of the flux density in the core may be very different from the computed value on the assumption of uniform flux distribution.

The curves of Fig. 196 show, for a particular brand of iron, that to obtain approximate uniform distribution of flux throughout the cross section of the plate, the thickness should not exceed about 0·5 mm. (0·014 in.) when the frequency is 50 cycles per second, and about 0·35 mm. (0·01 in.) when the frequency is 100 cycles per second.

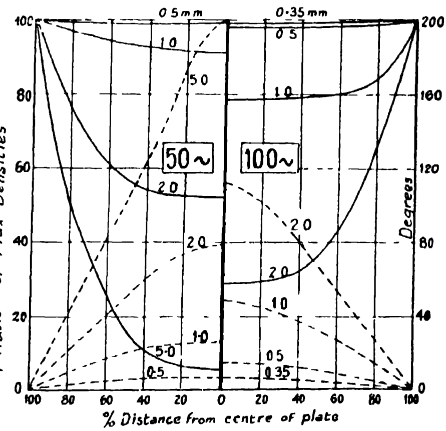


FIG. 196. Distribution of Flux Density over Cross Section of Iron Plates of Various Thicknesses. Dotted Curves Show Phase Displacement of Flux Density in Space

Calculation of flux distribution in iron core-plates in which eddy currents circulate. The calculation, though involving the solution of a differential equation, is effected by the application of elementary principles. Thus, the change in flux density over the cross-section of a core-plate is related to the change, with respect to the cross-section, of M.M.F. due to eddy currents, and the latter is proportional to the change in the E.M.F. induced in the plate. In the calculations we shall assume that the flux in the core varies sinusoidally with respect to time; that the individual core plates are insulated from one another, and that the thickness of each plate is small in comparison with the width of the plate.

Then, considering a single plate of thickness δ cm., let two elementary parallel layers, each of thickness dx, be chosen in the cross-section at distances x cm. from the centre line, as represented in Fig. 194, p. 339. Let B_{mx} denote the maximum value of the flux density (assumed to be uniform*) over these layers, and Φ_{mx} the maximum value of the flux enclosed in unit width of plate by the inner surfaces, AB, CD, of the layers. Hence the maximum value of the E.M.F. induced in unit width of plate at a distance x from the centre line (i.e. along the inner surfaces) is given by

$$E_{mx} = \omega\Phi_{mx} \times 10^{-8} = 2\pi f\Phi_{mx} \times 10^{-8}.$$

* This assumption is justifiable if the thickness dx is very small in comparison with δ.

Similarly, the induced E.M.F. at a distance $(x + dx)$ from the centre line (i.e. along the outer surfaces) is given by

$$E_{m(x+dx)} = 2\pi f(\Phi_{mx} + 2B_{mx}dx) \times 10^{-8}.$$

Whence

$$E_{mx} - E_{m(x+dx)} = dE_{mx} = 2\pi f(2B_{mx}dx) \times 10^{-8} = 4\pi f B_{mx}dx \times 10^{-8}.$$

If ρ denotes the specific resistivity of the plate, the resistance of the circuit formed by the elementary layers, per unit width and depth of plate, is equal to $2\rho/dx$. Hence the eddy current circulating in this circuit is given by

$$I_{mx} = E_{mx}/(2\rho/dx) = E_{mx}dx/2\rho.$$

The magneto motive force (maximum value) due to this current is equal to $0{\cdot}4\pi I_{mx}$, and this represents the increment in magneto-motive force due to eddy currents for the elementary layers under consideration. Therefore the increment in flux density for the elementary layers is given by

$$dB_{mx} = \mu(0{\cdot}4\pi I_{mx}) = (0{\cdot}2\pi E_{mx}\mu/\rho)dx$$

where μ is the permeability.

Expressing dB_{mx} and dE_{mx} in symbolic notation, and taking the flux density as the quantity of reference, we have

$$\frac{d\dot{B}_{mx}}{dx} = \frac{0{\cdot}2\pi\mu}{\rho}\dot{E}_{mx} \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

$$\frac{d\dot{E}_{mx}}{dx} = -j\,(4\pi f\dot{B}_{mx} \times 10^{-8}) \quad . \quad . \quad (\beta)$$

Whence $\dfrac{d^2\dot{B}_{mx}}{dx^2} = \dfrac{0{\cdot}2\pi\mu}{\rho}\cdot\dfrac{d\dot{E}_{mx}}{dx}$

Substituting for $d\dot{E}_{mx}/dx$ from equation (β), we have

$$\frac{d^2\dot{B}_{mx}}{dx^2} = -j\left[\dot{B}_{mx}\left(0{\cdot}8\pi^2\frac{\mu}{\rho}f \times 10^{-8}\right)\right]$$

$$= -j\,2c^2\dot{B}_{mx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

where $\quad c^2 = 0{\cdot}4\pi^2 f\mu/10^8\rho,$

or $$c = \frac{2\pi}{10^4}\sqrt{\frac{f\mu}{10\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

The solution of the differential equation (γ) is

$$\dot{B}_{mx} = \dot{C}_1\varepsilon^{x\sqrt{(-j2c^2)}} + \dot{C}_2\varepsilon^{-x\sqrt{(-j2c^2)}}$$

where $\dot{C}_1$ and $\dot{C}_2$ are complex constants. In the present case these constants have equal values, since the flux density has the same value, but is of opposite sign, at points on opposite sides of, and equidistant from, the centre line of the plate. Whence

$$\dot{B}_{mx} = \dot{C}(\varepsilon^{xc\sqrt{-j2}} + \varepsilon^{-xc\sqrt{-j2}})$$

Denoting the flux density at the surface layers of the plate (for which x has the value $\frac{1}{2}\delta$) by $B_{m\delta}$, we have

$$\dot{B}_{m\delta} = \dot{C}\,(\varepsilon^{\frac{1}{2}\delta c\sqrt{-j2}} + \varepsilon^{-\frac{1}{2}\delta c\sqrt{-j2}})$$

Whence $$\frac{\dot{B}_{mx}}{\dot{B}_{m\delta}} = \frac{\varepsilon^{xc\sqrt{-j2}} + \varepsilon^{-xc\sqrt{-j2}}}{\varepsilon^{\frac{1}{2}\delta c\sqrt{-j2}} + \varepsilon^{-\frac{1}{2}\delta c\sqrt{-j2}}}$$

or $$\dot{B}_{mx} = \dot{B}_{m\delta}\frac{\varepsilon^{xc\sqrt{-j2}} + \varepsilon^{-xc\sqrt{-j2}}}{\varepsilon^{\frac{1}{2}\delta c\sqrt{-j2}} + \varepsilon^{-\frac{1}{2}\delta c\sqrt{-j2}}}$$

which ultimately reduces to*

$$B_{mx} = B_{m\delta}\,\frac{(\varepsilon^{cx}+\varepsilon^{-cx})\cos cx - j(\varepsilon^{cx}-\varepsilon^{-cx})\sin cx}{(\varepsilon^{\frac{1}{2}\delta c}+\varepsilon^{-\frac{1}{2}\delta c})\cos\frac{1}{2}\delta c - j(\varepsilon^{\frac{1}{2}\delta c}-\varepsilon^{-\frac{1}{2}\delta c})\sin\frac{1}{2}\delta c} \qquad (173)$$

$$= B_{m\delta}\,\frac{\cosh cx\cos cx - j\sinh cx\sin cx}{\cosh\frac{1}{2}\delta c\cos\frac{1}{2}\delta c - j\sinh\frac{1}{2}\delta c\sin\frac{1}{2}\delta c} \quad . \quad . \quad . \quad (173a)$$

The denominator of this expression represents a complex number having a constant value for given conditions. The numerator is equal to unity when $x = 0$, and becomes equal to the denominator when $x = \frac{1}{2}\delta$. For values of x intermediate between 0 and $\frac{1}{2}\delta$ the numerator represents a complex number having a value intermediate between unity and the value of the denominator. Hence the equation (173) to the flux density at any point in the cross-section of the plate is of the form

$$B_{mx} = B_{m\delta}(+\boldsymbol{a}+j\boldsymbol{b})$$

where $+\boldsymbol{a}+j\boldsymbol{b}$ represents the value of the quotient in equation (173).

Therefore at any instant the flux densities at different portions of the cross section of the plate not only differ in magnitude but also have a space displacement with respect to one another, i.e. these flux densities have different directions (in space) with respect to the direction of the flux density at the surface layers (which coincides with that of the magnetizing ampere-turns). A reversal in direction (i.e. a space displacement of 180°) occurs when $cx = \pi$, or when $x = \frac{10^4}{2}\sqrt{\frac{10\rho}{f\mu}}$. For example, if $\rho = 10^{-5}$, $\mu = 1000$, $f = 50$, then $x = 0{\cdot}2235$ cm. Hence, for these conditions, the flux density at the centre of a plate 4·47 mm. thick has a direction opposite to that of the flux density at the surface layers. The manner in which the magnitude and space displacement of the flux density varies over the cross section of a similar plate 5 mm. thick is shown in the worked example which follows.

Example. Calculation of flux distribution in an iron plate 5 mm. thick when subjected to an alternating magnetization of 50 frequency. Specific resistance = 10^{-5} ohm per cm. cube; permeability = 1000 (assumed to be constant).

From equation (172) the space displacement between the flux densities at centre and surface layers is

$$\theta = \tfrac{1}{2}\times 0{\cdot}5\times\frac{2\pi}{10^4}\sqrt{\frac{50\times 1000\times 10^5}{10}} = 3{\cdot}51 \text{ radians} = 201°$$

Hence it will be convenient to calculate the flux densities at space intervals corresponding to displacements of $\frac{1}{6}\pi$ radians, or 30°, i.e. values of cx in equation (173a) will be taken at intervals of $\frac{1}{6}\pi$ radains.

The evaluation of equation (173a) is effected with the aid of tables of hyperbolic functions and presents no difficulties. The values of $\cosh cx$, $\sinh cx$, $\cos cx$, $\sin cx$ required for the calculations are given in Table XIV.

Since	cosh 3·51	= 16·82 ;	cos 201°	= −0·933
	sinh 3·51	= 16·78 ;	sin 201°	= −0·359

the denominator of equation (173a) reduces to

$$16{\cdot}82\times(-0{\cdot}933)+j16{\cdot}78\times 0{\cdot}359 = -15{\cdot}7+j6{\cdot}02$$

* The reduction depends upon the theorems—

(α) . . . $\varepsilon^{x\sqrt{-j2}} = \varepsilon^{x(1-j1)} = \varepsilon^{x}\,.\,\varepsilon^{-jx} = \varepsilon^{x}(\cos x - j\sin x)$

(β) . . . $\varepsilon^{-x\sqrt{-j2}} = \varepsilon^{-x(1-j1)} = \varepsilon^{-x}\,.\,\varepsilon^{jx} = \varepsilon^{-x}(\cos x + j\sin x)$

The steps in the reduction of the exponent $x\sqrt{-j2}$ are—

$$\begin{aligned} x\sqrt{(0-j2)} = \sqrt{(0-j2x^2)} &= \sqrt{(2x^2\varepsilon^{-j\frac{1}{2}\pi})} \\ &= (x\sqrt{2})\varepsilon^{-j\frac{1}{4}\pi} \\ &= x\sqrt{2}(\cos\tfrac{1}{4}\pi - j\sin\tfrac{1}{4}\pi) \\ &= x(1-j1). \end{aligned}$$

The values of the numerators and quotients are calculated in the usual manner and the results are given in tabular form in Table XIV, and are plotted in Fig. 197*a*, from which it will be observed that the central portion of the plate for a distance of about 1·5 mm. on each side of the centre line. is almost entirely ineffective.

It is of interest to calculate the flux which is transmitted per cm. widtb plate when the flux density at the surface (i.e. $B_{m\delta}$) is 13,000 lines per sc cm., and thence deduce the thickness of plate required to transmit thi. at a uniform flux density of 13,000 lines per square cm. (i.e. on the assump of no eddy currents).

The calculation may be effected quite easily by dividing the cross-section of the plate into narrow strips and determining for each strip the component

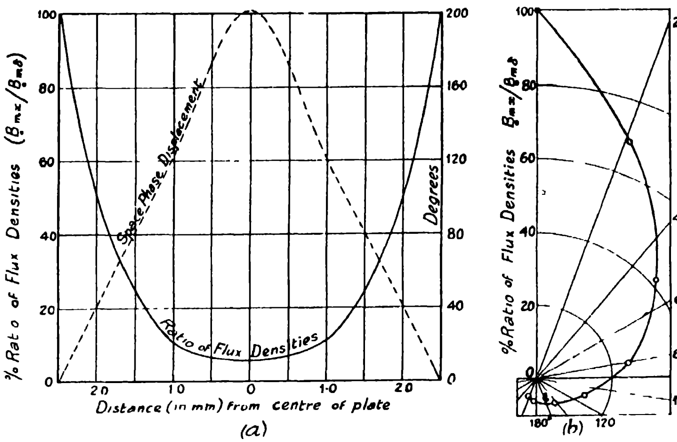


FIG. 197. Distribution of Flux Density over Cross Section of Iron Plate 5 mm. Thick

of the mean flux density in the direction parallel to the axis of magnetization Probably the simplest method is to plot the ratio $B_{mx}/B_{m\delta}$ and θ in polar co-ordinates, as in Fig. 197*b*, divide this into twenty parts by radii-vectores spaced 10° apart, draw the mid-point radii vectores, and measure their projections on the axis of reference (which is vertical in Fig 197*b*). Thus the first mid-point radius-vector is drawn at an angle of 5° from the vertical (corresponding to a point in the cross-section of the plate distant [2·5(5/201) =] 0·0622 mm. from the surface), and the remaining radii-vectores are drawn at angles of 10° from one another (corresponding to successive points 0·1244 mm. apart in the cross section of the plate along the direction perpendicular to the surface of the plate). The values of the projections, taken in order and expressed as a percentage of $B_{m\delta}$, are 92, 74, 52·6, 43·5, 32, 22·1, 13·7, 7·2, 2, – 1·5, – 3·6, – 5, – 5·8, – 6·2, – 6·4, – 6·6, – 6·6, – 6·4, – 6·2, – 5·6; and their algebraic sum is 279·2.

Whence the average flux density over the cross-section of the plate is (13,000 × 279·2/2000 =) 1815 lines per square cm., and the flux transmitted per cm. of width is (1815 × 0·5 × 1 =) 907 lines.

Hence the thickness of plate required to transmit this flux at a uniform flux density of 13,000 lines per square cm. (i.e. on the assumption of no eddy-currents) is (907/13000 =) 0·0698 cm., or about two-thirds of a millimetre, which is only 14 per cent of the thickness of the 5 mm. plate.

In consequence of the non-uniform distribution of the flux in thick laminations, the maximum value of the flux density may be considerably higher than the value calculated on the assumption of uniform flux distribution Therefore the eddy-current loss in such plates will be higher than that calculated from equation (171).

Effect of frequency and wave-form on hysteresis and eddy-current losses in iron cores. *Effect of frequency on hysteresis loss.* Experimental results of hysteresis tests, by the "ballistic" method, on magnetic materials show that the hysteresis loss in a given material is proportional to nearly the 1·6th power of the flux density. Thus the energy expended per cubic cm. of material per magnetic cycle is expressed by

$$W_h = \eta B_m^{1.6}$$

where η is a coefficient the value of which depends upon the quality of the iron and the units in which W_h and B_m are expressed. For iron laminations such as are used in electrical machinery, the value of η varies from 0·001 to 0·003 when W_h and B_m are expressed in C.G.S. units.

Assuming the hysteresis loop to be the same with alternating magnetization as when determined ballistically, the hysteresis loss, with alternating magnetization, will be proportional to the first power of the frequency and to the 1·6th power of the flux density. The proportionality of hysteresis loss and frequency at constant flux density holds only for low frequencies, as, for frequencies of the order of 100 cycles per second and above, the width of the hysteresis loop, corresponding to a given maximum flux density, increases with increasing frequency, and, therefore, the hysteresis loss per magnetic cycle becomes larger as the frequency increases.

Effect of wave-form of impressed E.M.F. on hysteresis loss. For low frequencies the hysteresis loss per cubic cm. is given by $P_h = \eta f B_m^{1.6}$.

Now the flux density B_m is given by Φ_m/A, where A is the magnetic cross-section of the core. Also $\Phi_m = E \times 10^8/(4k_f f N)$, where E is the R.M.S. value of the E.M.F. induced in the magnetizing winding, k_f the form-factor of this E.M.F., f the frequency, and N the number of turns in the magnetizing winding. Whence

$$P_h = \eta f[E \times 10^8/(4k_f fNA)]^{1.6}$$

$$= \eta\left(\frac{(E \times 10^8)}{4NA}\right)^{1.6} \frac{\eta}{k_f^{1.6} f^{0.6}} \quad . \tag{174}$$

Hence, if the resistance of the magnetizing winding is negligible, and the impressed E.M.F. and frequency are constant, the hysteresis loss is inversely proportional to the 1·6th power of the form-factor

of the impressed E.M.F. Therefore the hysteresis loss with peaked E.M.F. waves will be lower than that with flat-topped E.M.F. waves of the same R.M.S. value.

The following values indicate the extent to which the form-factor of the impressed E.M.F. affects the hysteresis loss—

Form-factor.	1·0	1·11	1·2	1·3	1·4
Relative hysteresis loss (constant E.M.F. (R.M.S. value) and frequency) .	1·18	1·0	0·88	0·78	0·69

The *eddy-current loss* in a given plate or wire has been shown to be proportional to the squares of the flux density, frequency, and form factor. Thus

$$P_e = \xi(k_f \,.\, f \,.\, B_m)^2$$

Substituting for B_m in terms of the induced E.M.F., frequency, etc., we have

$$P_e = \xi[k_f \,.\, f \,.\, E \times 10^8/(4k_f f NA)]^2$$
$$= \xi[E \times 10^8/(4\,NA)]^2 \quad . \quad . \quad . \quad . \quad (175)$$

Hence, in a given magnetic core magnetized by alternating current the eddy-current loss is proportional to the square of the impressed E.M.F. (R.M.S. value) and is independent of the frequency and wave-form of the latter. This statement, however, holds only for cases where the flux density is uniformly distributed over the cross-section of the core (e.g. when the laminations are very thin and the frequency is low).

The *total loss* in a magnetic core magnetized by alternating current is, therefore, dependent upon the form factor of the impressed E.M.F., when the frequency and R.M.S. value of impressed E.M F. are constant.

When the impressed E.M.F. is constant and the frequency is varied, the total loss decreases as the frequency increases, since the eddy-current loss is constant and the hysteresis loss decreases with increasing frequency. But when the flux, or flux density, is constant and the frequency is varied, the total iron loss increases as the frequency increases, since the hysteresis loss varies directly as the frequency and the eddy-current loss varies as the square of the frequency, the form-factor of the impressed E.M.F. being assumed to be constant.

In practice the total iron loss is usually expressed in the form of

curves which show the loss (in watts per lb., or kg.) at various flux densities and constant frequency. Typical curves are given in Fig. 198.

Calculation of magnetizing and exciting currents for an iron-cored magnetic circuit with alternating flux. *Case I, in which the distorting effect of hysteresis and magnetic saturation on the wave-form of the magnetizing current is ignored.* If both flux and magnetizing current are sinusoidal, the calculation of the ampere turns, or magnetizing

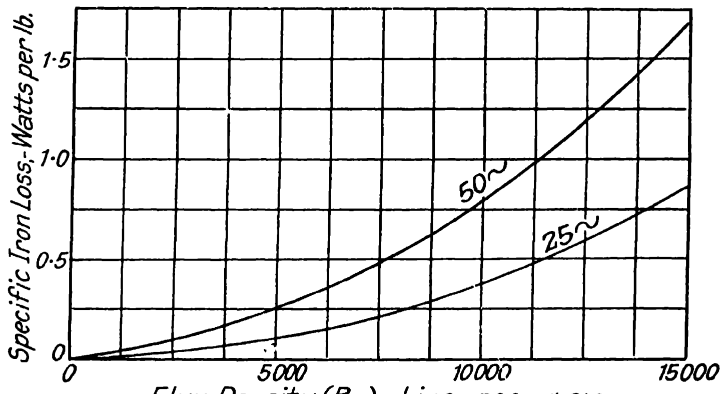


FIG. 198.—Specific Iron Loss for 0·018 in. (0·45 mm.) Alloyed Iron Laminations

current, required to produce a given flux is very simple. The procedure is the same as that when the magnetization is produced by direct current except that, with alternating magnetization, the *maximum* value of the flux, or flux density, is usually given (or is obtained by calculation), and the R.M.S. value of the magnetizing current to produce this flux has to be determined.

For example, if the magnetization, or *B-H*, curve determined by the ballistic method for a magnetic material, shows that F_{cm} ampere turns per cm. of magnetic length are necessary to obtain a flux density equal to B_m, then, with alternating (sinusoidal) magnetization and the same flux density (*maximum* value in this case), the *maximum* value of the ampere-turns will be equal to F_{cm}, and the R.M.S. value of the ampere-turns will be equal to $F_{cm}/\sqrt{2}$.

Hence, if l is the length (in cm.) of a magnetic circuit formed of this material, then, if the flux density B_m is the same at all parts of the circuit, the magnetizing current will be given by

$$I_m = lF_{cm}/(\sqrt{2} \cdot N).$$

If the magnetic circuit consists of a number of parts having different cross-sections and magnetization curves, the problem is treated by determining the ampere-turns required for each part of the circuit and adding these to obtain the total ampere turns. Thus, if l_1, l_2, l_3, . . . denote the several magnetic lengths, F_{1cm}, F_{2cm}, F_{3cm}, . . . the ampere turns per cm. required to obtain the requisite flux densities, corresponding to a given flux, in the several parts of the circuit, then the magnetizing current is given by

$$I_w = \frac{1}{N\sqrt{2}}(l_1F_{1cm} + l_2F_{2cm} + l_3F_{3cm} + \ . \ . \ .)$$

The exciting current may be determined when the losses in the iron core and magnetizing winding are known. Thus, if P denotes the total losses and E the impressed E.M.F., the power component of the exciting current is given by $I_p = P/E$, and, therefore, the exciting current is given by

$$I_0 = \sqrt{(I_p^2 + I_w^2)}.$$

Example. A closed magnetic circuit is built up of 0·35 mm. laminations and consists of two cores which are magnetically connected by yokes. The magnetic cross-section of each core is 100 square cm., and that of each yoke is 108 square cm. The magnetic length of each core is 20 cm., and that of each yoke is 11 cm. Each core is wound with a magnetizing coil having 75 turns and a resistance of 0·18 Ω. The two coils are connected in series and excited from a 50 cycle circuit to give a maximum flux of $1{\cdot}2 \times 10^6$ magnetic lines in the cores and yokes. Calculate the exciting current.

Data of the magnetic properties of the laminations are :—

Maximum flux density (B_m) (lines per square cm.)	11,100	12,000
Magnetizing ampere turns per cm. of length (F_{cm})	3·9	6
Core loss at 50 frequency (watts per lb.). .	0·76	0·88

The flux density (maximum value) in each core is $(1{\cdot}2 \times 10^6/100) = 12{,}000$ lines per square cm., and that in each yoke is $(1{\cdot}2 \times 10^6/108) = 11{,}100$ lines per square cm.

Hence the magnetizing ampere turns (maximum value) required are equal to

$$2 \times 11 \times 3{\cdot}9 + 2 \times 20 \times 6 = 326,$$

and the R.M.S. value of the magnetizing current is

$$326/(150 \times \sqrt{2}) = 1{\cdot}535 \text{ A}.$$

The weight of laminations in the two cores is

$$2 \times 20 \times 100 \times 0{\cdot}28/2{\cdot}54^3 = 68{\cdot}3 \text{ lb.},$$

and that in the two yokes is

$$2 \times 11 \times 108 \times 0{\cdot}28/2{\cdot}54^3 = 40{\cdot}5 \text{ lb.}$$

Whence the total iron loss is

$$0{\cdot}88 \times 68{\cdot}3 + 0{\cdot}76 \times 40{\cdot}5 = 90{\cdot}8 \text{ watts.}$$

The E.M.F. induced in the magnetizing coils is

$$4{\cdot}44 \times 1{\cdot}2 \times 10^{4} \times 50 \times 2 \times 75 = 400 \text{ V.}$$

Hence a first approximation to the exciting current is given by

$$\sqrt{[1{\cdot}535^2 + (90{\cdot}8/100)^2]} = \sqrt{(1{\cdot}535^2 + 0{\cdot}227^2)} = 1{\cdot}552 \text{ A.},$$

and the angle by which this current leads the flux is equal to $\tan^{-1} 0{\cdot}227/1{\cdot}535$ or 8°.

Since the pressure drop in the magnetizing coils due to the exciting current is $1{\cdot}552 \times 2 \times 0{\cdot}18 = 0{\cdot}56$ V., and has a phase difference of 81° with respect to the induced E.M.F. in these coils, the impressed E.M.F. is practically equal to the induced E.M.F. and the first approximation to the exciting current is sufficiently accurate for practical purposes.

Case II, in which the distortion of the wave-form of the magnetizing current is considered. If the magnetic circuit is a simple one (i.e. the magnetic cross-section and material is the same throughout the circuit) the magnetizing current corresponding to a given maximum sinusoidal flux may be determined without difficulty when the hysteresis loop for the given magnetic conditions is available. The procedure is similar to that given on p. 333 for the determination of the wave-form of magnetizing current.

Having determined the wave-form of the current necessary to carry the magnetization through a cycle, the R.M.S. value of this current is deduced, together with the R.M.S. values of the magnetizing and power components. To the power component is added the component of the supply current which supplies the eddy-current loss. The exciting current is then obtained from the resultant power component and the magnetizing component.

This method of procedure becomes too involved when complex magnetic circuits have to be calculated, as an equivalent hysteresis loop for the complete magnetic circuit would have to be deduced before the wave-form of the current could be determined. Moreover, even for simple magnetic circuits, the above method is too tedious for practical purposes. A shorter method, however, is available which, although not possessing the same accuracy as the preceding method, nevertheless possesses sufficient accuracy for practical purposes and, moreover, takes into account wave-form distortion due to magnetic saturation.

For this method, magnetization curves corresponding to alternating magnetization are required, and when such curves are available the method of procedure in calculating the magnetizing current is similar to that for a magnetic circuit excited with direct current.

The magnetization curves are obtained experimentally on a sample of the material by measuring the exciting current and the

power supplied to the magnetizing winding at various impressed voltages and constant frequency. The flux and flux density in the specimen are calculated from the induced E.M.F., the number of turns in the magnetizing winding, and the magnetic cross-section of the specimen. The power supplied, when corrected for the I^2R loss in the magnetizing winding, represents the hysteresis and eddy-current losses in the specimen. If the exciting current is split up into power and wattless components, the latter represents the equivalent magnetizing current and takes into account the effects of hysteresis and eddy currents.

The magnetizing ampere turns per cm. of magnetic length ($F_{w \cdot cm}$) are calculated from the wattless component (I_w) of the exciting current, the number of turns (N) in the magnetizing winding and the mean length of magnetic path (l). Thus

$$F_{w \cdot cm} = NI_w/l.$$

If the exciting current is denoted by I_0, the hysteresis and eddy-current losses by P_t, and the impressed E.M.F. by E, then the power component of the exciting current which supplies the hysteresis and eddy-current losses is given by $I_p = P_t/E$, and the wattless component is given by

$$I_w = \sqrt{(I_o^2 - I_p^2)} = \sqrt{[I_o^2 - (P_t/E)^2]}.$$

Whence the magnetizing ampere turns per cm. of magnetic length are

$$F_{w \cdot cm} = NI_w/l = \sqrt{[(NI_o/l)^2 - (NI_p/l)^2]}$$
$$= \sqrt{(F_{cm}^2 - F_{p \cdot cm}^2)}$$

where F_{cm} denotes the exciting ampere turns per cm. of magnetic path, and $F_{p \cdot cm}$ the ampere turns per cm. of magnetic length for supplying the hysteresis and eddy-currents; these ampere turns having a phase difference of 90° with respect to the magnetizing ampere turns.

If curves are plotted, showing the relationship between the maximum flux density, B_m, and $F_{w \cdot cm}$ and $F_{p \cdot cm}$, we have available a simple means for obtaining the two components of the exciting ampere turns.

CHAPTER XII

COMMERCIAL MEASURING INSTRUMENTS (AMMETERS, VOLTMETERS, WATTMETERS, WATT-HOUR METERS)

Types of instruments. In commercial alternating-current measurements the principal electrical quantities concerned are: current, potential difference, power, energy, power-factor, and frequency. Accordingly, commercial alternating-current measuring instruments comprise ammeters, voltmeters, wattmeters, energy (or watt-hour) meters (also called "supply" and "house-service" meters), power-factor indicators, and frequency indicators. Other types of instruments may be required for special purposes; for example, in electric power stations an instrument (called a synchroscope) is required for indicating the instant of phase coincidence of the E.M.Fs. of alternating-current generators which are to be connected in parallel. Again, in cable testing at high voltages it is desirable to be able to measure the peak, or maximum, value of the applied voltage.

For laboratory testing, instruments are required for detecting and measuring currents of small magnitudes, for measuring the phase differences between currents and E.M.Fs., etc., while for investigating transient and periodic phenomena, and for determining wave-forms, an instrument (called an oscillograph) is required, which will indicate and give a record of instantaneous values of current and E.M.F.

Classification and principles of operation. Commercial measuring instruments are classified according to both the quantity measured by the instrument and the principle of operation. Three general principles of operation are available: (1) electromagnetic, which utilizes the magnetic effects of electric currents; (2) electro-thermic, which utilizes the heating effect; (3) electrostatic, which utilizes the forces between electrically charged conductors.

Electromagnetic instruments may be subdivided according to the nature of the movable system and the method by which the deflecting, or operating, torque is produced. The sub-classes are: (*a*) moving-iron instruments, (*b*) electro-dynamic, or dynamometer, instruments, (*c*) induction instruments.

In *moving-iron instruments* the movable system consists of one

or more pieces of specially-shaped soft iron, which are so pivoted as to be acted upon by the magnetic field produced by the current, or currents, in one or more fixed coils.

In *electro-dynamic instruments* the operating torque is due to the interaction of the magnetic fields produced by currents in a system of fixed and movable coils. The movable system of these instruments consists of one or more coils, which are so pivoted as to move in the magnetic field produced by the fixed coils. The currents in both fixed and movable coils are obtained from a common source.

In *induction instruments* the movable system consists of a pivoted non-magnetic conducting disc or drum, a portion of which moves in the magnetic field produced by an electromagnet, or a system of electromagnets, excited with alternating current. The magnetic field induces currents in the movable disc, or drum, and the magnetic reaction of the induced currents and the alternating magnetic field produces the operating torque.

Electro-thermic instruments may be subdivided into two distinct types: (*a*) the "expansion" type (commonly called "hot-wire," or "hot-strip" instruments) in which the operation depends upon the linear expansion of a wire, or strip, heated by a current; (*b*) the "thermo-couple" type (commonly called "thermo" instruments), in which one or more thermo-couples are heated, either directly or indirectly, by the current to be measured, and the thermo-E.M.F. is caused to operate a direct-current instrument of the permanent-magnet moving-coil type.

Application of operating principles: relative advantages and disadvantages. The *electromagnetic principle of operation* may be employed for all kinds of commercial and laboratory instruments; it is, moreover, the only operating principle which can be employed for energy meters, power-factor indicators, frequency indicators, and synchroscopes.

The *electro-dynamic form of construction* is applicable to all classes of measuring instruments, as well as to synchroscopes, galvanometers, and oscillographs. Well-designed electro-dynamic measuring instruments possess the advantage that the instrument readings are practically independent of the wave-form and frequency* of the current passing through the instrument. The instruments can therefore be used on either alternating- or direct-current circuits without re-calibration.

The *moving-iron form of construction* is applicable to ammeters,

* This statement applies only to frequencies within the range adopted in electric lighting and power supply systems.

voltmeters, power-factor meters, frequency meters, synchroscopes, and galvanometers. With measuring instruments, however, the effects of hysteresis and eddy-currents in the iron elements will cause the instrument readings to be affected by the wave-form and frequency of the operating current, but by suitable design the errors due to these causes may be made almost negligible for commercial frequencies and wave-forms. Moving-iron instruments are, in general, inferior to electro-dynamic instruments with respect to accuracy and power consumption, but their first cost is lower than that of either electro-dynamic or induction instruments of similar range and size.

It is important to note that, unless the electromagnetic portion of the mechanism of electro-dynamic and moving-iron instruments is magnetically shielded, the indications are liable to be seriously affected by external magnetic fields, due, for example, to heavy currents in neighbouring conductors.

The *induction form of construction* is applicable to ammeters, voltmeters, wattmeters, energy meters, and frequency meters. Induction measuring instruments are more susceptible to errors due to frequency and wave-form than other forms of measuring instruments, and, in consequence, their application is limited to circuits of constant frequency. These instruments are used principally on switchboards, and are almost immune from the effects of external magnetic fields.

Induction watt-hour meters have entirely superseded the electro-dynamic form for the measurement of energy on commercial electric lighting and power circuits.

The **electro-thermic principle** is applicable to current-measuring instruments; it has been applied to oscillographs of the hot-wire type and to galvanometers of the thermo-electric type. Electro-thermic ammeters and voltmeters possess the advantage that the instrument readings are independent of the wave-form and frequency of the current passing through the instrument; they can, therefore, be used on either alternating- or direct-current circuits. The instruments, however, possess the disadvantages of sluggishness, relatively high power consumption (particularly in the "expansion" type of instrument), and liability to damage with small overloads.

The **electrostatic principle** is usually confined to instruments for measuring potential difference, although the principle has recently been applied to oscillographs. Electrostatic voltmeters possess two important advantages over other types of alternating-current voltmeters, viz. (1) the power-consumption is negligible; (2) the instruments can be constructed for direct connection to

high-voltage circuits. Electrostatic voltmeters also possess the advantage that the instrument readings are independent of frequency and wave-form.

AMMETERS AND VOLTMETERS

General requirements. The indications of alternating-current ammeters and voltmeters must represent the R.M.S. values of the current, or potential difference, respectively, applied to the instruments. The scale divisions, therefore, must be proportional to the mean squared values of the corresponding currents or potential differences. Hence, in the case of an ammeter or a voltmeter, the mean deflecting torque corresponding to a given deflection of the pointer must be proportional to the mean squared value of the current or potential difference applied to the instrument.

Moreover, if the readings are to be correct when the instruments are used on both direct- and alternating-current circuits, the torque must be uninfluenced by both the frequency and the wave-form of the applied current or potential difference.

Hot-wire ammeters and voltmeters. In these instruments the deflection of the pointer is produced by the linear expansion of a wire, or strip, heated by the current (or a definite fraction thereof) to be measured. Under ideal conditions the expansion is proportional to the square of the current, and if the angular deflections of the pointer are proportional to the expansion, the scale divisions will follow a parabolic, or square, law. Moreover, in such a case the instrument will read correctly with direct and alternating currents. But in practice a number of variable factors enter into the relationship between expansion and current, and the scale divisions have to be determined by calibration. If, however, an instrument is calibrated with direct current it will read correctly with alternating current provided that during a period, or half-period, the temperature of the wire remains sensibly constant.

A commercial instrument must be compact, and therefore a relatively short hot wire must be employed, the use of which requires a magnifying device to obtain a suitable deflection of the pointer corresponding to the small change in the actual length of the wire. A magnifying device, which has a large application in modern instruments, utilizes the principle that a small change in length of a wire stretched between fixed supports is accompanied by a relatively large change in sag.

Theory of sag magnifying device. If a fine wire is stretched between two fixed points, A, B (Fig. 199a), and is maintained taut by a transverse wire under tension applied to the mid-point C, then the sag, δ_0, is given by

$$\delta_0 = \tfrac{1}{2}\sqrt{(L_0^2 - l^2)},$$

where L_0 denotes the length of wire between the supports and l the distance between the supports. If the length of the wire increases by a small amount Δ, the new sag δ_1, is given by

$$\begin{aligned}\delta_1 &= \tfrac{1}{2}\sqrt{[(L_0+\Delta)^2 - l^2]}\\ &= \tfrac{1}{2}\sqrt{(L_0^2 + 2\Delta L_0 - l^2)}\\ &= \sqrt{(\tfrac{1}{2}\Delta L_0 + \delta_0^2)}\end{aligned}$$

since the term Δ^2 may be neglected.

Whence the increase in sag corresponding to the expansion Δ is

$$\delta_1 - \delta_0 = [\sqrt{(\tfrac{1}{2}\Delta L_0 + \delta_0^2)}] - \delta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

Observe that for given conditions the increase in sag will be the greater the smaller the initial sag, and that in the extreme case, where the initial sag is zero, the increase in sag is proportional to the square root of the expansion.

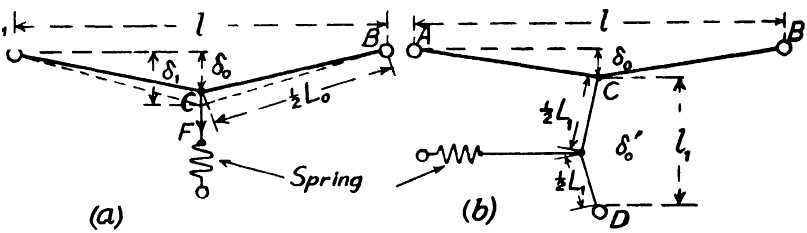


FIG. 199.—Sag Magnifying Devices for Hot wire Instrument

For example, if the original length and sag are 15 cm. and 0·1 cm. respectively, the increase in sag when the length of the wire increases by 0·025 cm. is

$$\begin{aligned}\delta_1 - \delta_0 &= [\sqrt{(\tfrac{1}{2} \times 0{\cdot}025 \times 15 + 0{\cdot}1^2)}] - 0{\cdot}1\\ &= 0{\cdot}344 \text{ cm.}\end{aligned}$$

Hence the increase in sag is $(0{\cdot}344/0{\cdot}025 =)$ 13·76 times the increase in length.

If, however, the initial sag were 0·2 cm. and the other conditions are unaltered, the increase in sag would be $[\sqrt{(\tfrac{1}{2} \times 0{\cdot}025 \times 15 + 0{\cdot}2^2)} - 0{\cdot}2] = 0{\cdot}277$ cm., and the magnification factor (i.e. change in sag/change in length) would be only $(0{\cdot}277/0{\cdot}025 =)$ 11·1.

The principle may be extended as shown in Fig. 199*b*, in which a lateral pull is now applied to the mid-point of the transverse wire, the ends of which are attached to the mid-point, C, of the hot wire and to a fixed support, D. Then if δ_0', δ_1', are the sags of the transverse wire corresponding to the sags δ_0, δ_1, respectively of the hot wire, l_1 is the initial distance between the points C and D, and L_1 the length of the transverse wire between these points, we have

$$\begin{aligned}\delta_0' &= \sqrt{[(\tfrac{1}{2}L_1)^2 - (\tfrac{1}{2}l_1)^2]}\\ \delta_1' &= \sqrt{\{(\tfrac{1}{2}L_1)^2 - [\tfrac{1}{2}l_1 - \tfrac{1}{2}(\delta_1 - \delta_0)]^2\}}\end{aligned}$$

Whence
$$\begin{aligned}\delta_1' - \delta_0' &= [\sqrt{\{(\tfrac{1}{2}L_1)^2 - (\tfrac{1}{2}l_1)^2 + \tfrac{1}{2}l_1(\delta_1 - \delta_0) - \tfrac{1}{4}(\delta_1 - \delta_0)^2\}}] - \delta_0'\\ &= [\sqrt{\{\delta_0'^2 + \tfrac{1}{2}l_1(\delta_1 - \delta_0) - \tfrac{1}{4}(\delta_1 - \delta_0)^2\}}] - \delta_0'\end{aligned}$$

Substituting for $\delta_1 - \delta_0$ from equation (176), and reducing, we obtain

$$\delta_1' - \delta_0' = [\sqrt{\{\delta_0'^2 + \tfrac{1}{2}(l_1 + \delta_0)(\sqrt{[\tfrac{1}{2}\Delta L_0 + \delta_0^2]} - \delta_0)} - \tfrac{1}{8}\Delta L_0\}] - \delta_0 \quad . \quad . \quad . \quad . \quad . \quad (176a)$$

For example, if the initial length and sag of the hot wire are 15 cm. and

0·1 cm. as above, the initial sag of the transverse wire is 0·3 cm., and the distance between the points C, D, Fig. 199*b*, is 10 cm., the change in sag of this wire due to an increase in length of 0·025 cm. in the length of the hot wire is

$$\delta_1' - \delta_0' = \sqrt{[0{\cdot}3^2 + \tfrac{1}{2} \times 0{\cdot}344(10 - 0{\cdot}344)]} - 0{\cdot}3$$
$$= 1{\cdot}03 \text{ cm.}$$

[NOTE. 0·344 is the change in sag of the hot wire due to its expansion.]

Hence the change in the sag of the transverse wire is (1·03/0·025 =) 41·2 times the increase in the length of the hot wire.

The above methods of magnification involve a relatively high initial stress in the hot wire when a large magnification is required (i.e. when the initial sag is very small).

The tension, T, in the hot wire can be easily calculated. Thus, if α is the inclination of the wire to the line joining its points of support, F, the force applied to the point of attachment of auxiliary wire (Fig. 199*a*), then, taking moments about one of the points of support and neglecting the weight of the hot wire, we have

$$\tfrac{1}{2}Fl = \delta_0 T \cos \alpha$$

Whence
$$T = \tfrac{1}{2}Fl/(\delta_0 \cos \alpha),$$

or, if the sag is small,
$$T = \tfrac{1}{2}Fl/\delta_0$$

For example, for the above conditions (i.e. $l = 15$ cm., $\delta_0 = 0{\cdot}1$ cm.)
$$T = \tfrac{1}{2}F(15/0{\cdot}1) = 75F,$$

and if $F = \frac{1}{50}$ lb. ($= 9$ grammes approx.) $T = 1\frac{1}{2}$ lb., which, with a wire 0·01 in. in diameter, results in a stress of $[1{\cdot}5/(0{\cdot}785 \times 0{\cdot}01^2) =]$ 19,100 lb. per sq. in.

Construction of hot-wire instrument (expansion type). The mechanism* of a typical hot-wire ammeter or voltmeter of the expansion type, in which the double sag principle is utilized as a multiplying device, is shown diagrammatically in Fig. 200*a*. The hot-wire, A, is usually of platinum-iridium (formerly platinum-silver was employed) and is stretched between supports which are fixed to a metallic base-plate, B, having approximately the same coefficient of expansion as the hot-wire, in order that changes of air temperature shall not affect the indications of the instrument.

The magnifying device consists of (1) a fine phosphor-bronze wire, C, stretched between the hot-wire, A, and an insulated pillar fixed to the base-plate; (2) a silk fibre, D, which is attached to C near its mid-point, lapped round a small grooved pulley, E, fixed to the spindle carrying the pointer, and is maintained in tension by a spring, F. The silk fibre is clamped to the pulley to prevent slipping. Hence any change in length of the hot-wire

* The term *mechanism* applied to instruments refers to the arrangement for producing and controlling the motion of the pointer. It includes all the essential parts necessary to produce this result, but does not include the base, cover, scale, or any other accessories, such as series resistances or shunts, the function of which is to make the readings agree with the scale markings.

relatively to the base-plate causes an angular movement of the spindle and a deflection of the pointer.

The spindle also carries a light aluminium damping disc, *G*, which moves in the narrow air gap of a horse-shoe magnet, *H*, fixed to the base of the instrument.

The functions of this device are somewhat different from those of the damping device employed with other types of instruments, as in the present case the moving system, under normal conditions,

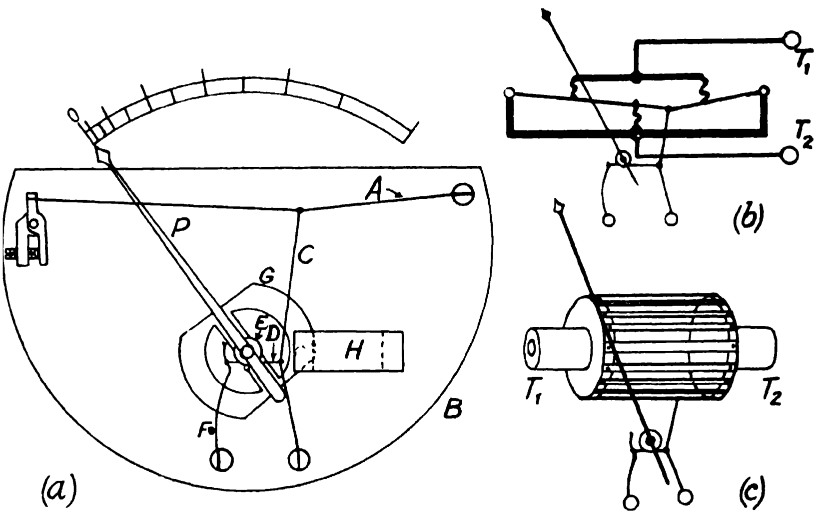


FIG. 200. —Hot-wire and Hot-strip Instruments

approaches its final position slowly. But in the event of a sudden change of current in the hot wire or vibration of the system due to mechanical shock, the inertia of the moving parts, if unchecked, would set up additional stresses in the hot wire which might cause fracture of the latter. The damping device tends to check sudden movements of the moving system and prevents excessive stresses in the hot wire due to the above causes.

Ammeter ranges up to 5 A. are obtained either by varying the diameter of the hot-wire or by dividing the wire into a number of sections connected in parallel. For example, for a range of 0·5 A. the wire has a diameter of 0·005 in., and the whole current traverses the wire ; but for a range of 5 A. the wire has a diameter of 0·01 in. and it is divided electrically into four sections, which are connected in parallel as shown at (*b*) in Fig. 200, so that the current in each section is one-fourth of the current being measured.

For currents exceeding 5 A. the wire has a diameter of about 0·01 in. and is connected in parallel with a non-inductive shunt contained in the case of the instrument. At radio frequencies, however, the shunt would introduce errors, and for these frequencies a special form of *hot-strip* instrument is employed. In this instrument the current to be measured passes through a number of narrow and thin strips of a material having a high specific resistance. The strips are arranged symmetrically with respect to one another and the terminals (see Fig. 200*c*), and are adjusted to have equal effective inductances and resistances. The distribution of current in the strips will, therefore, be independent of frequency, and the linear expansion of one strip will be a measure of the current in the circuit. This expansion is indicated by a pointer and scale in a manner similar to that adopted in the hot-wire instrument. In order that the effective resistance of the strips shall be the same at high and low frequencies, the material must be thin and must have a high specific resistance.

Voltmeter ranges up to about 400 V. are obtained by the use of a fine wire (having a diameter of about 0·0015 in.) and a non-inductive series resistance, the current required for full-scale deflection being about 0·25 A. Hence the power expended in the instrument at full-scale deflection is directly proportional to the range.

The principal advantages of hot-wire instruments are: (1) they can be used on direct- and alternating-current circuits without re-calibration; (2) the readings are unaffected by frequency, wave-form, or external magnetic fields. Their disadvantages have already been noted (p. 355).

Form of scale. An approximation to the form of scale of a hot-wire instrument can be obtained by assuming that the mean temperature of the hot wire during a brief interval of time is proportional to the mean squared value of the current during that period. The changes in the length of the wire and the sag can then be calculated, but these will not be strictly proportional to the change in temperature owing to the tension in the wire. The manner in which the tension affects the change in the length of the wire with change of temperature can be calculated easily when the wire is stressed within its elastic limit. Thus, if L_0 is the length of wire between the supports corresponding to the initial temperature, Θ, and a tension T; $L_0 + \Delta$ the length of wire corresponding to a temperature Θ_1, and a tension T_1; and L_0' is the unstretched length of the wire at its initial temperature; then the extension due to the tension T is $L_0 - L_0'$. Hence, if the stress is within the elastic limit, we have

extension/unstretched length = stress/modulus of elasticity

or
$$\frac{L_0 - L_0'}{L_0'} = \frac{T}{a\xi}$$

where a is the cross-section of the wire and ξ the modulus of elasticity.

Whence $$L_0' = L_0/(1 + T/a\xi)$$

Similarly if L_0'' is the unstretched length of the wire at the temperature Θ_1, we have

$$L_0'' = (L_0 + \Delta)/(1 + T_1/a\xi)$$

But $L_0'' = L_0'(1 + \alpha(\Theta_1 - \Theta))$, where α is the coefficient of linear expansion for the wire.

Hence $$(L_0 + \Delta)/(1 + T_1/a\xi) = L_0[1 + \alpha(\Theta_1 - \Theta)]/(1 + T/a\xi)$$

Whence $$\Delta = L_0 \left\{[1 + \alpha(\Theta_1 - \Theta)]\left(\frac{1 + T_1/a\xi}{1 + T/a\xi}\right) - 1\right\}$$
$$= [L_0/(T + a\xi)]\{kaI^2(T_1 + a\xi) + T_1 - T\}$$

where k is a constant involving the proportionality between $(\text{current})^2$ and temperature rise.

If this expression for Δ is substituted in equation (176) or (176a), we obtain an equation connecting $\delta_1 - \delta_0$ (or $\delta_1' - \delta_0'$) and the current in terms of the tensions T, T_1, and known quantities, such as L_0, δ_0, l_1, δ_0', a, k.

Now the increase in sag ($\delta_1 - \delta_0'$ or $\delta_1' - \delta_0'$) is directly proportional to the deflection, θ, of the pointer. Hence, for the single-sag form of magnification (Fig. 199a), we have

$$\theta = k'\left\{\sqrt{\left[\frac{\frac{1}{2}L_0^2}{T + a\xi}\{kaI^2(T_1 + a\xi) + T_1 - T\} + \delta_0^2\right]} - \delta_0\right\} \quad . \quad (177)$$

and for the double sag form of multiplication (Fig. 199b), we have

$$\theta = k''\left[\sqrt{\left\{\delta_0'^2 - \frac{\frac{1}{8}L_0^2}{T + a\xi}[kaI^2(T_1 + a\xi) + T_1 - T] - \tfrac{1}{2}(l_1 + \delta_0)\left(\delta_0 - \sqrt{\left[\frac{\frac{1}{2}L_0^2}{T + a\xi}(kaI^2(T_1 + a\xi) + T_1 - T) + \delta_0^2\right]}\right)\right\}} - \delta_0'\right] \quad . \quad (177a)$$

To determine the form of scale from these equations it is best to assume values for θ and to solve for kI^2, as the tension T_1 is then known from data of the sags and the tensions in the hot wire at zero and full-scale deflections, these tensions being determined from the known tensions in the silk fibre due to the tension-spring.

Example of hot-wire instrument. A hot-wire voltmeter of the type shown in Fig. 200a, having a maximum scale reading of 200 V., has a platinum iridium hot wire 0·002 in. (0·0508 mm.) in diameter, for which Young's modulus of elasticity is $1·36 \times 10^6$ kg. per cm.^{2} and the coefficient of linear expansion, per 1° C., is 9×10^{-6}.

The distance between the supports A, B, Fig. 199b, is 16·25 cm., the initial length of the hot wire is 16·3 cm. and the initial sag is 0·5 cm. The point of attachment, C (Fig. 199b), of the phosphor-bronze transverse wire is distant 10·05 cm. from the end A of the hot wire. The transverse wire has a length of 9·8 cm. and an initial sag of 0·4 cm. The increase in sag of the transverse wire corresponding to the full-scale deflection is 0·4 cm., and the corresponding increase in the sag of the hot wire is 0·1 cm.

The silk fibre is attached to the transverse wire at a point 5·7 cm. from the point C, and the initial tension in the fibre is 0·7 gramme, the corresponding tension in the hot wire being 32 gm. The calculated tension in the hot wire at full-scale deflection is 13·15 gm.

The operating current for the full-scale deflection of 87·5° is 0·24 A, and the corresponding temperature of the hot wire is about 150° C. The operating currents corresponding to intermediate deflections are given, together with

the scale readings, in the accompanying table, from which data the form of scale can be obtained.

Scale reading (volts).	0	50	75	100	125	150	175	200
Deflection (degrees) .	0	5	13	24	37	52	69	87·5
Operating current (amp.)	0	0·06	0·09	0·12	0·15	0·18	0·21	0·24

The resistance of the hot wire is 17·8 O. (hot), and the value of the non-inductive resistance connected in series with it is 815·2 O., giving a total resistance of 833 O.

Hence the power expended in the instrument at full-scale is ($200^2/833$ =) 48 W.

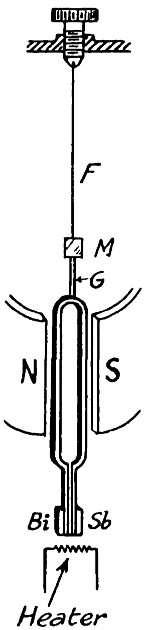


FIG. 201. Principle of Thermo-couple Galvanometer

Thermo-couple ammeters and galvanometers. In these instruments the heating effect of the current is utilized indirectly to deflect the pointer. The mechanism of the ammeter consists essentially of: (1) a moving coil of fine copper wire which carries a pointer and is pivoted in the magnetic field of a permanent magnet, the ends of the coil being connected to a thermo-couple; (2) a heater, of the resistance type, which is located close to the thermo-couple and through which passes the current to be measured, or a definite fraction thereof. In the case of the galvanometer (which is of the reflecting type) the coil consists of a single turn and is suspended by a quartz fibre. The principle of the construction is shown in Fig. 201.

In the pivoted instrument the pivots are arranged inside the moving coil to allow the thermo-couple and heater to be mounted at the end of the coil remote from the pointer. The control consists of a flat spiral spring and the damping is electromagnetic, as in the ordinary type of direct current moving-coil instrument.

The heater consists either of a short filament of wire or a grid (having an area of about 0·2 sq. cm.) of platinized mica, according to the range of the instrument.

Ammeter ranges from 10 mA. to 100 mA. are obtained by heaters of different resistances, but higher ranges are obtained by shunting the heater with non-inductive shunts. In the galvanometer the sensitivity, with a given heater, may be varied by altering the distance between the thermo-couple and the heater.

The power taken at full-scale deflection by an unshunted instrument is very small (about 0·015 watt), and the instrument will withstand safely an overload of about three times the normal current.

Due to the small dimensions of the heaters, the unshunted instrument and the galvanometer possess extremely small self-inductance and capacity. They are, therefore, particularly suitable for high-frequency measurements. Moreover, the deflections are practically proportional to the mean squared value of the current passing through the heater, and are independent of wave-form and frequency.

The instruments are standardized and calibrated on a direct-current circuit with the aid of standard direct-current instruments.

Form of scale. In an ideal instrument the whole of the heat produced by the current passing through the heater is radiated to the thermo-junction, and the temperature of the latter is proportional to the square of this current. Now the E.M.F. of a thermo-junction is proportional to its temperature,

and therefore the current in the moving coil of the indicator will be proportional to the square of the current in the heater. Hence, since for a permanent magnet moving-coil instrument the deflection is proportional to the current in the moving coil, the deflection in the case of the thermo-ammeter will be proportional to the square of the current in the heater. Conversely, the current (I) in the heater is proportional to the square-root of the deflection (θ) of the indicator, i.e. $I = k\sqrt{\theta}$, where k is a constant. Thus the scale must be divided according to a square law. For example, if the full-scale deflection is 70° and corresponds to a current of 120 mA., the intermediate scale divisions and the corresponding deflections are as follow—

Current (mA.)	120	100	80	60	40	20	0
θ (= 70 × current^2/120^2)	70	48·6	31·1	17·5	7·78	1·945	0

In the practical forms of instrument as manufactured by the Cambridge Instrument Co. the design is such that the above ideal conditions are approached very closely, and the scales of these instruments follow a square law.

Thermo-electric ammeters and galvanometers with independent thermo-junctions. Instead of the special construction, described above, in which the thermo-couple and heater form an integral part of the instrument, these parts may be constructed as a separate unit and may then be used in conjunction with an ordinary direct-current moving-coil galvanometer, milli-voltmeter, or micro-ammeter. In this case a greater thermo-electric E.M.F has usually to be provided by the thermo-junction, and either a number of thermo-couples are connected in series and heated by a common heater, or a single thermo-couple is employed and the temperature of the heater is raised to about 200° C., both heater and thermo-couple being enclosed in a highly-exhausted glass bulb. With these independent thermo-junctions currents exceeding 1 A. may be measured without the use of shunts.

Moving-iron ammeters and voltmeters. A variety of forms of these instruments are in commercial use, all of which may be divided into two groups, viz. (1) those in which the deflecting torque is due to the magnetic *attraction* of the moving-iron element either by the fixed coil carrying the current to be measured, or by a fixed iron element located therein ; (2) those in which the deflecting torque is due to the magnetic *repulsion* between the moving-iron element and a fixed iron element, both of which are located inside a fixed coil and are similarly magnetized.

The controlling torque may be due either to a flat spiral spring or to the action of gravity on a weighted lever attached to the moving system.

Air damping is employed universally, two forms of which are in common use : (1) an air dashpot, consisting of a fixed curved cylinder in which moves a loosely-fitting piston carried by the moving system ; (2) an air chamber, consisting of a practically closed chamber, fixed concentrically with respect to the spindle of the instrument, and a vane (attached to the spindle) which moves in the chamber with very small clearances.

Construction. Figs. 202, 203 show the principles of construction

of two forms of instruments. In both cases the fixed or magnetizing coil, A, carries the current to be measured. It is wound with relatively few turns of thick wire for an ammeter and a large number of turns of thin wire for a voltmeter, the number of ampere-turns being approximately constant in a given instrument for all ranges.

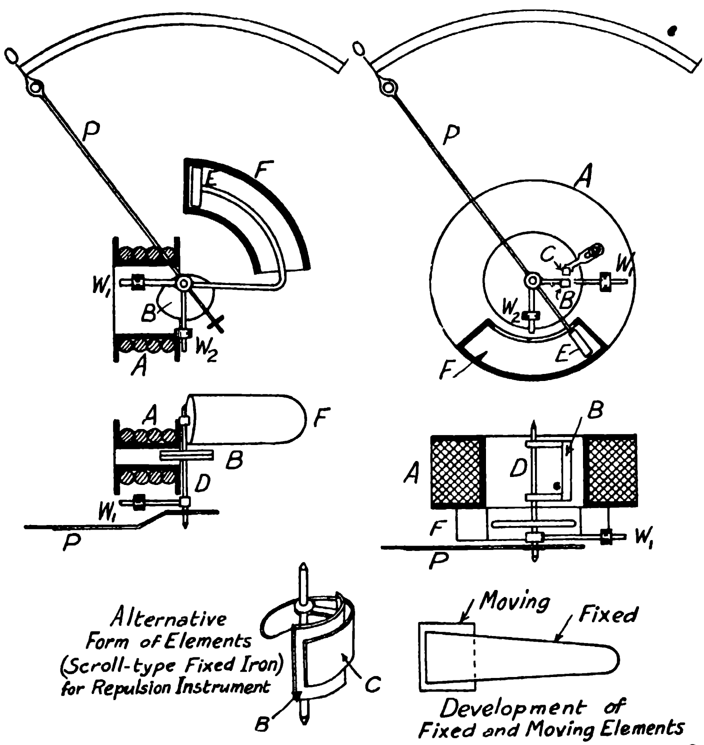


FIG. 202 FIG. 203

Attraction and Repulsion Forms of Moving Iron Ammeters and Voltmeters

In the instrument operating on the attraction principle (Fig. 202) the coil is of flat shape and is fixed with its magnetic axis parallel to the base of the instrument.

The moving-iron element consists of a few discs, B, of soft iron, which are fixed to a spindle, D, pivoted in jewelled bearings. The spindle also carries a pointer, P, a balance weight, W_1, a controlling weight, W_2, and a damping piston, E, which moves in a curved

fixed cylinder, *F*. The special shape of the moving-iron discs is for the purpose of obtaining a scale of suitable form.

In the repulsion form of instrument (Fig. 203) the coil, *A*, is cylindrical and is fixed with its magnetic axis perpendicular to the base of the instrument. The moving-iron element, *B*, is a small bar or rod of soft iron fixed parallel to, and at a small distance from, the spindle, which also carries the pointer, *P*, and damping vane, *E*, and is pivoted in jewelled bearings. The fixed iron element, *C*, is supported by a non-magnetic framework which carries the damping chamber and bearings and forms a clamp for the coil.

Considerable variations in the shape and arrangement of the moving and fixed iron elements are possible and are to be found in commercial instruments. For example, the fixed iron element may be a tongue-shaped piece of thin soft sheet iron bent into cylindrical form and mounted concentric with the spindle. The moving iron may consist of a small piece of similar sheet iron bent to form a cylindric segment and fixed to the spindle so as to move concentrically with respect to the fixed iron tongue. The fixed and moving irons are so arranged that when the pointer is on zero the moving iron is parallel to the broadest part of the surrounding iron tongue, and that as the movement is deflected it moves towards the narrowest part of the tongue. The form of scale depends upon the shape of the fixed iron tongue, and by varying the latter a variety of scale forms are possible.

The instruments may be effectively *shielded* from the influence of external magnetic fields by enclosing the working parts, except the pointer, in a laminated iron cylinder with laminated iron end covers. More generally, however, the complete instrument is enclosed in a cast-iron case which usually gives sufficient shielding for practical purposes.

An interesting form of moving-iron ammeter (recently developed by Crompton & Co. for switch panels of the cubicle type) is shown in Fig. 204, a unique feature being that the operating forces are produced by the current in a conductor external to the instrument. The instrument, therefore, has no internal electrical parts or connections, and no terminals. It is designed for mounting on the sheet-steel panel of the cubicle, with the dial in a horizontal position and projecting from the front of the panel. The body of the instrument projects into the interior of the cubicle, and one of the connecting cables in the cubicle is passed through the space *C*, Fig. 204.

The essential parts of the instrument comprise: (1) a laminated-iron magnetic circuit *A*, *B*, which is provided with an air gap, *D*,

shaped to accommodate the moving-iron elements, E_1, E_2; (2) a pivoted moving system consisting of two specially shaped elements, E_1, E_2, of soft iron, a pointer P, control spring, and damping vane.

The magnetization of the magnetic circuit is produced by current in a single conductor located in the space C, and the rear portion, B, of the magnetic circuit is removable in order that this conductor may be conveniently placed in position. The magnetic circuit is so proportioned that a minimum range of 50 amp. can

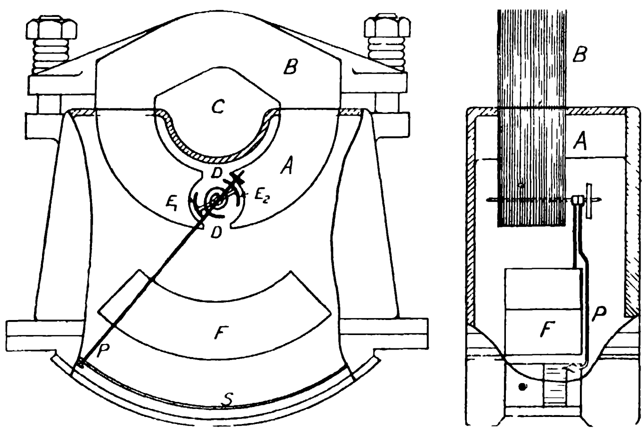


FIG. 204 —Crompton's Form of Moving-iron Ammeter, in which Operating Torque is Produced by Current in an External Conductor (Horizontal Edgewise Pattern)

be obtained with a single conductor in the space C. Higher ranges are obtained by changing the control spring.

The action of the instrument is very simple. Thus when the magnetic circuit, A, B, is excited, the iron elements, E_1, E_2, tend to move to positions bridging the air gaps, D, and so diminishing the reluctance of the magnetic circuit. The form of scale depends upon the shape of these elements relative to the air gaps D.

Ranges of ammeters and voltmeters. For a given moving-iron instrument the ampere-turns necessary to produce full-scale deflection are constant. Hence the ranges of **ammeters** are altered by changing the number of turns and size of conductor in the magnetizing coil. Obviously the maximum range is reached when the coil is wound with one turn. This range is of the order of 300 A.,

but it varies with different instruments. In the construction of the coils for ranges of 100 A. and above, the conductors must be laminated in order to avoid errors due to eddy currents.

With **voltmeters** the range may be altered by changing the number of turns, but with a given instrument the range may be increased by connecting a resistance in series with it. Hence the same coil-winding specification may be employed for a number of ranges.

Conditions determining value of series resistance. Considerations of space and temperature rise necessitate the coil being wound with copper wire, and the size of the conductor must be so chosen that the pressure drop across the coil at full-scale deflection is only a small fraction of the potential difference at the terminals of the instrument. A non-inductive resistance, the value of which is several times that of the coil, is connected in series. This resistance is wound of a material, such as constantan or manganin, having a negligible temperature coefficient of resistivity, in order that the instrument readings shall be practically unaffected by normal variations of temperature. [On account of this feature the non-inductive series resistance is often called a "swamping" resistance.] Moreover, under these conditions the impedance of the instrument will change very little with normal variations in frequency, wave-form, and position of the moving system.

The change of impedance, at constant frequency, with change of position of the moving system is due to the change of inductance of the magnetizing coil in consequence of the change in position of the moving-iron element with respect to the coil. The inductance increases as the deflection increases, and the change of inductance corresponding to a full-scale deflection may be of the order of 5 per cent (or more, or less, according to design) of the inductance when the moving system is in its zero position. As is shown later (p. 372), the rate of change of inductance with position of the moving system is related to the torque.

The change of impedance with change of frequency may be approximately compensated by connecting a condenser of suitable capacity in parallel with the non-inductive "swamping" resistance, as shown on p. 373.

Data of moving-iron instruments. (1) A commercial moving-iron **voltmeter** of the type shown in Fig. 202 has a range of 120 volts. The operating coil is wound with 3000 turns of No. 35 S.W.G. copper wire (diameter = 0·0084 in.), the resistance being 140 ohms at 20° C., and the series resistance—of Eureka (constantan) wire—has a resistance of 1060 ohms, giving a total resistance for the instrument of 1200 ohms at 20° C.

The inductance of the operating coil is 0·141 H. with the moving system in its zero position, and 0·1533 H. when the moving system is deflected to the full scale position. The inductances corresponding to intermediate

positions of the moving system are given in the following table, together with data from which the form of scale can be determined.

Scale reading (volts)	0	40	60	80	100	120
Deflection (degrees)	0	11·7	29	47·1	61	75
Inductance (henries)	0·141	0·142	0·1455	0·1495	0·1517	0·1533

The following calculated data refer to the operation of the instrument when used on a 50-cycle, 100-volt, circuit of sinusoidal wave-form.

Power loss in instrument $= 100^2/1200 = 8{\cdot}33$ W.

Reactance of instrument $= 314 \times 0{\cdot}1517 = 47{\cdot}6$ O.

Ratio: reactance/resistance $= 47{\cdot}6/1200 = 0{\cdot}0397$

Impedance of instrument at 20° C. $= \sqrt{(1200^2 + 47{\cdot}6^2)} = 1200{\cdot}9$ O

Power factor $= 1200/1200{\cdot}9 = 0{\cdot}99885$

Operating current at 20° C. $= 100/1200{\cdot}9 = 0{\cdot}083237$ A.

$$\text{Ratio}: \frac{\text{operating current at 100 V., 50 frequency}}{\text{operating current at 100 V., zero frequency}} = 0{\cdot}99885$$

Power expended in operating coil at 20° C. $= 140 \times 0{\cdot}083^2 = 0{\cdot}97$ W.

Resistance of instrument at 50° C $= 1060 + 140[1 + (50 - 20) \times 0{\cdot}0039]$
$= 1216{\cdot}5$ O.

Thus the frequency and temperature errors are negligible for practical purposes

(2) A commercial moving-iron *ammeter*, for a range of 10 amp. and of the type shown in Fig. 202, has the operating coil wound with 29 turns of No. 14 S.W.G. copper wire (diameter = 0·08 in.), the resistance of which at a temperature of 20° C is 0·015 O.

The inductances of the instrument with the moving system in a number of positions are given in the following table, together with the corresponding scale markings and angular deflections–

Scale reading (amp.)	0	2	4	6	8	10
Deflection (degrees)	0	6	16	36	56	73
Inductance (μH.)	16·4	–	16·6	17·1	18·1	19·7

The following calculated data refer to the operation of the instrument, at full-scale reading (10 A.), on an alternating-current circuit of 50 frequency and sinusoidal wave-form—

Power loss	= 1·5 W.	Power factor	= 0·93
Reactance	= 0·006 O.	Pressure drop	= 0·15 V.
Impedance	= 0·0161 O.		

Theory of moving-iron ammeters and voltmeters. With all forms of these instruments the instantaneous force acting upon the moving system is proportional to the product of two magnetic field strengths (viz. the field strengths of the fixed and moving elements), both of which are due to the same current. Hence, if the iron elements are free from magnetic saturation the instantaneous deflecting force will be proportional to the square of the instantaneous value of the current in the magnetizing coil, and, therefore, the mean deflecting force will be proportional to the mean squared value of the current taken over a period, i.e. $F = \frac{1}{T}\int_0^T ki^2dt$, where F denotes the mean force acting upon the moving system, T, the period, and k, a constant, connecting magnetic force and current in magnetizing coil.

Again, if the iron elements were entirely free from hysteresis and eddy currents, the mean deflecting force, corresponding to a given current in the magnetizing coil, would have the same value whether the current were steady

or alternating (provided that the mean square value of the current was the same in the two cases).

When, however, the effects of hysteresis and eddy currents are considered, the mean deflecting force, corresponding to a given R.M.S. value of current, will vary with both the frequency and wave-form of the current.

Consider first the *effects of hysteresis*, and, for simplicity, an instrument in which the moving-iron element is attracted by a fixed coil carrying a current. Then, if B denotes the flux density in the iron element due to a current i in the coil, the instantaneous force will be proportional to iB, and the mean force taken over a period of the current will be given by $F = \frac{1}{T}\int_0^T kiB\,dt$.

Now if the iron possesses hysteresis, the wave-form of the flux density will have a time phase displacement with respect to the current wave-form (see Fig. 188, p. 335), and therefore the force will become negative (i.e. the moving iron will be repelled) during a portion of each half-period. Hence, the mean deflecting force taken over a period will be smaller than that for the corresponding case when the iron is free from hysteresis.

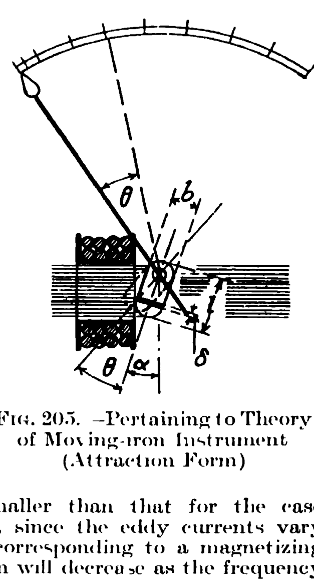


FIG. 205.—Pertaining to Theory of Moving-iron Instrument (Attraction Form)

Again, with currents of different wave-forms, but of the same R.M.S. value, the corresponding hysteresis loops for the moving iron element will vary in size, and the wave-forms of the flux density will be dissimilar. Hence, the value of the mean deflecting force taken over a period will vary with the current wave-form, and will have a lower value for peaked wave-forms than for flat-topped wave-forms.

The *effects of eddy currents* in the moving-iron element cause a further phase displacement between the flux density and the magnetizing current, and also a diminution of flux density. Hence, for given conditions, the mean force acting upon the element will be smaller than that for the case when eddy currents are absent. Moreover, since the eddy currents vary directly as the frequency, the mean force corresponding to a magnetizing current of given R.M.S. value and wave-form will decrease as the frequency increases.

Similar effects will be produced by eddy currents in the magnetizing coil and other solid metal adjacent thereto.

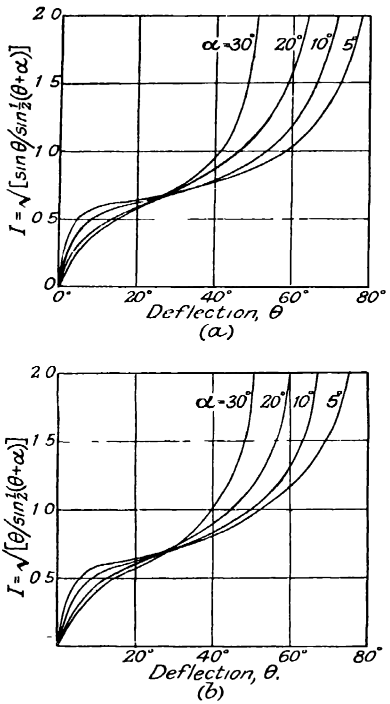
But, with careful design, the use of high-resistance material for the coil supports and adjacent parts, and the use of alloyed iron having a low hysteresis and eddy-current loss, the changes in the mean deflecting force with change in frequency and wave-form may, especially with ammeters, be made very small, so that over the commercial range of frequency in electric power plants the errors due to these causes are almost negligible.

Form of scale. The form of scale of a moving-iron instrument depends upon the law of variation of the deflecting force with variation of current and position of the moving system, as well as, to some extent, upon the system of control.

The law of variation of deflecting force with position of moving system can be investigated theoretically in cases where the iron elements are of simple shape and are free from magnetic saturation, hysteresis, or eddy currents, but even in these cases the investigations are only approximately

correct on account of the assumptions which must be made to obtain a simple treatment.

Thus in the **attraction form of instrument** shown in Fig. 202, the essential elements of which are represented in Fig. 205, the moving element may be regarded as a soft-iron magnetic needle, pivoted eccentrically in the magnetic field due to the current in the magnetizing coil. If this field is assumed to be uniform and of strength H corresponding to a current I, then the magnetization of the needle, when deflected $\theta°$ from its zero position, will be



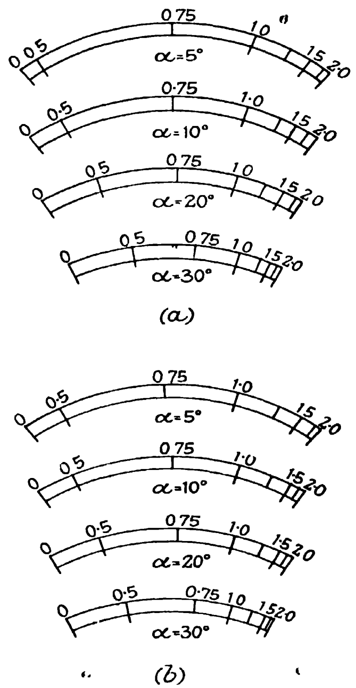


Fig. 206 Fig. 207

Forms of Scale for Moving-iron Instrument (Attraction Form)
a, Gravity Control ; *b*, Spring Control

proportional to $H \sin(\theta + \alpha)$, where α is the complement of the angle between the axis of the magnetic field and the axis of the needle in the zero position of the moving system. Hence the deflecting torque will be proportional to $H^2 \sin(\theta + \alpha) \cos(\theta + \alpha)$, or to $H^2 \sin 2(\theta + \alpha)$, and may be expressed by $\mathfrak{T}_d = k_1 I^2 \sin 2(\theta + \alpha)$. If the control is due to gravity, the controlling torque is given by $\mathfrak{T}_c = k_2 \sin \theta$. Therefore, for equilibrium, we must have $\mathfrak{T}_d = \mathfrak{T}_c$, or

$$k_1 I^2 \sin 2(\theta + \alpha) = k_2 \sin \theta$$

Whence

$$I = k\sqrt{[\sin\theta/\sin 2(\theta + \alpha)]} \quad . \quad . \quad (178)$$

where k, k_1, k_2, are constants.

If the control is due to a flat spiral spring, $\mathfrak{T}_c = k_3\theta$, and, for equilibrium, we must now have

$$k_1 I^2 \sin 2(\theta + \alpha) = k_3\theta.$$

Whence

$$I = k'\sqrt{[\theta/\sin 2(\theta + \alpha)]} \quad . \quad . \quad . \quad (178a)$$

Curves connecting I and θ for selected values of α, and for the particular cases when k and k' are both equal to unity, are given in Fig. 206, and a comparison of the scales is given in Fig. 207.

With the *repulsion instrument*, Fig. 208, we will assume that the two similarly magnetized elements are identical and that their distance apart is always small in comparison with their length, so that the magnetic force between them may be considered as being due to adjacent poles only. Then if H_1, H_2, are the pole strengths corresponding to a current, I, in the magnetizing coil the force between the elements is $F = H_1H_2/d^2$, where d is the distance apart of adjacent poles. If r is the distance of each of these poles from the axis, α the angular displacement for the zero position of the moving system (see Fig. 208), and θ the angular deflection corresponding to the current I, then $d = 2r\sin\frac{1}{2}(\theta + \alpha)$, and

$$\begin{aligned} F &= H_1H_2/[2r\sin\tfrac{1}{2}(\theta + \alpha)]^2 \\ &= k_1 I^2/\sin^2\tfrac{1}{2}(\theta + \alpha), \end{aligned}$$

since both H_1, H_2 are proportional to the current.

Hence, the deflecting torque is given by

$$\begin{aligned} \mathfrak{T}_d &= Fr\cos\tfrac{1}{2}(\theta + \alpha) \\ &= k_1' I^2 \cos\tfrac{1}{2}(\theta + \alpha)/\sin^2\tfrac{1}{2}(\theta + \alpha). \end{aligned}$$

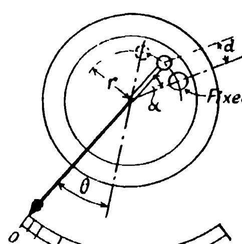


FIG. 208.- Pertaining to Theory of Moving-iron Instrument (Repulsion Form)

The controlling torque is given by the same expressions as in the preceding case.

Therefore, for equilibrium, we have, with gravity control,

$$k_1' I^2 \frac{\cos\frac{1}{2}(\theta + \alpha)}{\sin^2\frac{1}{2}(\theta + \alpha)} = k_2 \sin\theta$$

Whence $$I = k\sin\tfrac{1}{2}(\theta + \alpha)\sqrt{\frac{\sin\theta}{\cos\frac{1}{2}(\theta + \alpha)}} \quad (179)$$

With spring control, we have

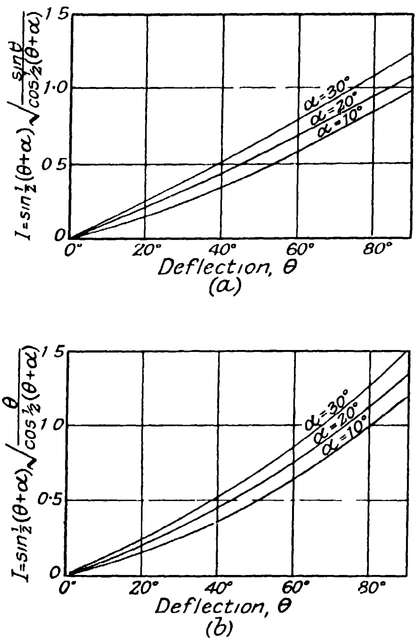
$$k_1' I^2 \frac{\cos\frac{1}{2}(\theta + \alpha)}{\sin^2\frac{1}{2}(\theta + \alpha)} = k_3\theta$$

Whence $$I = k'\sin\tfrac{1}{2}(\theta + \alpha)\sqrt{\frac{\theta}{\cos\frac{1}{2}(\theta + \alpha)}} \quad (179a)$$

Curves connecting I and θ for selected values of α and for the particular cases when k and k' are both unity are given in Fig. 209, and a comparison of the scales is given in Fig. 210, which should be compared with Fig. 207. It should be observed, however, that the actual scales of repulsion instruments are not so open in the lower portion as those shown in Fig. 210—owing chiefly to friction and the demagnetizing effect of the poles upon each other. Moreover, as explained on p. 365, the scale shape may be varied by specially shaping the elements or by fixing the spindle eccentric with the magnetic axis of the fixed coil.

Relation between torque and $dL/d(\alpha + \theta)$. The change of inductance of the magnetizing coil due to the deflection of the moving system will be investigated for the simplified form of instrument (Fig. 205) operating on the attraction principle.

The inductance of the magnetizing coil is proportional to the flux linked with the coil per ampere of magnetizing current, i.e. the inductance is proportional to the permeance of the magnetic circuit. The magnetic circuit consists partly of air and partly of iron, the magnetic length and cross-section of the



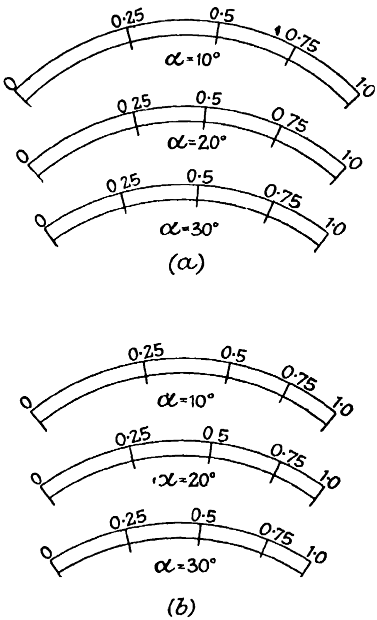


FIG. 209 FIG. 210

Forms of Scale for Moving-iron Instrument (Repulsion Form)
a, Gravity Control ; *b*, Spring Control

latter being variable ; but the reluctance of the iron portion is always small in comparison with that of the air portion of the circuit. Hence, the circuit may be considered as the equivalent of two magnetic paths in parallel, the reluctance of one path being constant and that of the other (iron) path being variable. The permeance of the circuit is then given by $\mathcal{P} = \mathcal{P}_a + \mathcal{P}_i$, where $\mathcal{P}_a$, $\mathcal{P}_i$, denote the permeances of the air and iron paths respectively.

If the paths of the magnetic lines through the moving-iron element are assumed to be straight, as represented in Fig. 205, the permeance of this portion of the magnetic circuit corresponding to a deflection θ from the zero position of the moving system is given by

$$\mathcal{P}_i = \frac{\mu\delta l \cos(\alpha + \theta)}{b/\cos(\alpha + \theta)} = \frac{\mu\delta l}{b}\cos^2(\alpha + \theta)$$

$$= \frac{\mu\delta l}{2b}[1 + \cos 2\,(\theta + \alpha)] = k_3[1 + \cos 2(\theta + \alpha)]$$

where l, b are the dimensions marked in Fig. 205, δ is the thickness of the moving iron and μ is the permeability.

Hence, for a deflection θ, the inductance of the magnetizing coil is given by

$$L = k_1\mathfrak{S} = k_1(\mathfrak{S}_a + \mathfrak{S}_t) = k_4\mathfrak{S}_a + k_3k_4[1 + \cos 2(\theta + \alpha)]$$

Whence

$$dL/d(\theta + \alpha) = -2k_3k_4 \sin 2(\theta + \alpha)$$
$$= -k_5 \sin 2(\theta + \alpha)$$

Now the expression for the torque (p. 370) is $\mathfrak{T}_d = k_1 I^2 \sin 2(\theta + \alpha)$, which, when substitution is made for $\sin 2(\theta + \alpha)$ from the preceding equation (the minus sign being ignored), reduces to

$$\mathfrak{T}_d = k_1' I^2 \frac{dL}{d(\theta + \alpha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

where the constant k_1' includes the constants k_1 and k_5.

Therefore the torque is proportional to the product of the current and the rate of change of inductance of the magnetizing coil with respect to the deflection corresponding to this current.

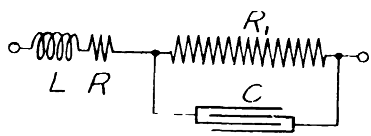


FIG. 211.—Connections for Method of Compensating Frequency Error in Moving-iron Voltmeter

Compensation for frequency error in voltmeters. The effect of changes in frequency upon the readings of moving-iron voltmeters as ordinarily arranged with a non-inductive resistance in series with the fixed coil—results in an error which increases as the frequency increases, the error being due to the change of impedance of the instrument with change of frequency.

The error may be compensated by connecting a condenser in parallel with the series resistance of the instrument, as shown in Fig. 211, the capacity being of such value that the numerical value of the impedance of the circuit is constant for all frequencies. Thus, if L, R denote the inductance (which should correspond to the normal, or other, scale reading at which compensation is desired) and resistance, respectively, of the fixed coil; R_1, the non-inductive series resistance, and C the capacity to be connected in parallel with the latter, the impedance of the circuit is given symbolically by

$$Z = R + j\omega L + \frac{1}{1/R_1 + j\omega C}$$
$$= R + \frac{R_1}{1 + \omega^2C^2R_1^2} + j\omega\left(L - \frac{CR_1^2}{1 + \omega^2C^2R_1^2}\right)$$

If the instrument is to read accurately at all frequencies, we must have

$$Z = R + R_1$$

i.e. $$R + R_1 = R + \frac{R_1}{1 + \omega^2C^2R_1^2} + j\omega\left(L - \frac{CR_1^2}{1 + \omega^2C^2R_1^2}\right)$$

The conditions to be fulfilled are, therefore,

(i) $$\frac{R_1}{1 + \omega^2C^2R_1^2} = R_1,$$

or $$\omega^2C^2R_1^2 = 0,$$

(ii) $$\omega\left(L - \frac{CR_1^2}{1 + \omega^2 C^2 R_1^2}\right) = 0$$

or $$L = CR_1^2/(1 + \omega^2 C^2 R_1^2)$$
$$= CR_1^2,$$

when condition (i) is imposed.

Obviously, condition (i) cannot be satisfied rigorously, but it can be satisfied approximately if the term $\omega^2 C^2 R_1^2$ is very small in comparison with unity, ie. if ωL is very small in comparison with R_1. Under these conditions the capacity required to compensate the frequency error is

$$C = L/R_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

In commercial moving-iron instruments, however, the reactance (ωL) at 50 frequency is of the order of from 5 to 10 per cent of the series resistance (R_1), and therefore the relationship given by equation (181) does not hold. But if an instrument is required to give accurate indications on an alternating-current circuit of a particular frequency as well as on a direct-current circuit, the condition to be satisfied is that the numerical value of the impedance of the instrument at this frequency shall be equal to the resistance, i.e.

$$R + R_1 = \sqrt{\left[\left(R + \frac{R_1}{1 + \omega^2 C^2 R_1^2}\right)^2 + \omega^2\left(L - \frac{CR_1^2}{1 + \omega^2 C^2 R_1^2}\right)^2\right]}$$

Difficulties arise in the solution of this equation for C, as the fourth power of this quantity is involved.

If, however, the resistance (R) of the operating coil is small in comparison with that of the series resistance (R_1), a close approximation to complete compensation is obtained when

$$R_1 = \sqrt{\left[\left(\frac{R_1}{1 + \omega^2 C^2 R_1^2}\right)^2 + \omega^2\left(L - \frac{CR_1^2}{1 + \omega^2 C^2 R_1^2}\right)^2\right]}$$

the solution of which gives

$$C = \frac{L(\sqrt{[2 - (\omega L/R_1)^2]} - 1)}{R_1^2 - \omega^2 L^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (181a)$$

Example. A moving-iron voltmeter with a maximum scale reading of 100 volts has a resistance of 2000 ohms and an inductance of 0·45 henry. The working coil is wound with 200 ohms of copper and the remainder of the circuit is a non-inductive resistance in series with it. With what capacity must this non-inductive resistance be shunted in order that the instrument shall read correctly on continuous-current circuits and on alternating-current circuits of frequency 50? (*L.U.* 1924.)

To fulfil the specified conditions the impedance of the instrument at 50 frequency must equal 2000 ohms. Since

$$\omega L/R_1 = 314 \times 0{\cdot}45/1800 = 0{\cdot}0786$$

the capacity of the condenser must be calculated from equation (181*a*). Substituting appropriate values for ω, L, R_1, we obtain

$$C = 10^6 \times \frac{0{\cdot}45(\sqrt{[2 - (314 \times 0{\cdot}45/1800)^2]} - 1)}{1800^2 - (314 \times 0{\cdot}45)^2} = 0{\cdot}0576\,\mu\text{F}.$$

As an **extension** of the problem we will calculate the error, at maximum scale reading, when the compensated instrument is used on an alternating-current circuit of 50 frequency, assuming that the instrument reads correctly on a direct-current circuit.

The true impedance of the compensated instrument is

$$Z = R + \frac{R_1}{1 + \omega^2C^2R_1^2} + j\omega\left(L - \frac{CR_1^2}{1 + \omega^2C^2R_1^2}\right)$$

Substituting appropriate values for ω, C, R_1, we have

$$1 + \omega^2C^2R_1^2 = 1 + (314 \times 0{\cdot}0576 \times 10^{-6} \times 1800)^2 = 1{\cdot}00106;$$

$$R_1/(1 + \omega^2C^2R_1^2) = 1800/1{\cdot}00106 = 1798{\cdot}1;$$

$$CR_1^2/(1 + \omega^2C^2R_1^2) = 0{\cdot}0576 \times 10^{-6} \times 1800^2/1{\cdot}00106 = 0{\cdot}1864;$$

$$\omega[L - CR_1^2/(1 + \omega^2C^2R_1^2)] = 82{\cdot}8.$$

Whence $\quad Z = 1998{\cdot}1 + j82{\cdot}8$

and $\quad Z = 2000$ ohms.

Hence the instrument is correctly compensated for 50 frequency with a condenser of the above value connected in parallel with the series resistance.

As a **further extension** of the problem it will be of interest to work out the error of the uncompensated instrument on a circuit of 50 frequency, assuming the instrument to read correctly on a direct current circuit.

The impedance of the instrument is now

$$Z = R + R_1 + j\omega L$$
$$= 2000 + j141{\cdot}5$$

Whence $\quad Z = 2005$ ohms.

If the scale divisions at the part of the scale with which we are concerned are proportional to the current in the instrument, then the reading at 50 frequency will be $100 \times 2000/2005 = 99{\cdot}75$. Hence the error is $\frac{1}{4}$ of 1 per cent (low).

Electro-dynamic ammeters and voltmeters. All electro-dynamic instruments depend for their action upon the dynamic force between adjacent conductors, or coils, carrying electric currents. The application of this principle to measuring instruments is due to Kelvin and Siemens, and their instruments—the current-balance and the electro-dynamometer—are of the non-deflectional, or zero type.

Construction. In deflectional electro-dynamic ammeters and voltmeters the moving coil is wound with fine wire and is mounted on a spindle which carries the pointer, control springs, and damping vane or piston. This coil is usually pivoted within a pair of coaxial fixed coils (Fig. 212*a*), which, in an ammeter, are wound with thick wire, and are connected in parallel with the moving coil. In a voltmeter the fixed coils are wound with thin wire and are connected in series with the moving coil and a non-inductive resistance.

With an alternative form of construction a single fixed coil is employed, which is arranged inside the moving coil (Fig. 212*b*).

In general, no iron is employed in the magnetic circuits of the coils, but instruments have been constructed in which a laminated-iron magnetic circuit for the operating coils forms an essential part of the instrument.

The controlling force is supplied by a pair of flat spiral springs, which also act as the leading-in connections for the moving coil.

The damping is generally pneumatic, as in moving-iron instruments, but eddy-current damping is employed by some manufacturers.

Shielding. Electro-dynamic instruments in which iron does not form an essential part of the operating mechanism may be effectively shielded from the effects of external magnetic fields by enclosing

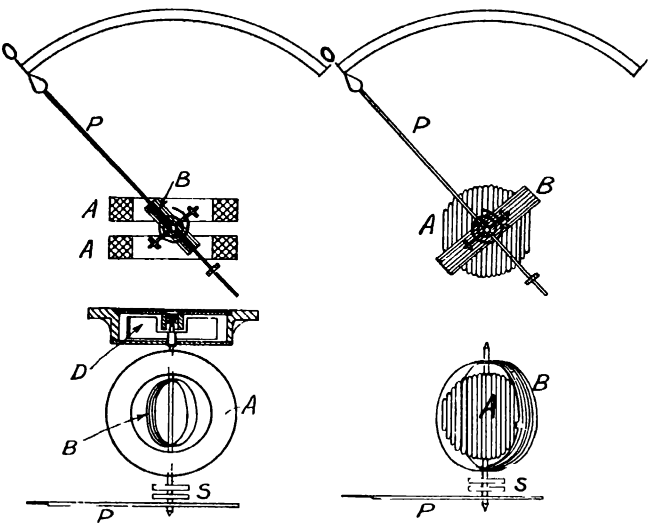


FIG. 212.—Forms of Electro-dynamic Instruments
(*a*) Internal Moving Coil ; (*b*) External Moving Coil

[*A*, fixed coils ; *B*, moving coil ; *D*, damping chamber and vanes , *P*, pointer , *S*, control springs]

NOTE.—Illustrations of the mechanism of actual instruments are given in Figs 225, 232

the mechanism (except the tip of the pointer) in a laminated iron hollow cylinder with closed ends. When using such instruments on direct-current circuits, the current through the instrument should be reversed and the mean of the two readings taken. In this manner the effect of the magnetic condition of the shield on the instrument readings is eliminated.

Theory of electro-dynamic ammeters and voltmeters. With instruments in which the moving coil is circular and is pivoted centrally in a short fixed solenoid or coil, as in Fig. 212*a*, the torque acting upon the moving coil can be easily calculated if the flux density is assumed to be constant throughout the space occupied by the moving coil. Denoting this flux density by B, the current in the moving coil by I_2, the force acting upon an element, of

length $rd\alpha$, subtending an angle $d\alpha$ with respect to the centre of the coil, Fig 213, is given by $dF = k_1 B I_2\, r d\alpha \sin \alpha$, where k_1 is a constant involving the system of units, and α denotes the angular position of the element with respect to the pivotal axis of the coil. Hence, the torque due to the element is given by

$$d\mathfrak{T}_d = dF\, r \sin \alpha \sin(\beta + \theta) = k_1 B I_2\, r^2\, d\alpha \sin^2 \alpha \sin(\beta + \theta),$$

where θ is the angular deflection of the coil from its zero position, and β is the angle between the plane containing the zero position of the moving coil and the plane which contains the pivotal axis and is perpendicular to the flux. Therefore the torque for the whole coil of n_2 turns is

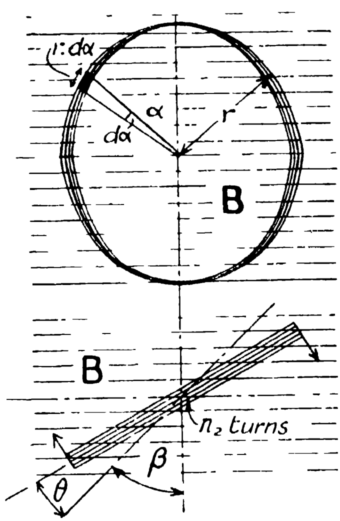


FIG. 213.—Pertaining to Theory of Electro-dynamic Instrument

$$\mathfrak{T}_d = 4 n_2 k_1\, B I_2\, r^2 \sin(\beta + \theta) \int_0^{\frac{1}{2}\pi} \sin^2 \alpha \,.\, d\alpha = k_1 n_2\, \pi r^2\, B I_2 \sin(\beta + \theta)$$

since $\int_0^{\frac{1}{2}\pi} \sin^2 \alpha d\alpha = \pi/4.$

Now $\pi r^2 B$ is equal to the flux (Φ) linked with the moving coil when its magnetic axis coincides with that of the fixed coil (i.e. when $(\beta + \theta) = 0$ and $(\beta + \theta) = \pi$) and if M_M denotes the mutual inductance under these conditions, $\Phi = M_M I_1 \times 10^8/n_2$.

Therefore the torque is given by

$$\mathfrak{T}_d = k_2 M_M I_1 I_2 \sin(\beta + \theta) \quad . \quad (182)$$

where $k_2 = k_1 \times 10^8$.

This equation may be expressed in another form. Thus, on the assumption of uniform flux density throughout the space occupied by the moving coil, the mutual inductance of the coils corresponding to a deflection θ is

$$M_{(\beta + \theta)} = - M_M \cos(\beta + \theta),$$

since, when $\beta + \theta = 0$, the mutual inductance $= - M_M$. Whence the rate of change of mutual inductance with respect to the deflection is

$$\frac{dM_{(\beta + \theta)}}{d(\beta + \theta)} = - M_M \frac{d \cos(\beta + \theta)}{d(\beta + \theta)} = M_M \sin(\beta + \theta)$$

Therefore equation (182) may be written in the form

$$\mathfrak{T}_d = k_2 I_1 I_2 \frac{dM_{(\beta + \theta)}}{d(\beta + \theta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182a)$$

which shows that the torque is proportional to the product of the rate of change of mutual inductance with deflection and the currents in the fixed and moving coils.

If the coils carry alternating currents, and eddy-current effects are negligible (i.e. the flux is proportional to and in phase with the current in the fixed coil), the instantaneous torque corresponding to the instantaneous values, i_1, i_2, of the currents in the fixed and moving coils will be given by

$$\mathfrak{T}_{d(inst)} = k_2 M_M\, i_1\, i_2 \sin(\beta + \theta).$$

Whence the mean value of the torque during a period is

$$\frac{1}{T}\int_0^T k_2 M_M \sin(\beta + \theta)\, i_1 i_2\, dt.$$

Therefore, for the assumed conditions and the case when $i_1 = i_2 = i$, the mean torque acting upon the moving coil is proportional to the mean squared value of the current passing through the instrument, and is independent of frequency and wave-form.

Form of scale. If the control is due to flat spiral springs, the controlling torque is proportional to the angle of deflection, θ, and for equilibrium we have

$$\mathfrak{T}_d = \mathfrak{T}_c$$

or $k_2 M_M I_1 I_2 \sin(\beta + \theta) = k_3 \theta$.

Whence $I_1 I_2 = k \dfrac{\theta}{M_M \sin(\beta + \theta)} = k \dfrac{\theta}{dM_{(\beta+\theta)}/d(\beta + \theta)}$

or, if the moving coil is connected in series with the fixed coil, i.e. $I_1 = I_2 = I$, we have

$$I = k' \sqrt{\frac{\theta}{\sin(\beta + \theta)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

$$= k' \sqrt{\frac{\theta}{dM_{(\beta+\theta)}/d(\beta + \theta)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183a)$$

Curves connecting I and θ for selected values of β, and for the case when

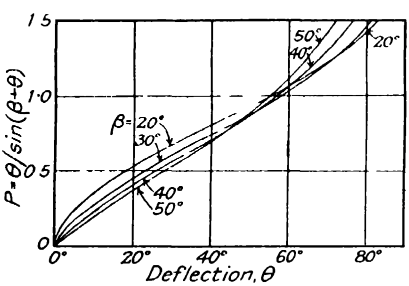


FIG. 214

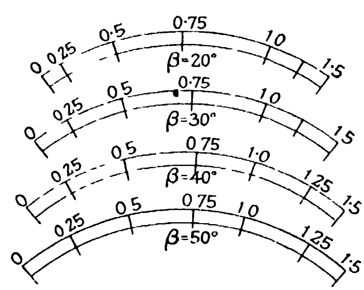


FIG. 215

Form of Scale for Electro-dynamic Ammeter

$k' = 1{\cdot}0$, are given in Fig. 214. The scales corresponding to these values of β are given in Fig. 215.

Conditions for instrument readings to be unaffected by frequency and wave-form. The principal conditions are : (1) the currents in the fixed and moving coils must either be equal or have a constant common ratio ; (2) eddy currents in the coil supports and conductors must be reduced to a minimum ; (3) the reactance of the instrument, when used as a voltmeter, must be very small relatively to its resistance, and the latter must be constant at all temperatures.

With *voltmeters* these conditions are satisfied by (i) connecting the fixed and moving coils in series ; (ii) designing these coils for a pressure drop which is only a small fraction of the range of the instrument ; and (iii) connecting in series with them a non-inductive resistance having a zero temperature-resistivity coefficient.

With *ammeters* for ranges above about 250 mA. the moving coil cannot be connected in series with the fixed coil (on account of the control springs being unsuitable for currents above about 250 mA.). Therefore the moving coil must be connected either in parallel with the fixed coils (Fig. 216*a*), or across a shunt which is connected in series with the fixed coils (Fig. 216*b*). In both cases the ratio of the currents in the fixed and moving coils must be unaffected by variations of either frequency or temperature.

In the instrument with a shunted moving coil (Fig. 216*b*), the shunt is designed for a relatively large pressure drop (about 0·5 V. in a 5 A. instrument) and only a fraction of this pressure is utilized for operating the moving coil, a non-inductive resistance being connected in series with the latter. Both shunt and non-inductive series resistance are constructed of materials having zero temperature-resistivity coefficients. For extreme accuracy the

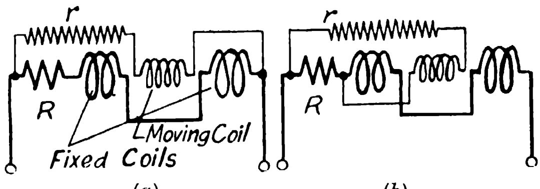


FIG. 216 —Alternative Methods of Connecting Fixed and Moving Coils in Electrodynamic Ammeter

shunt should be adjusted to have the same ratio of resistance to inductance as the moving-coil circuit.

With the instrument in which the moving coil is connected in parallel with the fixed coils (Fig. 216*a*) the conditions which must be fulfilled are: (1) the ratio (resistance/reactance) must have the same value for each branch; (2) the percentage change of resistance with temperature must be the same for the two branches.

To satisfy the first condition we must have

$$\omega L_1/R_1 = \omega L_2/R_2, \text{ or } L_1/L_2 = R_1/R_2,$$

where L_1, L_2, are the effective inductances of the fixed- and moving-coil circuits, respectively, and R_1, R_2, the resistances of these circuits.

Now

$$L_1 = L_f \pm M, \quad L_2 = L_m \pm M,$$

where L_f, L_m, denote the true self-inductances of the fixed- and moving-coil circuits, and M denotes their mutual inductance.

Hence for the ratio L_1/L_2 to be constant when $L_f \neq L_m$, M must be either zero or constant. Alternatively, if M is variable, L_f and L_m must be equal.

With all deflectional instruments, however, the mutual inductance varies with the relative positions of the moving and fixed coils. When the axes of these coils are perpendicular to each other the mutual inductance is zero; for other positions, to the right or left of this position, the mutual inductance increases as the angular displacement between the coils increases, and may have either positive or negative values. Therefore, the first condition above is satisfied rigorously only in the special case when the self-inductance of the moving-coil circuit is equal to that of the fixed-coil circuit.

In order that the ratio of currents in the fixed and moving coils shall be unaffected by temperature variations, the percentage change of resistance with temperature must be the same for both circuits. This result is best obtained by connecting in series with the coils (which are wound with copper conductors), resistances having zero resistivity-temperature coefficients, the value of each series resistance being several times that of the coil to which it is connected.

Thus, both forms of electro-dynamic ammeters must have relatively large losses (5 to 7 W. at full scale) if the readings are to be unaffected by variations of frequency and temperature.

Ranges of ammeters and voltmeters. A given size of instrument requires a definite number of ampere-turns to be supplied by the fixed and moving coils to obtain a full-scale deflection. Hence, with *milli-ammeters*, in which the fixed and moving coils are connected in series, the ranges are altered by changing the number of turns and size of conductor in the fixed and moving coils.

With *ammeters* in which only a fraction of the rated current is carried by the moving coil, the range is altered by changing the fixed coils, and in instruments in which two fixed coils are employed (Fig. 212*a*), a double-range instrument may be obtained by connecting these coils either in series or in parallel, the internal connections (which are so arranged that the changes may be effected by either plugs or links) being shown in Fig. 217. The maximum range for which ammeters are usually constructed is 200 A.

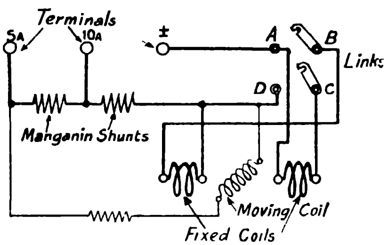


FIG. 217.- Connections of Double-range Electrodynamic Ammeter [For higher range, connect *A–B*, *C–D* by links; for lower range, connect *B–C* by links]

With *voltmeters* the range is altered by changing the number of turns in the coils and the value of the series resistance, but the range of a given instrument may be increased by connecting additional resistance in series with it. For example, the range of a given voltmeter may be doubled by connecting in series with it a non-inductive resistance equal in value to the original resistance of the instrument. The loss in the instrument and series resistance is thereby doubled for a given scale reading.

Data of electro-dynamic instruments. A **voltmeter** of the type illustrated in Fig. 212*b*, has a range of 120 volts and a resistance of 1550 ohms at 20° C., of which 77 ohms is due to the resistances of the fixed and moving coils (which are wound with copper wire) and the remainder -1473 ohms -is a non-inductive resistance of Eureka (constantan).

The inductances of the instrument (as measured by the Anderson Bridge at 50 frequency) for a number of positions of the moving system are given in the following table, together with the corresponding scale markings and angular deflections.

Scale reading (volts)	0	40	60	80	100	120
Angular deflection (degrees)	0	7	13·8	24	37·1	54
Inductance (mH.)	70·1	72	74 8	78·3	82·8	88·6

The inductance of the fixed coil is 74·5 mH., and that of the moving coil is 2·2 mH.

The mutual inductance of the fixed and moving coil circuits may be calculated from the above data. Thus, if the mutual inductance is denoted by M, the self-inductances of the fixed and moving coils by L_f, L_m, respectively, and the self-inductance of the instrument by L, we have

$$L = L_f + L_m + 2M$$

Whence $\quad M = \frac{1}{2}[L - (L_f + L_m)]$

The calculated values of M for different positions of the moving system are

Scale reading (volts)	0	40	60	80	100	120
Mutual inductance (mH.)	−3·3	−2·1	−0·95	+0·8	+3·05	+5·95

These values, when plotted against angular deflection, give a straight line (i.e. $dM/d\theta$ is constant), and theoretically the instrument should have a "square-law" scale. The actual scale, except for the smaller deflections, closely follows the square law.

The following calculated data refer to the operation of the instrument when used on a 100-volt direct-current circuit and a 50-cycle, 100-volt alternating-current circuit.

Power loss in instrument $100^2/1550 = 6·45$ W.

Reactance of instrument $314 \times 0·0828 = 26$ O.

Ratio: reactance/resistance $26/1550 = 0·0168$

Impedance of instrument at 20° C. $\sqrt{(1550^2 + 26^2)} = 1550·22$ O.

Power factor $= 1550/1550·22 = 0·99986$

Operating current at 20° C., 50 frequency 0·064491 A.

Operating current at 20° C., zero frequency 0·0645 A.

Ratio : $\dfrac{\text{operating current at 100 V., 50 frequency}}{\text{operating current at 100 V., zero frequency}}$ 0·99986

Power expended in operating coils at 20° C. 0·316 W.

Resistance of instrument at 50° C. 1559 O.

Hence when the instrument is used on the alternating current circuit the error is only −0·011 per cent.

A 5 A. precision *ammeter* of the type illustrated in Fig. 212 (*a*), with the internal connection arranged as in Fig. 216 (*b*), has a resistance between terminals of 0·188 O. at 20° C., of which 0·073 O. is due to the fixed coils, and the remainder is the joint resistance of the parallel circuit formed by the moving coil and the manganin shunt. The moving coil itself has a resistance of 0·7 O., and the manganin resistance connected in series with it has a resistance of 4·1 O. The resistance of the shunt is 0·118 O.

The self-inductance of the fixed coils is 0·16 mH., and that of the moving coil is about 1 μH. The mutual inductance of the fixed and moving coils is of the order of 1 μH. Data from which the form of scale may be determined are as follow—

Scale reading (amp.)	0	1	2	3	3·5	4	4·5	5
Deflection (degrees) .	0	2·5	12·7	32·8	45·5	59·5	74	86

The following calculated data refer to the operation of the instrument at full-scale reading (5 A.) on an alternating-current circuit of 50 frequency and sinusoidal wave-form—

Power loss	. . . .	4·7 W.
Reactance	(assuming $L = 0·16$ mH.) . .	0·05 O.
Impedance	(,, ,,) . .	0·1945 O.
Power factor	(,, ,,) . .	0·967
Pressure drop	. . . .	0·94 V.

Induction ammeters and voltmeters. In these instruments the deflecting torque is due to eddy currents induced in a pivoted disc, or drum, by a shifting (e.g. rotating or travelling) magnetic field produced by an alternating-current electromagnet; a "travelling" field being employed with "disc" instruments, and a rotating field being usually employed with "drum" instruments. With both forms of instrument, spring control and electromagnetic

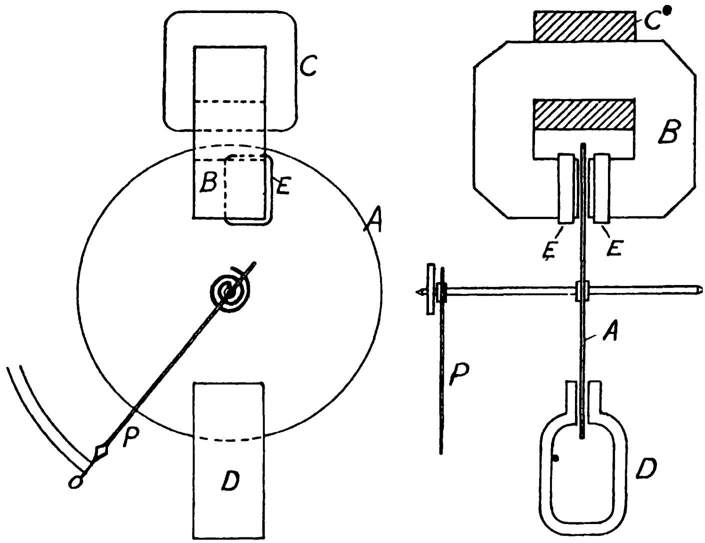


FIG. 218.—Principle of Disc Type Induction Ammeter or Voltmeter

(eddy-current) damping are employed the magnetic field for damping purposes being supplied by permanent magnets.

Construction of disc instrument.* The arrangement of the essential parts of a disc instrument is shown in Fig. 218. The aluminium disc, *A*, is fixed to the spindle which carries the pointer, and is pivoted in jewelled bearings. The operating electromagnet, *B* (the core of which is laminated), is of the shielded, or split-phase, type, and its exciting winding, *C*, is arranged for connection either in series with, or as a shunt to, the main circuit, according to whether the instrument is required to indicate current or pressure. The damping magnet, *D*, is usually fixed opposite the operating magnet

* The drum form of instrument has been developed principally on the Continent (by Siemens and Halske) and in America (by the Westinghouse Electric Co.). For details of the Westinghouse instrument, see *Transactions of American Institute of Electrical Engineers* vol. 31, p. 1565.

in order to avoid demagnetization from the stray magnetic field of the latter.

The travelling magnetic field is obtained by providing a short-circuited coil, or band, E, of copper, around a portion of each pole face of the magnet, the band usually encircling about half of the pole face. Hence, when the magnet is excited, the currents induced in the short-circuited coils cause the flux in the shielded portion (i.e. the portion encircled by the short-circuited coil) of the pole face to lag about 50° with respect to that in the unshielded portion, and therefore a continuous transverse movement of the flux occurs in the air gap.

The portion of the disc situated in the air gap is cut by the travelling flux and the reaction of the induced currents with the flux produces a torque, which, provided that the disc is not specially shaped, has a constant value (for a given exciting current) for all positions of the disc relative to the operating magnet. The disc, therefore, tends to rotate in the same direction as the flux is travelling (i.e. a given point on this portion of the disc moves from the unshielded portion of the pole face to the shielded portion). If unrestrained, the motion of the disc would be one of continuous rotation, the speed of rotation being limited by friction, windage, and braking action due to the damping magnet, D. For an indicating instrument a controlling torque must be provided by a flat spiral spring, and the rotation must be limited to less than one complete revolution. In practice, it is possible to obtain a full-scale deflection of about 330° with the attendant advantage of a long and open scale.

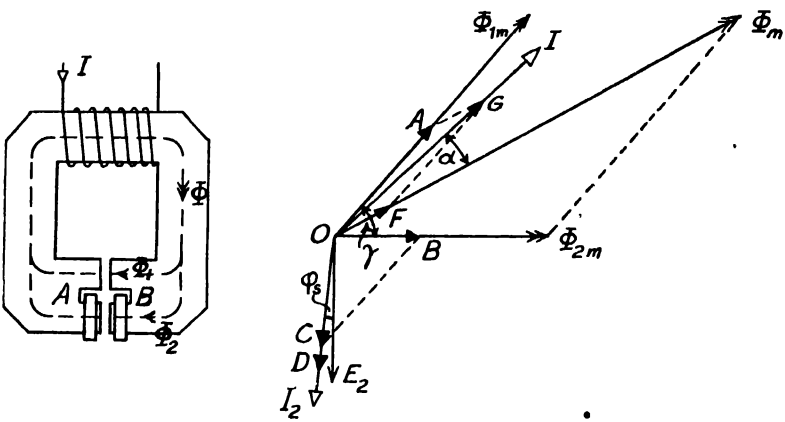
It can be shown that, for ideal conditions and constant frequency, the torque is proportional to the mean squared value of the exciting current of the operating magnet, and, since spring control is employed, the angular deflections of the pointer are proportional to the mean squared values of the exciting current.

In order to obtain a more uniformly-divided scale, the disc may be so shaped that the resistance offered to the induced currents increases as the deflection increases. (See Fig. 221.)

Ranges of ammeters and voltmeters. Since the deflecting torque is due to an electromagnet, a definite number of ampere-turns are required, with a given instrument, to obtain a full-scale deflection. Hence, with ammeters, the number of turns and size of conductor in the exciting coil must be chosen with reference to the range required. The maximum range for a direct-connected instrument is about 200 A., above which low range (5 ampere) instruments must be employed in conjunction with current transformers.

With voltmeters, the exciting coil is wound for a lower voltage than that corresponding to the maximum scale reading, and a non-inductive resistance is connected in series with the winding for the purpose of reducing errors due to variations of frequency and temperature (see p. 388).

Theory of disc type induction ammeters and voltmeters. The theory of these instruments may be developed fairly simply by assuming the fluxes in the shielded and unshielded portions of the pole face to vary sinusoidally with respect to time; by neglecting the effects of magnetic saturation,



(*a*) (*b*)

FIG. 219.—Circuit and Vector Diagrams for Shielded-pole Electromagnet

hysteresis, and core loss in the operating electromagnet; and by assuming that the paths of the current in the disc are unrestricted by its dimensions. Then, if the flux in the unshielded portion of the pole face (Fig. 219*a*) at the instant t is given by $\Phi_1 = \Phi_{1m} \sin \omega t$, and that in the shielded portion by $\Phi_2 = \Phi_{2m} \sin(\omega t - \gamma)$, where γ is their phase difference, the instantaneous value of the flux in the core of the magnet is given by

$$\begin{aligned} \Phi = \Phi_1 + \Phi_2 &= \Phi_{1m} \sin \omega t + \Phi_{2m} \sin(\omega t - \gamma) \\ &= (\Phi_{1m} + \Phi_{2m} \cos \gamma) \sin \omega t - \Phi_{2m} \sin \gamma \cos \omega t \\ &= \sqrt{(\Phi_{1m}^2 + \Phi_{2m}^2 + 2\Phi_{1m}\Phi_{2m} \cos \gamma} \sin (\omega t - \alpha)) \\ &= \Phi_m \sin (\omega t - \alpha), \end{aligned}$$

where $\Phi_m = \sqrt{(\Phi_{1m}^2 + \Phi_{2m}^2 + 2\Phi_{1m}\Phi_{2m} \cos \gamma)}$

and $\tan \alpha = \Phi_{2m} \sin \gamma / (\Phi_{1m} + \Phi_{2m} \cos \gamma)$.

The **vector diagram** for these conditions is shown in Fig. 219*b*, $O\Phi_{1m}$ representing the flux in the unshielded portion of the pole face, $O\Phi_{2m}$ that in the shielded portion, and $O\Phi_m$ the flux in the core.

The *exciting current* of the magnet can be readily obtained from the *magneto-motive force, or ampere-turn, diagram*, which is constructed in the following manner: Let OA, OB (Fig. 219*b*), represent the magnetizing ampere-turns necessary to pass the fluxes Φ_{1m}, Φ_{2m}, through the unshielded

and shielded portions, respectively, of the pole pieces and air gap, the magnetic reluctances of these paths being S_1, S_2, respectively. These ampere-turns are in phase with the respective fluxes (since magnetic saturation and effects of hysteresis are not being considered) and their maximum values are equal to $\Phi_{1m} S_1/0{\cdot}4\pi$, and $\Phi_{2m} S_2/0{\cdot}4\pi$, respectively.

Now the magnetizing ampere-turns for the shielded portion are the resultant of the ampere-turns, OA, acting across this portion of the magnetic circuit (i.e. between the points A, B, Fig. 219a) and the ampere-turns produced by the shielding coil. Hence, if there were no magnetic leakage the ampere-turns to be produced by the shielding coil would be represented by OC. But the actual ampere-turns are represented by OD, the ratio OD/OC being equal to the leakage factor or dispersion coefficient, ν. The current in the shielding coil is therefore represented by OI_2, which, owing to the inductance of this coil, lags γ with respect to the induced E.M.F. OE_2, the latter lagging 90° with respect to $O\Phi_{2m}$.

The total magnetizing ampere-turns for the electromagnet are represented

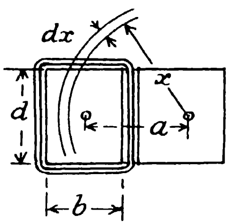


FIG. 220.—Pertaining to Calculation of Currents in Disc of Induction Instrument

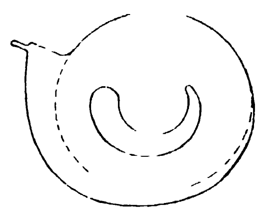


FIG. 221. Cam-shaped Disc for Induction Ammeter or Voltmeter

by OG, and are obtained by adding to OA the ampere-turns, OF, required to pass the flux Φ_m through the core and yokes. These ampere turns, OF, are in phase with the resultant flux $O\Phi_m$; they will usually be small, since the reluctance of this portion of the magnetic circuit will be much lower than that containing the poles and air gap. Hence the exciting current is represented by OI.

In determining the exciting current for an actual induction instrument the magnetic reaction effects due to the currents in the disc would also have to be considered.

Torque. To obtain an expression for the torque, it is first necessary to determine the resultant force acting upon the disc. This force is the difference between the forces due to the interaction of (1) the flux in the shielded pole face and the induced current due to the flux in the unshielded pole face, and (2) the flux in the unshielded pole face and the induced current due to the flux in the shielded pole face. Obviously, only the currents in the portion of the disc under the combined pole face need be considered.

The E.M.F. induced in the disc by the flux of the unshielded pole is

$$e_1 = -10^{-8}\, d\Phi_1/dt = \omega\Phi_{1m} \times 10^{-8} \sin(\omega t - \tfrac{1}{2}\pi),$$

and that due to the flux in the shielded pole is

$$e_2 = -10^{-8} \times d\Phi_2/dt = \omega\Phi_{2m} \times 10^{-8} \sin(\omega t - \gamma - \tfrac{1}{2}\pi).$$

Each of these E.M.Fs. may be considered to act independently in producing currents in the disc. Assuming each current to circulate in a circular path concentric with the pole face at which the inducing flux is produced, and

neglecting the inductance of these paths, the current in an element of the disc (see Fig. 220) under the shielded pole is

$$di_1 = e_1/(2\pi x\rho/\delta\, dx),$$

where dx is the width of the element, x its radius, δ the thickness of the disc (assumed to be uniform), and ρ the specific resistance. Hence the current in the portion of disc under the shielded pole is given by

$$\begin{aligned} i_1 &= \int_{a-\frac{1}{2}b_2}^{a+\frac{1}{2}b_2} di_1 = \frac{e_1\delta}{2\pi\rho}\int_{a-\frac{1}{2}b_2}^{a+\frac{1}{2}b_2}\frac{dx}{x} \\ &= \frac{e_1\delta}{2\pi\rho}\log_e\frac{a+\frac{1}{2}b_2}{a-\frac{1}{2}b_2}, \end{aligned}$$

where a is the distance between the centres of the unshielded and shielded poles and b_2 is the breadth of the shielded pole.

Similarly the current in the portion of the disc under the unshielded pole is

$$i_2 = \frac{e_2\delta}{2\pi\rho}\log_e\frac{a+\frac{1}{2}b_1}{a-\frac{1}{2}b_1},$$

where b_1 is the breadth of this pole.

In a number of shielded pole electromagnets for ammeters the unshielded and shielded portions of the pole face are approximately equal width. Hence, for these cases, $b_1 = b_2 = a$, and the above expressions for the currents simplify to

$$i_1 = \frac{e_1\delta}{2\pi\rho}\log_e 3 = \frac{1\cdot 1\, e_1\delta}{2\pi\rho}$$

$$i_2 = \frac{e_2\delta}{2\pi\rho}\log_e 3 = \frac{1\cdot 1\, e_2\delta}{2\pi\rho}$$

since $\log_e 3 = 1\cdot 1$, approximately.

Substituting for e_1 and e_2 in terms of Φ_{1m} and Φ_{2m}, we have

$$i_1 = \frac{1\cdot 1\delta\omega\,\Phi_{1m}}{2\pi\rho\times 10^8}\sin(\omega t - \tfrac{1}{2}\pi) = \frac{kf\Phi_{1m}}{\rho}\sin(\omega t - \tfrac{1}{2}\pi),$$

$$i_2 = \frac{1\cdot 1\delta\omega\,\Phi_{2m}}{2\pi\rho\times 10^8}\sin(\omega t - \gamma - \tfrac{1}{2}\pi) = \frac{kf\Phi_{2m}}{\rho}\sin(\omega t - \gamma - \tfrac{1}{2}\pi),$$

where f is the frequency $(= \omega/2\pi)$ and $k = 1\cdot 1\delta \times 10^{-8}$.

The instantaneous values of the forces due to the interaction of these currents and the fluxes are

$$F_1 = \tfrac{1}{10}\, i_2\, \Phi_1/b_1 d$$

for the portion of the disc under the unshielded pole, and

$$F_2 = \tfrac{1}{10}\, i_1\, \Phi_2/b_2 d$$

for the portion of the disc under the shielded pole, d being the transverse width of the combined pole face.

Hence for the case where $b_1 = b_2 = a$, we have

$$F_1 = \tfrac{1}{10}i_2\Phi_1/ad, \quad F_2 = \tfrac{1}{10}i_1\Phi_2/ad,$$

and, on substituting for the currents and fluxes, we have

$$F_1 = k_1(f/\rho)\Phi_{1m}\,\Phi_{2m}\sin\omega t\sin(\omega t - \gamma - \tfrac{1}{2}\pi),$$

$$F_2 = k_1(f/\rho)\Phi_{1m}\,\Phi_{2m}\sin(\omega t - \tfrac{1}{2}\pi)\sin(\omega t - \gamma),$$

where $\quad k_1 = \tfrac{1}{10}k/ad.$

Whence the resultant force acting upon the disc is

$$\begin{aligned} F_r = F_2 - F_1 &= k_1(f/\rho)\Phi_{1m}\,\Phi_{2m}[\sin(\omega t-\gamma)\sin(\omega t - \tfrac{1}{2}\pi) - \sin\omega t\sin(\omega t - \gamma - \tfrac{1}{2}\pi)] \\ &= k_1(f/\rho)\Phi_{1m}\,\Phi_{2m}[-\sin(\omega t-\gamma)\cos\omega t + \sin\omega t\cos(\omega t-\gamma)] \\ &= k_1(f/\rho)\Phi_{1m}\,\Phi_{2m}[-\cos\omega t\,(\sin\omega t\cos\gamma - \cos\omega t\sin\gamma) \\ &\qquad + \sin\omega t\,(\cos\omega t\cos\gamma + \sin\omega t\sin\gamma)] \\ &= k_1(f/\rho)\Phi_{1m}\,\Phi_{2m}\sin\gamma \end{aligned}$$

Therefore the torque acting upon the disc is

$$\mathfrak{F}_d = k'(f/\rho)\Phi_{1m}\,\Phi_{2m}\sin\gamma \quad . \quad . \quad . \quad (184)$$

where the constant k' involves k_1 and the radius of the resultant force with respect to the pivotal axis of the disc.

Observe that the time angle ωt does not appear in this expression, and therefore the resultant force does not alternate or pulsate but has a constant value for given conditions. Also that this force tends to move the disc towards the shielded pole face.

The effect upon the torque of the inductance of the current paths in the disc may be allowed for in the following manner: Let L_3 denote the inductance corresponding to the mean current path in each of the zones already considered. Then, for the case where the shielded and unshielded poles are of equal width, the impedance of each current path is equal to

$$\sqrt{(R_3^2 + \omega^2 L_3^2)} = R_3\sqrt{(1 + \omega^2 L_3^2/R_3^2)} = R_3\sqrt{(1 + \omega^2\tau_3^2)},$$

where $\tau_3 = L_3/R_3$, and R_3 is the resistance as calculated above (i.e. $R_3 = 2\pi\rho/1{\cdot}1\delta$). Hence if φ is the phase difference between these currents and the E.M.Fs. e_1, e_2, induced in the disc, the expressions for the currents i_1, i_2, must now take the form

$$i_1' = \frac{1{\cdot}1 e_1\delta\cos\varphi}{2\pi\rho\sqrt{(1 + \omega^2\tau_3^2)}} = \frac{1{\cdot}1 e_1\delta}{2\pi\rho(1 + \omega^2\tau_3^2)}$$

$$i_2' = \frac{1{\cdot}1 e_1\delta\cos\varphi}{2\pi\rho\sqrt{(1 + \omega^2\tau_3^2)}} = \frac{1{\cdot}1 e_2\delta}{2\pi\rho(1 + \omega^2\tau_3^2)}$$

since $\cos\varphi = 1/\sqrt{(1 + \omega^2\tau_3^2)}$

Therefore the torque is now given by

$$\mathfrak{F}_d = \frac{k'f}{\rho(1 + \omega^2\tau_3^2)}\,\Phi_{1m}\,\Phi_{2m}\sin\gamma \qquad (184a)$$

The value of τ_3, however, is usually such as to make the term $\omega^2\tau_3^2$ small in comparison with unity for the range of commercial supply frequencies, and under these conditions the inductance of the current paths in the disc has little effect upon the torque. Thus, except at high frequencies, the torque, corresponding to given values of Φ_{1m}, Φ_{2m}, and γ, varies directly as the frequency and inversely as the specific resistance of the disc. Since a good conducting material must be employed for the disc, the specific resistance (ρ) will increase at the rate of approximately 0·4 per cent per 1° C. increase in temperature. Hence, in the above case, a 10° C. increase in temperature will result in a 4 per cent reduction in torque.

Compensation for frequency and temperature errors in induction ammeters and voltmeters. These errors are usually greater in the ammeter than the voltmeter, and their compensation is more difficult in ammeters of the disc type than those of the drum type. Since, however, the principal application of these instruments is for switchboards of power plants, where the frequency variation is usually small, the compensation for frequency error is not so important as the compensation for temperature error.

The method employed for temperature compensation in commercial ammeters is to shunt the operating coil so that the ratio of currents in operating coil and shunt increases as the temperature increases, thereby increasing the fluxes. For example, the shunt may be non-inductive, of a material having a higher temperature coefficient of resistivity than the operating coil, and located inside the case of the instrument, or, alternatively, the shunt may be inductive with its inductance varying with temperature, the variation of inductance being obtained by means of a movable iron tongue controlled by a thermostat located inside the case of the instrument.

The compensation, however, is by no means perfect in an ammeter, but may be made approximately correct in a voltmeter in which the operating

coil is of high impedance and is shunted with a much lower resistance of copper (or material having a high temperature coefficient of resistivity); a resistance of equal value, but having a zero temperature coefficient of resistivity, being connected in series with the combination.

Form of scale. Considering the case of constant frequency and temperature, the deflecting torque, as given by equation (184), may be written

$$\mathfrak{T}_d = k_d \Phi_{1m} \Phi_{2m} \sin \gamma$$

Since the controlling torque is due to a spring, we have $\mathfrak{T}_c = k_c \theta$, where θ is the angle of deflection. Hence, for equilibrium, $\mathfrak{T}_d = \mathfrak{T}_c$, or

$$k_d \Phi_{1m} \Phi_{2m} \sin \gamma = k_c \theta.$$

Now when the effects of magnetic saturation, hysteresis, and eddy currents in the operating magnet are ignored, the exciting current is proportional to Φ_{1m}, and both the ratio Φ_{1m}/Φ_{2m} and their phase difference, γ, are constant, since, under these conditions, the shape of the vector diagram of Fig. 219*b* remains constant when the exciting current is varied. Hence, if I is the exciting current, we may write $\Phi_{1m} \Phi_{2m} \sin \gamma = kI^2$, whence, from the preceding equation,

$$\theta = kI^2$$

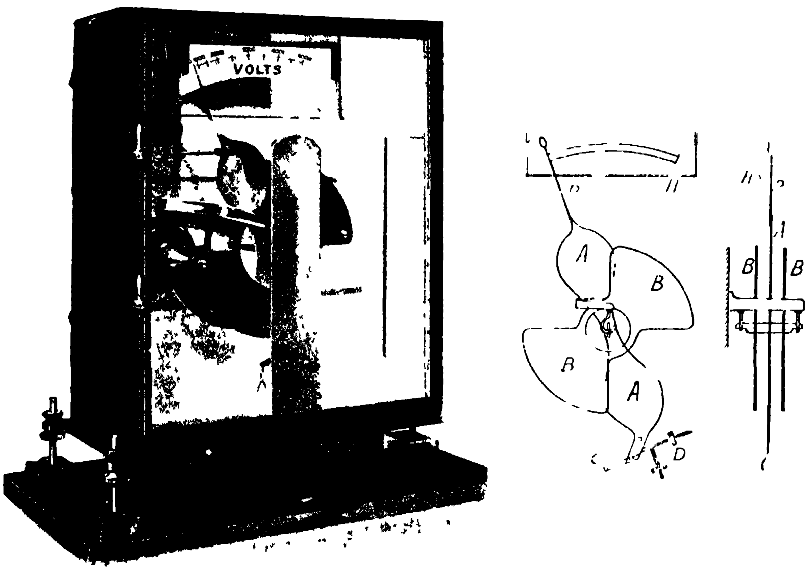
$$I = k\sqrt{\theta}.$$

Thus the scale must be divided according to a parabolic law, and, in consequence, the divisions at the upper limit of the scale are considerably more extended than those at the lower limit. In order to obtain a more uniform scale, the disc is cam-shaped, so that the surface of the portion acted upon by the operating magnet decreases as the deflection increases (Fig. 221).

Electrostatic voltmeters. In these instruments the deflecting torque is due to the electric force (attraction or repulsion) between two charged conductors. The instruments, therefore, are free from errors due to magnetic and heating effects, frequency, and wave-form. Moreover, by suitable design, electrostatic instruments may be employed for the direct measurement of high voltages with a high degree of accuracy and with an almost negligible expenditure of energy. Electrostatic instruments, particularly those for low-voltage circuits, are characterized by the smallness of the operating torque in comparison with that of electromagnetic instruments, and, in consequence, special features are necessary to avoid errors due to pivot friction; for example, the moving system may be supported by a unifilar or bifilar suspension instead of being pivoted, or, alternatively, knife-edge supports may be employed instead of pivots.

Construction. The majority of electrostatic voltmeters are virtually modified forms of the Kelvin electrometers (quadrant and attracted-disc, or absolute, types). The simplest form of Kelvin instrument (Fig. 222), which is suitable for pressures from about 800 to 10,000 volts, is a modified form of quadrant electrometer. It consists essentially of a single flat paddle-shaped aluminium "needle," A, which is connected to one terminal and is

supported on knife edges so as to swing between two vertical quadrant plates, *B*, which are both connected to the other terminal of the instrument. A pointer, *P*, is attached to the upper extremity of the needle, and the lower extremity is extended into a curved arm, *C*, to which the lever, *D*, carrying the control and balance weights is attached. Carefully adjusted weights (supplied with the instrument) may also be hung from the extremity of the arm so



(*a*) (*b*)

Fig. 222 Kelvin Electrostatic Voltmeter

(*a*) View of Instrument showing Tinfoil Screen; (*b*) Diagram showing Essential Parts

[Kelvin, Bottomley & Baird]

as to increase the controlling force and thereby extend the range of the instrument.*

The damping device is operated by the observer and consists of a stiff horizontal wire, *H*, which can be brought lightly against the back of the pointer and thereby damp the movements by mechanical friction. This wire is suspended so as to hang normally clear of the pointer and is operated from the outside of the case by an insulated lever.

* The instrument is designed for test-room work.

For lower voltages (from about 500 to 3000 volts) the **multi-cellular form of construction** is adopted. Two or more "needles" are fixed to a common horizontal axis, and a corresponding number of fixed quadrants, which are electrically connected, are arranged vertically between the needles, symmetrically about the axis of rotation. The lever carrying the control and balance weights is fixed at right angles to the spindle on which the needles are mounted, and no provision is made for extending the range. The working parts, except the pointer and control lever, are screened from external electrostatic fields by circular plates, which are connected to the (metal) case and the moving system.

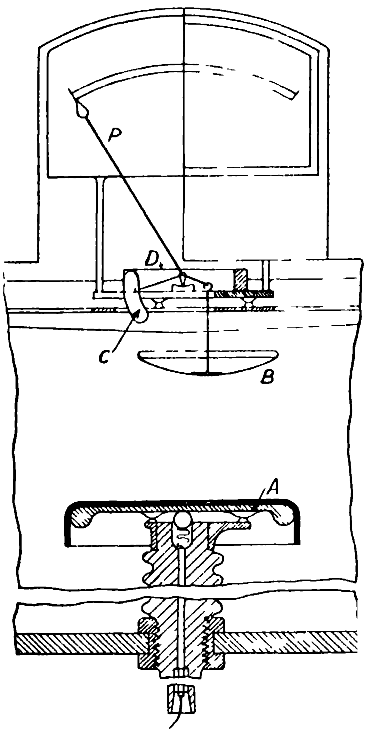


FIG. 223.—Extra High-voltage Electrostatic Voltmeter [Kelvin, Bottomley & Baird]

Low-voltage instruments, in which flat needles are employed, are of the multi-cellular type with a suspended moving system. Twelve or more needles, and a corresponding number of pairs of fixed quadrants, may be necessary, according to the range. The suspension usually consists of a fine phosphor-bronze wire or strip which supplies the control. Liquid damping is employed and takes the form either of a disc immersed in oil carried horizontally from the spindle to which the needles are fixed, or of a rectangular vane, pierced with holes and immersed in oil, carried vertically from the spindle.

An alternative construction (due to Aryton and Mather) employs a cylindrical "needle" and fixed cylindrical segments concentric with the needle. The needle is mounted vertically in pivots which enable a very small clearance to be adopted between the moving and fixed parts. Spring control and eddy-current damping are employed.

Extra-high voltage instruments (for measuring voltages above 20,000 V.) are usually modifications of the absolute, or attracted disc, electrometer. Special precautions must be taken in the design to shield the working parts from the influence of neighbouring conductors, to avoid the formation of brush discharges, and to secure adequate insulation.

The essential parts of one form of instrument are shown diagrammatically in Fig. 223. The fixed disc, *A*, is of aluminium, the edges being turned-up and rounded. It is mounted horizontally upon an insulating pillar fixed to a substantial slate base. The moving disc, *B*, is dished to a spherical shape and is suspended from a lever fixed to a horizontal spindle which carries the pointer, *P*, and an aluminium sector, *C*, which moves between the poles of a permanent magnet, *D*, and provides the damping. Gravity control is employed, and the whole of the working parts are enclosed in a metal case which is provided with a small window for observing the scale.

Theory of electrostatic voltmeters. Consider an instrument of the quadrant type having a single needle and a pair of double quadrants arranged symmetrically with respect to the needle. The instrument is then equivalent to a condenser the capacity of which varies with the deflection of the needle. Let C be the capacity when the deflection is θ and the voltage between needle and quadrants is E. Then the energy of the system is given by $W = \frac{1}{2}CE^2$. Let the deflection be now increased by $\delta\theta$ due to an increase of voltage δE, and let δC be the change in capacity. Then the increment in energy is

$$\delta W = \tfrac{1}{2}\delta C(E + \delta E)^2.$$

Hence if $\mathfrak{T}_d$ is the torque corresponding to the deflection θ and $\delta\mathfrak{T}_d$ the increment in torque, the work done in increasing the deflection of the system is

$$\delta W = (\mathfrak{T}_d + \delta\mathfrak{T}_d)\delta\theta.$$

Whence

$$(\mathfrak{T}_d + \delta\mathfrak{T}_d)\delta\theta = \tfrac{1}{2}\delta C(E + \delta E)^2,$$

$$\mathfrak{T}_d + \delta\mathfrak{T}_d = \tfrac{1}{2}(E + \delta E)^2\,\delta C/\delta\theta,$$

and, in the limit,

$$\mathfrak{T}_d = \tfrac{1}{2}E^2 dC/d\theta$$

Thus the torque is proportional to the product of the square of the applied voltage and the rate of change of the capacity with respect to the angular deflection.

Now the capacity of a parallel-plate condenser with air dielectric is given by $C = A/4\pi D$, where A is the opposed surface area of the plates and D their distance apart. In the present case, A represents the portion of surface area of the needle which opposes the quadrants and D represents the clearance between needle and quadrants. If the needle is of the double sector shape and r is the radius of its edge, the increment in the opposed surfaces of the needle and quadrants corresponding to an increment, $d\theta$, in the deflection is $4(\frac{1}{2}r^2 d\theta) = 2r^2 d\theta$, since both sides of the needle are effective. Hence

$$dC/d\theta = 2r^2 d\theta/(4\pi\delta\,.\,d\theta) = r^2/2\pi\delta.$$

Substituting this value in the above equation for the torque, we obtain

$$\mathfrak{T}_d = E^2 r^2/4\pi D$$

or

$$\mathfrak{T}_d = k_1 E_1^2$$

where

$$k_1 = r^2/4\pi D.$$

In the attracted-disc type of instrument the force of attraction between the disc and plate, when the electric field between them is uniform, is

$$F = \kappa A E^2 / 8\pi D^2,$$

where A is the area of the disc opposed to the plate, E the voltage between disc and plate, D their distance apart, and κ the dielectric constant of the medium between them. Thus, in this case, the force acting upon the moving system is directly proportional to the square of the applied voltage and inversely proportional to the square of the distance between the moving and fixed plates.

Form of scale. In the quadrant-electrometer form of instrument the equation to the deflecting torque is $\mathfrak{T}_d = k_1 E^2$. Hence with gravity control

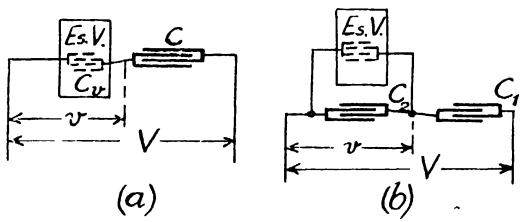


FIG. 224.—Alternative Forms of Condenser multiplier for Electrostatic Voltmeter

we have $\mathfrak{T}_c = k_2 \sin\theta$, where $\mathfrak{T}_c$ is the controlling torque and θ the angle of deflection. Therefore, for equilibrium, we must have $\mathfrak{T}_d = \mathfrak{T}_c$, or

$$k_1 E^2 = k_2 \sin\theta.$$

Whence $$E = k\sqrt{\sin\theta} \qquad (185)$$

If spring control is employed,

$$E = k'\sqrt{\theta} \qquad (185a)$$

In the latter case the scale is divided according to a parabolic law. In both cases, however, the lower scale divisions are cramped and the upper divisions are extended.

Multipliers for electrostatic voltmeters. The range of a low-voltage instrument may be extended by means of a **resistance multiplier** (or potential divider), in which the voltmeter is connected across a definite fraction of a high non-inductive resistance (of from 10,000 to 100,000 O.), which is connected across the supply circuit. Thus, if R is the total resistance and R_1 the resistance of the portion across which the voltmeter is connected, the voltage of the supply circuit is equal to reading of voltmeter $\times R/R_1$, provided that the reactance of the voltmeter is large in comparison with the resistance R_1.

This form of multiplier is employed in connection with multicellular 120 V. instruments for circuits up to 600 V.

For high-voltage circuits **condenser-multipliers** are employed. Two methods of connection are possible; in one case (Fig. 224a), a condenser of suitable capacity is connected in series with the

voltmeter; in the other case (Fig. 224*b*), two or more condensers are connected in series across the supply circuit and the voltmeter is connected across one of them.

With the first method, the additional capacity required is extremely small and may only be a fraction of the capacity of the voltmeter itself. In consequence, the multiplying factor will vary with the deflection of the voltmeter owing to the variation of the capacity of the latter.

The capacity of the condenser-multiplier is easily determined when the capacity of the voltmeter and the voltages are known. Thus, if the reading, v, of a voltmeter is to represent the voltage, V, across the supply circuit, and the capacity of the voltmeter at this reading is C_v, the capacity, C', of the condenser-multiplier is given by the relationship

$$\omega C_v v = \omega\left(\frac{1}{1/C' + 1/C_v}\right)V$$

or

$$\frac{v}{V} = \frac{C'}{C' + C_v}$$

whence

$$C' = C_v/(V/v - 1)$$

For example, if the "100" reading of a 120 V. voltmeter is to represent 10 000 volts, and the capacity of the voltmeter at this reading is $70\,\mu\mu$F., the capacity required for the condenser-multiplier is equal to $[70/(100-1)] = 0{\cdot}707\,\mu$F.

With the alternative method shown in Fig. 224*b*, the effect of the variation of the capacity of the voltmeter may be made extremely small by arranging that the capacity of the condenser to which the voltmeter is connected is large in comparison with that of the voltmeter, i.e. the joint capacity of this condenser and the voltmeter is sensibly constant for all deflections of the latter. Under these conditions, if C_2 is the capacity of the condenser shunting the voltmeter and C_1 is the capacity of the series condenser, or condensers, then the relationship deduced in the previous case is applicable to the present case, i.e.

$$C_1 = C_2/(V/v - 1).$$

For example, if $C_2 = 7\,\mu$F. (i.e. 100 times the capacity of the voltmeter in the preceding case), and $V/v = 100$,

$$C_1 = 7/(100 - 1) = 0{\cdot}707\,\mu\text{F}.$$

With all condenser-multipliers it is highly important that the **condensers have low dielectric losses and high insulation resistance.**

WATTMETERS

Commercial wattmeters are of the deflectional type and operate on either the electro-dynamic or the induction principle.

Standard wattmeters for laboratory use are of the non-deflectional or torsion-head type, and operate on the electro-dynamic principle, but the electrostatic principle has been utilized to obtain a standard instrument of the deflectional type. Electrostatic "wattmeters" are virtually quadrant electrometers used in conjunction with standard resistances in the manner explained later (p. 475).

Electro-dynamic single-phase wattmeters. The mechanism of a *commercial electro-dynamic wattmeter* closely resembles that of an electro-dynamic ammeter, but the moving coil of the wattmeter has a high non-inductive resistance connected in series with it and is provided with separate terminals. The fixed coil is connected either directly in series with the circuit, or in the secondary circuit of a "series" or "current" transformer, the primary of which is connected in series with the main circuit. The moving coil, together with a non-inductive series resistance, is connected across the main circuit in order that the current in this coil shall be proportional to the voltage of the circuit.

The instruments may be constructed with either a non-magnetic (air-path) circuit or an iron-cored magnetic circuit, the former being employed in the majority of instruments in practice. Instruments with iron-cored magnetic circuit, however, possess the advantage of a higher torque and are immune from the influence of external magnetic fields, so that magnetic shielding is unnecessary. The iron portion of the magnetic circuit, however, must be confined to the fixed coils, as the principles of operation of the electro-dynamic wattmeter require the moving-coil circuit to be as free as possible from inductance. The principal objections to the iron-cored magnetic circuit are (1) that hysteresis may cause the flux to have a slight phase displacement and a slightly different wave-form from the current in the fixed coil, and (2) that the inductance of the moving coil is higher than that of a similar instrument with air-path magnetic circuit. By suitable design, however, the inaccuracies due to these causes may be made sufficiently small for ordinary commercial purposes.

The mechanism of a typical instrument with air-path magnetic circuit is shown in Fig. 225.

Standard wattmeters have a suspended moving-coil system and the instruments are constructed on the astatic principle in order to eliminate errors due to external magnetic fields. Moreover, to

avoid eddy currents, no metal parts, except the coils, are situated in the magnetic field, and the fixed coils are wound with stranded cable, the separate strands being insulated from one another. The coil supports are of ivory or ebonite, and the working parts are enclosed in a glass sided case with a wooden framework

The arrangement of the coils is shown in Fig 226*a* and differs from that for a commercial instrument, as the magnetic axis of

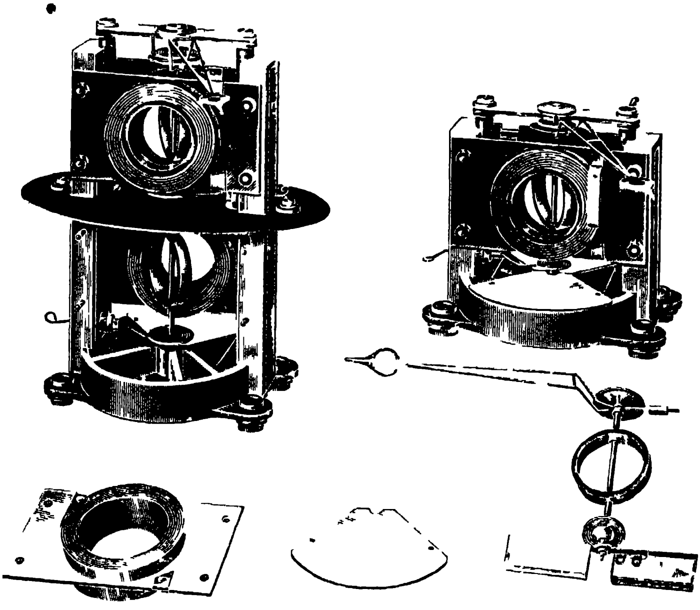
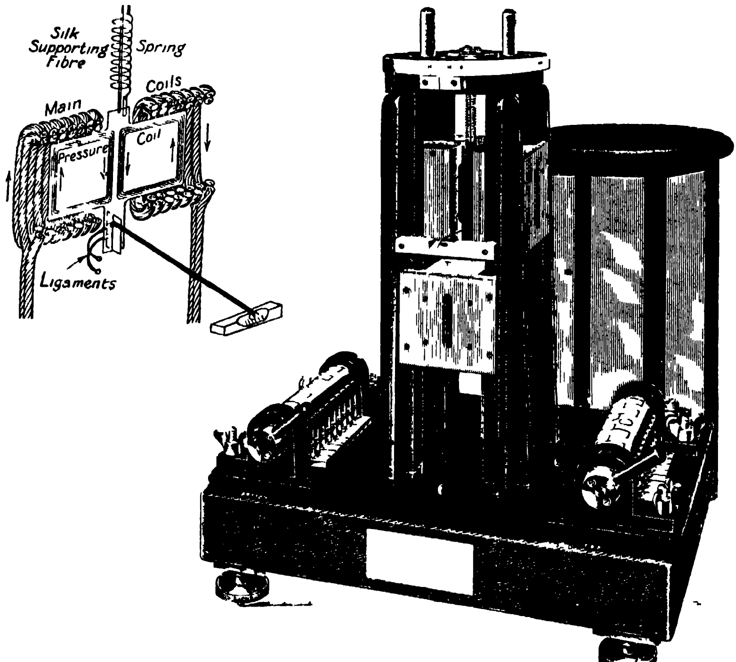


FIG 225 Mechanism of Electro-dynamic Wattmeters
[Nalder Bros & Thompson]

(*a*) Assembled mechanism of single phase wattmeter, (*b*) moving system, showing damping vanes and control springs, (*c*) mechanism of polyphase wattmeter (p 406) with one of lower current coils and cover of one of damping chambers removed, these being shown separately at (*d*) and (*e*) respectively

the moving coil is maintained normally perpendicular to that of the fixed coil. The moving coil is wound in two equal portions on a flat mica vane, the two portions being connected in series so that the current circulates in opposite directions, as indicated by the arrows in Fig. 226*a*. The current is led to the coil by flexible ligaments. The coil is suspended by a silk fibre and the spiral torsion spring is arranged concentric with the suspension, the lower end being fixed to the coil support, and the upper end to the torsion head.

The fixed coil is of rectangular shape and is also wound in two equal portions, which are connected in series so that the current circulates in opposite directions. The coil is wound with a multi-core stranded cable having usually ten separate stranded cores, which are insulated from one another and are connected to a commutator in order that a number of current ranges may be obtained



(a) (b)

FIG. 226—(a) Arrangement of Fixed and Moving Coils in Standard Wattmeter (Fixed Coils shown in Section); (b) General View of Polyphase Standard Wattmeter, with Cover removed to show Torsion Head [H. Tinsley & Co.]

by grouping the individual cores in series, series-parallel, or parallel.

The correct operating position of the moving coil is indicated by a pointer fixed to the coil support and index marks on a fixed part of the instrument.

The **method of using the instrument** is as follows: After being levelled and with no current in either fixed or moving coils, the torsion head is adjusted to bring the pointer to the index mark,

and the reading on the torsion head noted. This reading normally should be zero. The instrument is then connected to the circuit in which the power is to be measured. The torque due to the currents in the fixed and moving coils will cause the pointer to deflect to one side or the other of the index mark, and the torsion head is rotated until the pointer returns to its initial position. The number of degrees through which the torsion head is moved, multiplied by the "constant" of the instrument, gives the power.

In order that errors due to inductance and capacity in the moving-coil circuit shall be as small as possible, the moving-coil is designed to have a low inductance and to require a large series resistance, while the latter is so constructed that its self capacity is extremely small. For example, in a typical instrument the inductance of the moving coil is 7 mH. (which is the self-inductance under operating conditions, since the mutual inductance of fixed and moving coils is zero) and a series resistance of 100,000 ohms is required when the operating pressure is 100 volts.

Ranges. The maximum current range of *commercial electro-dynamic wattmeters* is from 100 to 200 A. For higher currents the fixed coils are usually wound for a maximum current of 5 A. and are supplied from the secondary winding of a current transformer of suitable ratio.

The moving coil is usually wound to carry a current of from 0·02 A. to 0·03 A., and therefore the resistance of the moving-coil circuit must be of the order of from 30 to 50 ohms per volt of pressure range.

For pressures up to 600 V. a series resistance is employed, but for higher pressures an instrument having a 100 V. pressure circuit is employed in conjunction with a potential, or shunt, transformer.

Standard wattmeters of the type described have a maximum current range of 500 A. For higher currents special designs have been developed, an interesting example being the tubular instruments of Agnew and Moore—which are suitable for currents up to 5,000 A.*

Theory of the electro-dynamic wattmeter. Let the current and pressure in the circuit to which the wattmeter is connected be given by $i = I_m \sin(\omega t - \varphi)$ and $e = E_m \sin \omega t$, respectively. Then if the inductance of the moving-coil circuit is ignored and the resistance of this circuit is denoted by R, the current in the moving-coil will be given by $i_2 = (E_m/R) \sin \omega t$.

If this current is small in comparison with the current in the main circuit, the current in the fixed coil will be equal to, and in phase with, the latter.

* For details, see *Transactions of American Institute of Electrical Engineers*, vol. 31, p. 1483; *Journal of the Institution of Electrical Engineers*, vol. 45, p. 380.

Hence, with an air-path magnetic circuit and no eddy currents in the coils or their supports, the flux due to the current in the fixed coil will be proportional to, and in phase with, the current in the circuit. Moreover, in the case of a *deflectional instrument*, if the flux density is assumed to be uniform throughout the space occupied by the moving coil, the torque at any instant will be given by

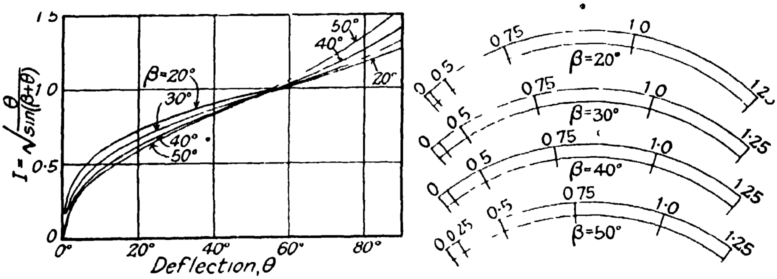
$$\begin{aligned}\mathfrak{T}_{d(inst)} &= k_2 M_M i_1 i_2 \sin(\beta + \theta) \\ &= (k_2 M_M \sin(\beta + \theta) I_m E_m/R) \sin \omega t \sin(\omega t - \varphi) \\ &= (\tfrac{1}{2} k_2 M_M \sin(\beta + \theta) I_m E_m/R) [\cos \varphi - \cos(2\omega t - \varphi)] \\ &= (k_2 M_M \sin(\beta + \theta) I E/R) [\cos \varphi - \cos(2\omega t - \varphi)]\end{aligned}$$

where k_2 is a constant, M_M the maximum mutual inductance of the fixed and moving coils, and $(\beta + \theta)$ the deflection of the moving coil from the position corresponding to the coincidence of the magnetic axes of the coils (see p. 377).

Therefore the mean torque during a period is

$$\begin{aligned}\mathfrak{T}_d = \frac{1}{T}\int_0^T \mathfrak{T}_{d(inst)}\, dt &= \frac{k_2 M_M \sin(\beta + \theta)}{R} I E \cos \varphi \\ &= \frac{k_2 M_M \sin(\beta + \theta)}{R} P\end{aligned}$$

where P $(= EI \cos \varphi)$ is the true power in the circuit.



(a) (b)

FIG. 227.—Forms of Scale for Electro-dynamic Wattmeter

The torque of an electro-dynamic instrument, however, is also proportional to the product of the currents in the coils and the rate of change of mutual inductance with deflection (p. 377), so that, for the electro-dynamic wattmeter the torque may be expressed in the form

$$\mathfrak{T}_d = k_3 \frac{dM_{\beta + \theta}}{d(\beta + \theta)} P$$

Form of scale. Since spring control is employed, the controlling torque is $\mathfrak{T}_c = k_1\theta$, where θ is the deflection from the zero position. Hence, for equilibrium, we have $\mathfrak{T}_d = \mathfrak{T}_c$, or, for a *deflectional instrument*,

$$\frac{k_2 M_M \sin(\beta + \theta)}{R} P = k_1\theta$$

Whence $$P = k\theta/\sin(\beta + \theta) \qquad (186)$$

where $k = k_1 R/k_2 M_M$.

Curves connecting P and θ for the special case when $k = 1{\cdot}0$ and for selected values of β are given in Fig. 227*a*, and the corresponding scale shapes are given in Fig. 227*b*.

In practical instruments, however, the scale divisions towards the ends of the scale may be made more uniform than is shown by equation (186) and Fig. 227, as, by a suitable choice of the dimensions of the coils in conjunction with the value of β and the angle corresponding to a full-scale deflection, the non-uniformity of the actual flux distribution in the interior of the fixed coils may be utilized to obtain higher values of torque at the upper and lower limits of the scale than those corresponding to the hypothetical case, considered above, in which the flux distribution is uniform. In such a case the rate of change of mutual inductance with deflection will be practically constant over the whole of the scale, instead of varying as a sine function of the angle $(\beta + \theta)$, as in the hypothetical case. The data, on p. 381, of electro dynamic instruments are of interest in this connection

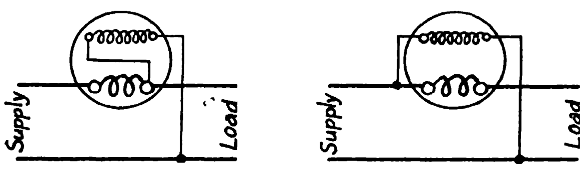
With a **non-deflectional,** or torsion-head, **instrument,** the moving coil is always brought back, by twisting the torsion head and control spring, to its initial position where the magnetic axes of the coils are perpendicular to each other, i.e. $\beta + \theta = 90°$. Hence, if θ' is the twist applied to the torsion head, we have

$$(k_2 M_M / R)\, P = k_1 \theta'$$

Whence $$P = k\theta' \qquad . \quad . \quad . \quad . \quad . \quad (186a)$$

Thus, in this case, the power is proportional to the angle through which the torsion head is twisted.

Correction for power expended in wattmeter. If the current (i_2) in the



(a) (b)

FIG. 228.—Alternative Connections for Wattmeter

moving-coil circuit is not small in comparison with the current in the circuit, and the wattmeter is connected in the manner shown in Fig. 228*a*, then if i denotes the current in the "load," the current i_1 in the fixed coil is $i_1 = i + i_2$. Hence the torque of a non-deflectional instrument (which is considered for the sake of simplicity of treatment) is now given by

$$\begin{aligned}
\mathfrak{T}_{d(inst)} &= k_2 M_M (i + i_2)\, i_2 \\
&= k_2 M_M (i\, i_2 + i_2^2) \\
&= k_2 M_M \left\{ \frac{I_m E_m}{R} \sin \omega t \sin(\omega t - \varphi) + \frac{E_m^{\ 2}}{R^2} \sin^2 \omega t \right\} \\
&= \frac{k_2 M_M}{R} \left[IE \{\cos \varphi - \cos(2\omega t - \varphi)\} + \frac{E_m^{\ 2}}{R} \sin^2 \omega t \right]
\end{aligned}$$

and the mean torque by

$$\begin{aligned}
\mathfrak{T}_d = \frac{1}{T}\int_0^T \mathfrak{T}_{d(inst)} \cdot dt &= \frac{k_2 M_M}{R} \left(IE \cos \varphi + \frac{E^2}{R} \right) \\
&= \frac{k_2 M_M}{R} \left(P + \frac{E^2}{R} \right)
\end{aligned}$$

Whence, equating deflecting and controlling torques, we have

$$\frac{k_2 M_M}{R} \left(P + \frac{E^2}{R} \right) = k_1 \theta'$$

and $$P = k\theta' - E^2/R \qquad . \quad . \quad . \quad . \quad . \quad (187)$$

Therefore, to obtain the power supplied to the load, the power expended in the pressure-coil circuit must be subtracted from the power ($k\theta'$) indicated by the wattmeter. If the wattmeter is connected as shown in Fig. 228*b*, the power expended in the current coils must be subtracted from the power indicated by the instrument.

In practice, this correction need only be applied to cases where small powers are to be measured. For example, if a wattmeter, the pressure circuit of which has a resistance of 3333 ohms, is connected as in Fig. 228*a*. and indicates a power of 15 watts, the power actually supplied to the load, assuming the pressure to be 100 volts, is

$$P = 15 - 100^2/3333 = 12 \text{ watts.}$$

Correction for inductance of moving-coil circuit. The error due to inductance in the moving-coil circuit is most easily calculated in instruments of the non-deflectional, or torsion-head, type, as the moving coil always occupies a standard position relative to the fixed coil, and, therefore, the mutual inductance, if any, is constant. If L, R denote the inductance and resistance, respectively, of the moving-coil circuit, the current in this circuit, when the impressed E.M.F. is $e = E_m \sin \omega t$, is now

$$i_2' = [E_m/\sqrt{(R^2 + \omega^2 L^2)}] \sin(\omega t - \alpha),$$

where $\tan \alpha = \omega L/R = \omega\tau$. Thus, the current in the moving coil is not in phase with the impressed E.M.F.; moreover, its value and phase difference are both affected by frequency.

The deflecting torque at any instant t is

$$\begin{aligned}
\mathfrak{T}_d'{}_{(inst)} &= k_2 M_{\mathrm{M}} i\, i_2' \\
&= k_2 M_{\mathrm{M}} I_m \frac{E_m}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \alpha) \sin(\omega t - \varphi) \\
&= \frac{k_2 M_{\mathrm{M}} I_m E_m}{R\sqrt{(1 + \omega^2\tau^2)}} \sin(\omega t - \varphi)\, [\sin \omega t \cos \alpha - \cos \omega t \sin \alpha] \\
&= \frac{k' I_m E_m}{1 + \omega^2\tau^2} \sin \omega t \sin(\omega t - \varphi) - \frac{k' I_m E_m \omega\tau}{1 + \omega^2\tau^2} \cos \omega t \sin(\omega t - \varphi)
\end{aligned}$$

where $k' = k_2 M_{\mathrm{M}}/R$, $1/\sqrt{(1 + \omega^2\tau^2)} = \cos \alpha$, and $\omega\tau/\sqrt{(1 + \omega^2\tau^2)} = \sin \alpha$.

Whence the mean deflecting torque during a period is

$$\begin{aligned}
\mathfrak{T}_d' &= \frac{1}{T}\int_0^T \mathfrak{T}_d'{}_{(inst)}\, dt = \frac{k'IE}{1 + \omega^2\tau^2} \cos \varphi + \frac{k'IE\omega\tau}{1 + \omega^2\tau^2} \sin \varphi \\
&= k'IE \cos \varphi \left(1 - \frac{\omega^2\tau^2}{1 + \omega^2\tau^2}\right) + k'IE \sin \varphi \frac{\omega\tau}{1 + \omega^2\tau^2} \\
&= k'\left\{IE \cos \varphi + \left(\frac{\omega\tau}{1 + \omega^2\tau^2} IE \sin \varphi - \frac{\omega^2\tau^2}{1 + \omega^2\tau^2} IE \cos \varphi\right)\right\}
\end{aligned}$$

Since the term $\omega^2\tau^2$ is usually very small indeed in comparison with unity, we have, to a very close approximation,

$$\begin{aligned}
\mathfrak{T}_d' &= k'(IE \cos \varphi + \omega\tau\, IE \sin \varphi) \\
&= k'(P + \omega\tau\, IE \sin \varphi)
\end{aligned}$$

If $k_1\theta''$ is the controlling torque, we have, for equilibrium,

$$k'(P + \omega\tau\, IE \sin \varphi) = k_1\theta''$$

Whence $\quad P = k\theta'' - \omega\tau\, IE \sin \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$

where $\quad k = k_1/k' = k_1 R/k_2 M_{\mathrm{M}}$

Thus the effect of inductance in the moving-coil circuit is to cause the wattmeter to read high (on lagging power factor) by the amount $\omega\tau\, IE \sin \varphi$, and therefore this quantity must be subtracted from the wattmeter reading

to obtain the true power. For leading power factors the wattmeter will read low, and the quantity $\omega\tau\, IE \sin\varphi$ must then be added to the wattmeter reading to obtain the true power.

It should be noted that the correction factor is zero at unity power factor (on the assumption that $\omega^2\tau^2$ is negligible in comparison with unity), and approaches its maximum value ($\omega\tau\, IE$) as the power factor approaches zero, so that when the wattmeter is used on circuits of very low power factor, the correction factor may require careful consideration. The percentage error

$$= 100\,\frac{\omega\tau\, IE \sin\varphi}{IE \cos\varphi} = 100\,\omega\tau \tan\varphi,$$

which, for a power factor of 1 per cent ($\varphi = 89.4°$), becomes equal to $9550\,\omega\tau$, and, for a power factor of 0.175 per cent ($\varphi = 89.9°$), becomes $57300\,\omega\tau$. For example, if the inductance of the moving coil circuit is 6 milli henries and the resistance is 3000 ohms, $\tau = L/R = 2 \times 10^{-6}$, and for 50 frequency, $\omega\tau = 314 \times 2 \times 10^{-6} = 6.28 \times 10^{-4}$. Hence, if this instrument is used on circuits having a power factor of 1 per cent, the error in the wattmeter reading is ($9550 \times 6.28 \times 10^{-4} =$) 6 per cent, and, if used on circuits having a power factor of 0.175 per cent, the error is ($57300 \times 6.28 \times 10^{-4} =$) 36 per cent.

It is apparent that if an electro dynamic wattmeter is to give high accuracy at very low power factors the time constant of the moving coil circuit must be very small. For example, if an error of 0.5 per cent is not to be exceeded when the wattmeter is used on a 50 cycle circuit of 1 per cent power factor ($\varphi = 89.4°$) we have

$$\begin{aligned} \tau &= 0.5/(100\,\omega \tan\varphi) \\ &= 0.5/(100 \times 314 \times 95.5) \\ &= 0.166 \times 10^{-6} \end{aligned}$$

Whence $R/L = 6 \times 10^{6}$

Thus the resistance of the moving coil circuit must be at least 6000 times as great as the inductance (expressed in milli henries) of this circuit. It is practicable to obtain, and even to exceed, this ratio in a standard instrument.

Series resistances for high-voltage circuits. When an electro dynamic wattmeter is to be connected directly to a high voltage circuit (as may be necessary for certain tests), the series resistance required for the moving coil circuit will have a very high value. This resistance must be wound non-inductively, but the method of winding must eliminate, as far as practicable, electrostatic capacity between the turns, as otherwise, due to the large number of turns, the series resistance will possess an appreciable capacity, which will cause errors similar (but of opposite sign) to those caused by inductance. A number of methods of winding have been developed*. In one method (due to Duddell and Mather) the resistance wire is woven with silk threads to form a gauze, in another method the series resistance is subdivided into a number of units, the resistance wire being wound on thin strips of mica.

Induction wattmeters. The induction principle may be applied to indicating wattmeters, but such instruments are only suitable for circuits in which the frequency and voltage are constant.

The travelling magnetic field is produced by the joint action of two electromagnets, one being series wound and the other shunt wound (Fig. 229). The exciting current of the series magnet is proportional to the current in the circuit in which the instrument

* For details, see *Alternating-Current Bridge Measurements*, Hague (Pitman).

is connected, and that of the shunt magnet is proportional to the voltage of this circuit.

In the hypothetical case when the resistance of the shunt magnet is zero and both magnetic circuits are free from saturation and hysteresis, the fluxes will have a phase difference of ($90 \mp \varphi$) degrees (φ being the phase difference between voltage and current of the main circuit), and will be proportional to the current and voltage, respectively, of the main circuit. Hence, if these magnets act upon a spring controlled pivoted disc (assumed to be free from inductance)

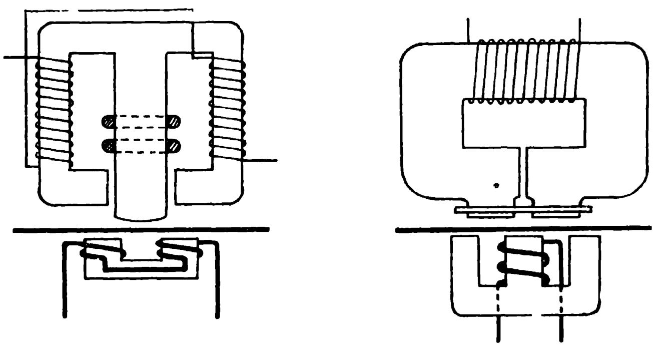


FIG. 229.—Forms of Electro-magnets for Induction Wattmeter

the deflecting torque (p. 387) will be proportional to the product of the fluxes and the sine of their phase displacement, i.e. to

$$EI \sin(90 \mp \varphi) = EI \cos \varphi,$$

and the angular deflections will be proportional to $EI \cos \varphi$, or to the power in the circuit.

If, however, the shunt winding possesses appreciable resistance, the flux of this magnet will not lag 90° with respect to the applied voltage, and, in consequence, the phase difference between the fluxes will be smaller than ($90 \mp \varphi$) degrees, as in the hypothetical case. Some means of increasing the natural lag of the flux of the shunt magnet is therefore necessary in a commercial instrument. A number of devices have been developed for this purpose, the majority of which employ shielding rings or a shielding plate for the poles of the shunt magnet.

Construction of disc-type wattmeter. The principal feature in which the construction of the induction wattmeter differs from that of the induction ammeter and voltmeter is the operating magnet, two forms of which are shown diagrammatically in Fig. 229

The shunt and series windings are usually placed upon separate laminated cores, which are arranged one on each side of the disc with their magnetic axes symmetrical, as shown in Fig. 229.

In the magnet shown in Fig. 229*a*, the shunt winding consists of two coils which are connected in series so as to produce a flux in the central limb. This limb is extended beyond the lower yokes, so that a portion of the flux shall pass through the disc to the core which carries the series winding. Air gaps are arranged between the central core and the lower yokes, in order to increase the magnetizing ampere-turns of the magnetic circuit and thereby increase the ratio reactance/resistance for the magnetizing coils.

The series winding also consists of two coils which are wound on a short U-shaped core and are connected in series to give poles of opposite polarity. The poles of the series magnet are arranged symmetrically with respect to the central limb of the shunt magnet, and, therefore, when both windings are excited a travelling magnetic field is produced in the air gap between these opposing poles. To obtain a phase difference of 90° between the shunt and series fluxes a number of short-circuited turns of thick copper wire are placed around the central limb of the shunt magnet, the correct adjustment being determined by test.

In the magnet shown in Fig. 229*b*, the shunt and series windings each consist of a single coil. The shunt magnet has an air gap in the direct path of the flux, and polar projections on each side of this gap divert a portion of the flux through the disc and the core of the series magnet.

The series magnet has a three-limbed core, the central limb carrying the exciting winding. This core is arranged symmetrically with respect to the polar projections of the shunt magnet, so that when the magnets are excited a travelling magnetic field is produced in the space between the opposing polar projections. The correct phase difference between the shunt and series fluxes is effected by an adjustable copper ring, which surrounds the pole faces of the shunt magnet.

Theory of induction wattmeter. The theory of the uncompensated wattmeter is developed in a manner similar to that (p. 384) of the induction ammeter. Thus, if magnetic saturation and hysteresis are absent, the flux due to the series winding is directly proportional to, and in phase with, the current in the main circuit, i.e. $\Phi_1 = k_1 I_m \sin(\omega t - \varphi)$, where φ is the phase difference between voltage and current in the main circuit. Similarly, the flux due to the shunt winding is

$$\begin{aligned}\Phi_2 &= [k_2 E_m / \sqrt{(R_2^2 + \omega^2 L_2^2)}] \sin(\omega t - (\tfrac{1}{2}\pi - \beta)) \\ &= -[k_2 E_m / \sqrt{(R_2^2 + \omega^2 L_2^2)}] \cos(\omega t + \beta)\end{aligned}$$

where R_2, L_2 denote the resistance and inductance, respectively, of the shunt

winding and $(\frac{1}{2}\pi - \beta)$ is the phase difference between the impressed E.M.F and the flux in the shunt magnet : β being equal to $\tan^{-1} R_2/\omega L_2$.

The E.M.F. induced in the disc by the flux Φ_1 is

$$e_1 = -10^{-8} d\Phi_1/dt = -k_1 \omega I_m \cos(\omega t - \varphi),$$

and that induced by the flux Φ_2 is

$$e_2 = -10^{-8}\, d\Phi_2/dt = -[k_2 \omega E_m/\sqrt{(R_2^2 + \omega^2 L_2^2)}] \sin(\omega t + \beta)$$

The currents in the disc are deduced in the manner given on pp. 385 to 387, from which follow the expressions for the instantaneous forces due to the interaction of these currents and the fluxes. Thus

$$\begin{aligned} F_1 &= k\, i_2' \Phi_1 \\ &= k\left(\frac{-k_2' \omega E_m}{\rho(1 + \omega^2 \tau_3^2)\sqrt{(R_2^2 + \omega^2 L_2^2)}} \sin(\omega t + \beta)\right)(k_1 I_m \sin(\omega t - \varphi)) \\ F_2 &= k\, i_1' \Phi_2 \\ &= k\left(\frac{-k_1' \omega I_m}{\rho(1 + \omega^2 \tau_3^2)} \cos(\omega t - \varphi)\right)\left(\frac{-k_2 E_m}{\sqrt{(R_2^2 + \omega^2 L_2^2)}} \cos(\omega t + \beta)\right) \end{aligned}$$

Whence the torque acting upon the disc is

$$\begin{aligned} \mathfrak{T}_d &= k'(F_2 - F_1) \\ &= \frac{K \omega E_m I_m}{\rho(1 + \omega^2 \tau_3^2)\sqrt{(R_2^2 + \omega^2 L_2^2)}} [\cos(\omega t + \beta)\cos(\omega t - \varphi) \\ &\qquad + \sin(\omega t + \beta)\sin(\omega t - \varphi)] \\ &= \frac{K \omega E_m I_m}{\rho(1 + \omega^2 \tau_3^2)\sqrt{(R_2^2 + \omega^2 L_2^2)}} \cos(\varphi + \beta) \\ &= \frac{K_1}{\rho L_2(1 + \omega^2 \tau_3^2)\sqrt{(1 + 1/\omega^2 \tau_2^2)}} EI \cos(\varphi + \beta) \end{aligned}$$

where $\tau_2 = L_2/R_2$.

Therefore, if the torque is to be proportional to the power in the main circuit, β must be zero. Under these conditions we have

$$\mathfrak{T}_d = \frac{K_1}{\rho L_2(1 + \omega^2 \tau_3^2)\sqrt{(1 + 1/\omega^2 \tau_2^2)}} EI \cos \varphi \qquad (189)$$

This equation shows that the indications of the induction wattmeter are less affected by variations of frequency than those of the induction ammeter.

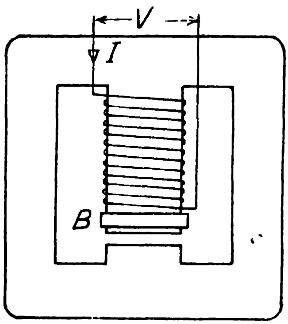
Method of obtaining correct phase difference between fluxes. From a comparison of equation (189) and equation (184*a*), p. 387, it follows that the correct phase difference between the fluxes at the pole faces of the series and shunt magnets is $(90 - \varphi)$ degrees. The natural phase difference between these fluxes, however, is $(90 - \varphi - \beta)$ degrees, where $\beta = \tan^{-1} R_2/\omega L_2$. Therefore to obtain the correct phase difference, either the flux at the pole face of the series magnet must lead the current by the angle β, or the flux at the pole face of the shunt magnet must lag 90° with respect to the impressed E.M.F. In the former case the series coil must be shunted inductively; in the latter case the pole face of the shunt magnet may be shielded, or, alternatively, the shunt coil may be shunted by a non-inductive resistance, an impedance coil being connected in series with the combination; in both cases some adjustment must be provided.

The shielding of the shunt magnet is the simpler method and is generally adopted in commercial instruments. An example of the application of this method is shown in Fig. 229, and the theory is given in the following section. In shielding the shunt pole, the shielding coil must encircle the whole of the pole core, as in Fig. 229; or, if a shielding plate is employed, it must be arranged symmetrically with respect to an axis perpendicular to the

common centre-line of the series and shunt pole faces, otherwise any unsymmetrical shielding effect along this centre line will produce a torque due to the shunt flux alone.

Theory of shielded-pole shunt magnet. Consider the simple case of the magnet shown in Fig. 230, in which the shielding coil B surrounds the pole face of the core which is magnetized by the shunt-excited coil. Then if Φ is the flux at the pole face, and the vector, $O\Phi$, of this quantity is taken as the reference vector in the vector diagram, Fig. 231, the saturation ampere-turns for the magnetic circuit are represented by OA, which is slightly in advance of the flux owing to hysteresis and eddy currents in the core.

The E.M.Fs. induced in the exciting and shielding coils are represented by OE_1 and OE_2, respectively, both lagging 90° with respect to the flux.



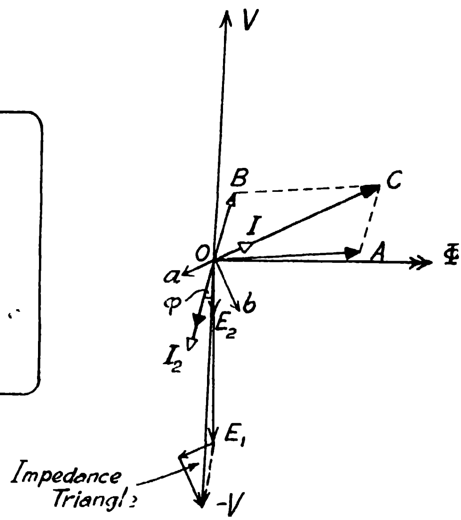


FIG. 230 FIG. 231

Circuit and Vector Diagrams for Shielded-pole Shunt Excited Electromagnet

The current in the shielding coil is represented by OI_2 which also represents the ampere-turns due to this coil. Hence the ampere-turns to be provided by the exciting coil are represented by OC, which is equal to the vector sum of the saturation ampere-turns, OA, and the ampere-turns, OB, balancing the ampere-turns due to the shielding coil. The exciting current is therefore represented by OI.

The voltage to be applied to the exciting coil is obtained by determining the internal E.M.Fs. in this coil. These E.M.Fs. comprise the E.M.F., OE_1, induced by the alternations of the flux Φ; the E.M.F., Oa, due to the resistance of the coil; and the E.M.F., Ob, due to leakage reactance. The resultant internal E.M.F. is therefore represented by O-V, and the applied E.M.F.—which balances the resultant internal E.M.F.— is represented by OV.

Now, in the induction wattmeter, OV must lead the flux vector by 90°. and an inspection of the vector diagram will show that this result can be obtained by suitably adjusting the ampere-turns of the shielding coil, this adjustment being effected either by alteration of the resistance of the shielding coil, or by altering its axial position.

Polyphase, or double, wattmeters. These instruments are intended for measuring power in polyphase circuits by the two-wattmeter method (p. 202), and consist of two single-phase wattmeter mechanisms combined into one case and having a single pointer and scale. The instruments may be of either the electro-dynamic or the induction type.

In the *electro-dynamic instrument* the two moving coils are fixed to a common spindle which carries the pointer, control springs, and

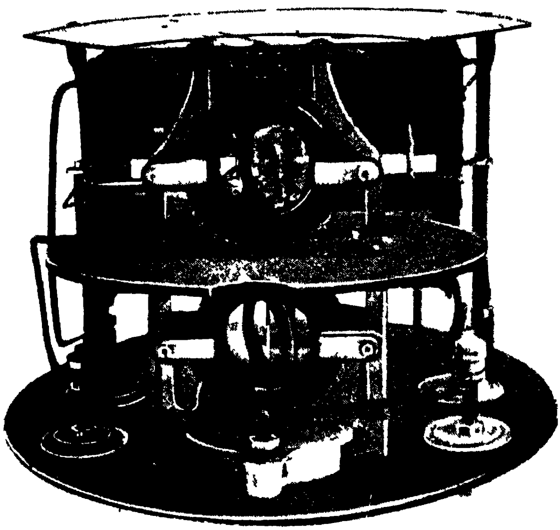


FIG 232 Interior of Weston Polyphase Switchboard Wattmeter, showing Laminated Magnetic Shield between Upper and Lower Elements

[NOTE An illustration of the mechanism of another switchboard polyphase wattmeter is given in Fig 225c]

damping vanes The coils may be arranged with their axes either in the same plane or perpendicular to each other, the former arrangement being usually adopted for switchboard instruments (Fig. 232) and the latter for standard instruments (Fig. 226*b*). The two sets of fixed coils are mounted one above the other (when viewed with the base of the instrument horizontal), and their axes are arranged either in the same plane or perpendicular to each other, according to the arrangement of the moving coils. Each set of fixed coils is provided with separate terminals and in three-phase

instruments (and also in two-phase instruments for three-wire systems) three terminals are provided for the moving-coil circuits, one terminal being common to both coils. With instruments for two-phase four-wire systems, four terminals are provided for the moving-coil circuits, as these cannot be interconnected in this case, the four ends of the moving coils being brought out, *via* the two control springs and two flexible ligaments, to the terminals and the series resistances.

With instruments in which the axes of the moving coils are arranged in the same plane, *mutual interference* may occur between the two systems, if unshielded, and precautions must be taken either to compensate such interference or to prevent its occurrence by magnetically shielding the systems from each other. The method of compensation for three-phase instruments consists of inserting a non-inductive resistance of suitable value in the common connection to the moving coils, thus forming a star-connected potential circuit whereby the magnitude and phase of the currents in the moving coils can be adjusted to compensate for the effects due to the mutual interference between the two systems of coils. The method is an application of the principle previously discussed in Chapter IX (p. 249). In modern instruments, however, the two systems are usually shielded from each other by means of a flat shield of laminated iron fixed between the upper and lower systems perpendicularly to the spindle (Figs. 225c, 232).

The *induction type* of polyphase wattmeter consists usually of two single-phase instruments, the two discs being fixed to a common spindle which carries the pointer and control spring. Each disc is provided with its operating and braking magnets, as in a single-phase instrument. In some instruments, however, only a single disc is employed, which is acted upon jointly by the two operating magnets, but with this form of construction interference between the operating magnets is liable to occur.

Marking of terminals of polyphase wattmeters. The terminals of the current and potential circuits of polyphase wattmeters require to be marked according to a definite scheme of connecting the instrument to the circuit. For example, in the Weston switchboard polyphase electro-dynamic wattmeter for three-phase circuits, the terminals of the upper fixed coils (i.e. those nearer to the pointer) are marked S_1, L_1, and the corresponding terminals of the lower set of fixed coils are marked S_2, L_2. The common terminal of the moving coils is marked V_3, and the terminals connected to the series resistances for the upper and lower moving coils are marked V_1 and V_2 respectively.

In connecting the instrument directly to a circuit the potential terminals having the subscripts 1 and 2 are connected to two of the line wires and the terminal V_3 is connected to the remaining line wire. The current terminals marked S_1, S_2 are connected to the *supply* side of the circuit, and those marked L_1, L_2 are connected to the *load* side of the circuit.

When instruments are used with current and potential transformers, a knowledge of the polarity of the transformers is necessary to enable the wattmeters to be connected correctly. Connection diagrams for a number of circuits are given in the author's *Power Wiring Diagrams*, pp. 114, 143, 144, 155, 156, 157.

WATT-HOUR OR ENERGY METERS

General requirements. In all cases of electric supply for power and lighting the charge to the consumer must be based upon the energy (in kilowatt hours) supplied. The measurement of this energy is effected by an integrating wattmeter operating on the induction principle, which principle is particularly suitable for house service meters on account of the lightness and robustness of the rotating element and the absence of rotating or moving contacts. Moreover, on account of the smallness of the variations of voltage and frequency in commercial supply systems* the accuracy of the induction meter is unaffected by such variations. The accuracy, however, is affected if the wave-form of the supply is badly distorted.

Theory. The induction energy meter may be derived from the induction wattmeter by substituting for the spring control and pointer an eddy current brake and a counting train, respectively. For the meter to read correctly, the speed of the disc must be proportional to the power in the circuit in which the meter is connected, and to fulfil this condition: (1) The torque due to the current generated in the disc by its rotation in the magnetic field of the operating magnets must be negligible in comparison with the operating torque; (2) the friction must be compensated at all speeds; and (3) the braking torque must be directly proportional to the speed of the disc.

Condition (1) is satisfied if the angular speed of the disc is very low in comparison with the angular speed of the travelling magnetic field, and in commercial meters the speed of the disc is of the order of 40 revolutions per minute at full load.

On account of the low speed of rotation of the moving system,

* The Board of Trade limits are: Voltage, $\pm$ 5 per cent of declared value; frequency, $\pm$ $2\frac{1}{2}$ per cent of declared value.

the friction after starting remains constant, and may be compensated by a constant torque acting in the same direction as the main driving torque. This compensating torque is obtained by producing a slight dissymmetry of the shunt flux, by means of an unsymmetrically placed shielding plate or coil.

With the friction compensated, and with correctly adjusted operating magnets, the resultant torque acting upon the disc will be proportional to the power in the circuit, and if the speed of the disc is to be proportional to this quantity, a braking torque varying directly as the speed, must be applied to the disc. The braking torque is produced by eddy-currents induced in the disc by its rotation in a magnetic field of constant intensity, the magnetic field being provided by one or two permanent magnets, so placed as to be unaffected by the alternating-current operating magnets.

For a given disc and brake magnet, the braking torque varies with the distance of the poles from the centre of the disc, the maximum torque occurring when the distance of the centre of the pole faces from the centre of the disc is equal to 83 per cent of the radius of the disc.* This feature is utilized when testing the meter to obtain a final adjustment of the speed to a definite value, corresponding to a given power.

Construction. Numerous forms of construction for house-service induction meters have been devised, the principal differences in construction being confined chiefly to the arrangement of the magnetic circuits of the operating magnets, the method of obtaining the correct phase difference between the fluxes of the series and shunt magnets, and the method of compensating for friction.

Two typical forms are illustrated in Figs. 233, 234. The operating magnets resemble those of the indicating wattmeters described on p. 402, but slight differences will be observed between the shunt magnets in Figs. 233, 234 and those in Fig. 229, these differences being due to the provision of devices for compensating friction. In the magnet shown in Fig. 233 this device consists of two adjustable loops of brass, *B*, *B*, which surround the lower polar projections of the shunt magnet; but in the magnet shown in Fig. 234 an unsymmetrically-placed shielding plate is employed, this plate being in the form of a small tongue of brass or copper placed under the central portion of the pole face of the shunt magnet and displaced slighty to the left of the position of symmetry.

In both instruments the rotating disc, *A*, of aluminium, is fixed to a vertical spindle, the lower end of which is supported by a

* *Electrical Measuring Instruments*, Drysdale and Jolley, Part I, p. 105. The theory of the eddy-current brake is given on p. 102 of this volume.

jewelled footstep bearing, and the upper end is supported by a guide bearing. To this end of the spindle is fixed a pinion, C, Fig. 233, which gears with an intermediate spindle fitted with a worm, from which the counting train is driven, the worm drive being necessary as the spindles of the counting train are horizontal. By fitting the worm to the intermediate spindle instead of the spindle on which the disc is mounted, the speed of the worm is reduced to a low value (about $3\frac{1}{3}$ revolutions per minute at full load), and therefore the friction is much less than that corresponding

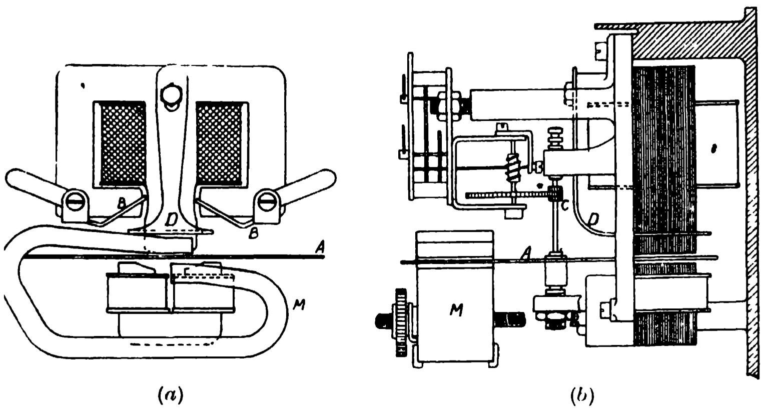


Fig. 233.—(a) Front Elevation, Showing Principal Parts of Induction Watt-Hour Meter. (b) Side Elevation of Meter, Showing Method of Driving Counting Train
[Metropolitan-Vickers Electrical Co.]

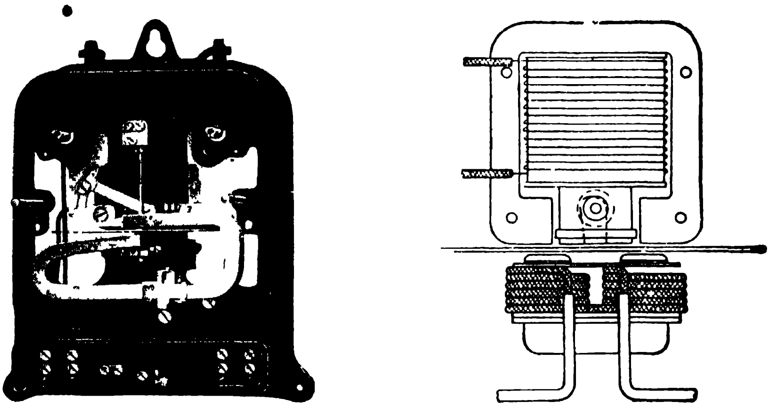
to a direct-driven worm. Moreover, with the worm fitted to the intermediate spindle, the jewelled bearings for the disc spindle are relieved from the thrust due to the worm.

The method by which the correct phase difference between the series and shunt fluxes is obtained is the same in both instruments, and consists of shielding the pole face of the shunt magnet by an adjustable loop of copper, D, Fig. 233. The loop is symmetrical with the shunt pole face, and the adjustment is in a direction perpendicular to the pole face.

The braking torque in both instruments is produced by a single permanent magnet, M, which is adjustable radially relative to the disc.

Adjustment of meter. The adjustments are effected when the meter is first tested, the method being as follows : Assuming correct voltage and frequency to be maintained, the meter is run at its full

load current alternatively on loads of unity power factor and a low lagging power factor, and the speed is adjusted to the correct value* by varying the positions of the brake magnet and the shielding loop. For example, if the meter runs fast on inductive load and correctly on non-inductive load, the shielding loop must be moved towards the disc. Again, if the meter runs slow on non-inductive load, the brake magnet is moved towards the centre of the disc.



(a) (b)

FIG. 234.—(a) Front View of Single-Phase Watt-Hour Meter with Dial Removed. (b) Detail of Magnets

[Ferranti, Ltd.]

[NOTE.—The adjustments for light load and inductive load can be seen in the front view of the meter]

The meter is next run on a light load (i.e. about $\frac{1}{20}$ of full load) and the friction compensating device is adjusted to give the correct speed. In making this adjustment, care must be exercised to see that friction is not over-compensated, otherwise the meter will run when only the shunt magnet is excited†. The adjustment should be such that: (1) The meter will start with a non-inductive load of 0·5 per cent of full-load; and (2) with a non-inductive load of

* For example, a 5 A. 200 V. meter may have a full-load speed (corresponding to 1,000 watts) of 40 revolutions per minute, in which case we have

correct speed of disc $= \frac{40}{1000} \times$ watts (by wattmeter) supplied to load.

† With some meters this is prevented by a small iron tongue, or vane, fitted to the disc in such a position that when the tongue is adjacent to the brake magnet the attractive force between tongue and magnet is just sufficient to prevent rotation of the disc with full shunt excitation and no current in the series coil.

$\frac{1}{50}$th of full-load the speed is within 2 per cent of the correct speed at this load.

Accuracy. In commercial meters manufactured to the specifications of the British Engineering Standards Association the permissible error is $\pm$ 2 per cent for all loads between $\frac{1}{20}$th of full

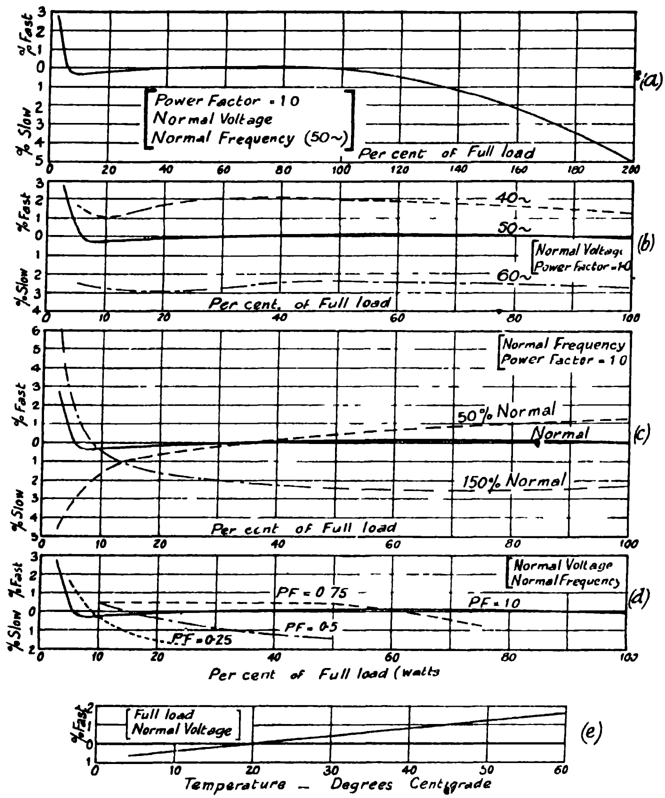


FIG. 235.—Typical Error Curves for Induction Watt-Hour Meter, Showing effect of (*a*) Normal Conditions ; Variation of—Frequency (*b*), Voltage (*c*), Power Factor (Current Lagging) (*d*), Temperature (*e*)

load and 25 per cent overload and at any power factor between unity and 0·5, lagging and leading. Typical error curves are shown in Fig. 235.

Effects of temperature variations. Energy meters are inherently almost free from errors due to temperature variations, as changes in the resistance of the disc affect both the driving and braking

torques in the same manner. Moreover, the error which would be caused by the diminution of the flux of the shunt magnet due to the increase in resistance of the shunt coil with increase of temperature is, to some extent, balanced by the decrease in the flux of the brake magnet. The resultant error at full-load, unity power factor, may be of the order + 0·05 per cent per 1° C. rise in temperature.

Although this error is practically unimportant under the conditions in which house service meters usually operate, it may become important in meters mounted upon switchboards and installed in factories; it is, of course, important in a precision instrument. A novel method of compensating the error consists in magnetically shunting the brake magnet by a magnetic alloy (of the iron-copper-nickel series) having a large temperature coefficient of permeability and an extremely low hysteresis loss.*

Polyphase watt-hour meters for unbalanced loads.† For the measurement of energy in polyphase circuits with unbalanced loads, a double-element watt-hour meter is usually employed. This meter consists essentially of two single-element meters with a common spindle and a single counting train. The series and shunt magnets of each element are connected according to the two-wattmeter method of measuring power in polyphase circuits, and each set of magnets acts upon a separate disc to avoid mutual interference. Each element is provided with the same adjustments as in a single-phase meter, and, in addition, one element is provided with an adjustment (usually a magnetic shunt fitted to the shunt magnet) for obtaining equality between the torques of the elements when the meter is operating with a balanced three-phase load of unity power-factor. A typical example is shown in Fig. 236(*a*).

Adjustments. The adjustment for equality of torque is usually effected on a single-phase circuit; the series coils of the elements are connected in series with each other and the shunt coils in parallel, one set of series coils being reversed in order that the torques may act in opposition. The magnetic shunt is then adjusted so that the torques due to the upper and lower elements balance each other when approximately full-load current is passing through the meter, the shunt excitation being normal.

The other adjustments for friction, non-inductive load, and

* See *Journal of the American Institute of Electrical Engineers*, vol. 44, p. 241. See also *Electrician*, vol. 92, p. 292, for an alternative method of temperature compensation.

† For the measurement of energy in three-phase circuits with balanced loads, a single-phase meter may be employed. See *Power Wiring Diagrams*, pp. 116, 118.

inductive load are usually effected by testing each element separately on a single-phase circuit, but when this procedure is adopted, normal shunt excitation must be maintained on *both* elements during the tests in order that the compensation for friction may be the same during the test as in service The methods of effecting the adjustments are similar to those employed with single-phase meters, but, since the braking-torque is due to the joint action of two brake magnets (one acting upon each disc), the adjustment for non-inductive load must be effected on each element alternately to determine the correct positions for both magnets.

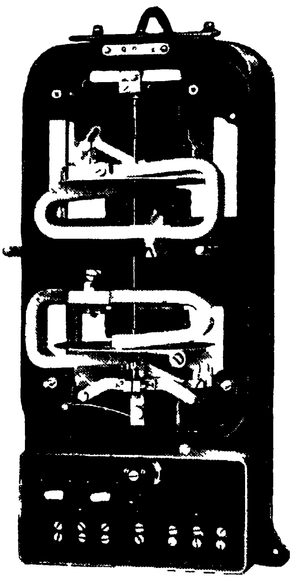
The correct inductive-load adjustment of each element is important. as any incorrect adjustment of either element may cause appreciable inaccuracies when the meter is operating on a three-phase circuit. For example, if the three-phase system is balanced and the power factor is unity, one element (No. 1) of the watt-hour meter is operating under conditions equivalent to a power factor of 0·866, leading, and the other element (No. 2) is operating under conditions equivalent to a power factor of 0·866, lagging. Hence, if element No. 1 is under-compensated (i.e. the element runs fast with a single-phase inductive (lagging) load), and element No 2 is over-compensated (i e. the element runs slow with a single-phase (lagging) inductive load), the meter will run slow on the three-phase circuit If, however, the elements are interchanged with reference to the phases, or line wires, of the three-phase system, the meter will run fast.

If the power factor of the three-phase system is $\cos\varphi$, lagging, one element of the meter will be operating under conditions equivalent to a power factor of $\cos(30^\circ - \varphi)$—leading or lagging, according to whether $\varphi < > 30^\circ$—and the other element will be operating under conditions equivalent to a power factor of $\cos(30^\circ + \varphi)$ lagging. But if the power factor is $\cos\varphi$, leading, the former element will be operating under conditions equivalent to a power factor of $\cos(30^\circ + \varphi)$, leading, and the latter element under conditions equivalent to a power factor of $\cos(30^\circ - \varphi)$ lagging or leading, according to whether $\varphi < > 30^\circ$.

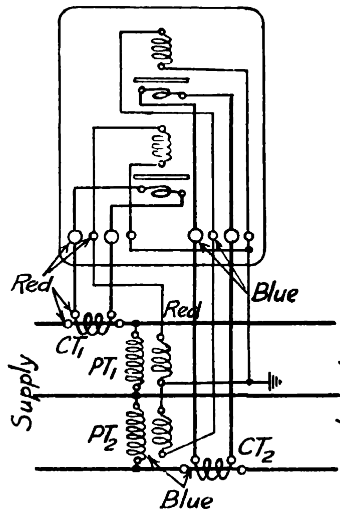
Marking of terminals of polyphase watt-hour meters. The necessity for a definite system of marking the terminals of polyphase watt-hour meters is apparent from the preceding section, and the British Engineering Standards Association have standardized the following scheme for marking not only the terminals of the meters but also the phases of the supply system, in order that no ambiguity shall exist about the manner in which the meter is to be connected in service. Moreover, the scheme also removes any ambiguity

concerning the connections of a meter when it is calibrated at the manufacturer's testing department.

The standard direction of phase rotation shall be counter-clockwise. The phase, or line wire, of reference of the supply system shall be numbered 1 and coloured white; the phase



(a)



(b)

FIG. 236 (a) Front View of Ferranti Polyphase Watt-Hour Meter for Unbalanced Loads (b) Connections for Polyphase Watt-Hour Meter when Used with Current and Potential Transformers

which lags 120° shall be numbered 2 and coloured blue, and the other phase (which lags 240°) shall be numbered 3 and coloured red.

In connection with power and energy measurements by the two-wattmeter method, the current coils of the instruments shall be connected in the "blue" and "red" phases, and the common terminal of the potential circuits shall be connected to the "white" phase. The "blue" phase is then the "leading" phase and the "red" phase is the "lagging" phase.*

* This designation of the "leading" and "lagging" phases follows from a consideration of the phase differences between the line currents and the voltages impressed upon the potential circuits of the instruments. For example, for the standard conditions and unity power factor, these phase differences are 30°, leading, for the element connected in the "blue" phase, and 30°, lagging, for the element connected in the "red" phase.

> The terminals of the element (of the meter) which is to be connected in the "leading" phase shall be numbered 2 and coloured blue; those of the element which is to be connected to the "lagging" phase shall be numbered 3 and coloured red.

When a meter has been calibrated with current and potential transformers, the terminals of these transformers are marked in a manner similar to those of the meter, to ensure that when the meter is put into service the connections between transformers and elements may be the same as when the meter was calibrated A diagram of connections for a typical case is given in Fig 236(*b*).

CHAPTER XIII

INSTRUMENTS FOR SPECIAL PURPOSES

(POWER FACTOR METERS, SYNCHROSCOPES, FREQUENCY METERS, GALVANOMETERS, OSCILLOGRAPHS)

In this chapter we shall consider a few of the instruments in general use which are employed for special purposes, such as the indication of power factor, phase coincidence and frequency, the detection of alternating currents, and the delineation of wave-form.

POWER-FACTOR METERS

General. The term power-factor meter (or phase meter) refers to an instrument for indicating directly, by a single reading, the power factor of the circuit to which it is connected. Instruments are available for indicating the power factor of either single-phase or polyphase circuits, under balanced and unbalanced loads.

Principles of operation. Power-factor meters operate on the electromagnetic principle, and are of either the moving-coil (electro-dynamic) or the moving-iron types. In general, both types have two electromagnetic systems; one system being supplied with current equal, or proportional, to the current in the circuit of which the power factor is to be measured, and the other being supplied with current proportional to the voltage of the circuit. The coils of one system are arranged to produce either a rotating magnetic field, or a number of alternating magnetic fields having a time-phase difference with respect to one another, the axes of the fields being displaced in space by angles equal to the time-phase angles. The coil, or coils, of the other system usually produce an alternating magnetic field, the axis of which is fixed in space.

The direction of the resultant magnetic field in space depends upon the relative positions of the axes of the rotating and the (single) alternating fields at the instant when the latter (i.e. the alternating field) attains its maximum value. Hence, since these fields are excited by currents proportional to the current and pressure of the circuit, the position of the resultant field in space will depend upon, and vary with, the phase difference between these quantities, i.e. with the power factor of the circuit.

The position of the resultant field is indicated by a pointer and scale, the pointer being fixed to a spindle, which, in the

moving-iron instrument, carries a set of moving-iron vanes, and in the electro-dynamic instrument carries the coils which produce the rotating magnetic field.

With both types of instrument the moving system is perfectly balanced and is free from controlling forces. Hence, when an instrument is disconnected from a circuit the pointer remains in the position which it occupied at the instant of disconnection. With these general remarks we will consider the theory of the moving-coil and moving-iron instruments.

Electro-dynamic single-phase power-factor meter. In this instrument the alternating magnetic field is produced by a fixed coil, or a pair of coaxial coils, supplied with current proportional to the current in the circuit.

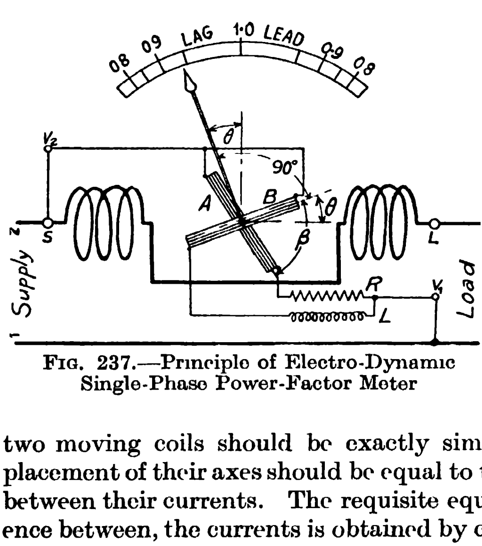


FIG. 237.—Principle of Electro-Dynamic Single-Phase Power-Factor Meter

The rotating magnetic field is produced by a pair of intersecting coils, which are fixed to a common spindle, the coils being supplied with equal currents, proportional to the pressure of the circuit, and having a phase difference of approximately 90°. The two moving coils should be exactly similar, and the angular displacement of their axes should be equal to the (time) phase difference between their currents. The requisite equality in, and phase difference between, the currents is obtained by connecting a non-inductive resistance in one circuit and a reactance, of the same ohmic value, in the other circuit, as shown in the connection diagram of Fig. 237. Obviously the currents in the moving coils will be affected by the frequency, since the ohmic value of the reactance will change with the frequency.

The current is led into and out of the moving coils by fine silver ligaments, which are so arranged as to exert no controlling force on the moving system.

Theory of single-phase electro-dynamic power-factor meter. The theory of this instrument is best developed by deducing expressions for the torque acting on each moving coil and obtaining the condition for equilibrium of the moving system. Thus, if the voltage of, and the current in, the circuit to which the instrument is connected are represented by $e = E_m \sin \omega t$, and $i = I_m \sin(\omega t - \varphi)$ respectively; the currents in the moving coils A, B

(Fig. 237) will be given by $i_A = (E_m/R)\sin\omega t$, and $i_B = (E_m/Z)\sin(\omega t - \gamma)$, respectively, where R denotes the resistance of the non-inductive, or "A," circuit, Z the impedance of the inductive, or "B," circuit, and γ the characteristic angle for this circuit (i.e. $\tan\gamma$ = reactance/resistance).

If the magnetic field produced by the current (i) in the fixed coil is assumed to be uniform in the space through which the moving coils are deflected, the instantaneous values of these torques, when the moving system is in equilibrium, are

$$\begin{aligned}\mathfrak{T}_{A(inst)} &= k i i_A \cos(\theta - \beta)\\ &= k\, I_m (E_m/R) \sin\omega t \sin(\omega t - \varphi)\cos(\theta - \beta)\\ &= kI\,(E/R)\cos(\theta - \beta)\,[\cos\varphi - \cos(2\omega t - \varphi)]\end{aligned}$$

for coil A, and

$$\begin{aligned}\mathfrak{T}_{B(inst)} &= -k\, i\, i_B \cos\theta\\ &= -k\, I_m\,(E_m/Z)\sin(\omega t - \gamma)\sin(\omega t - \varphi)\cos\theta\\ &= -k\, I\,(E/Z)\cos\theta\,[\cos(\gamma - \varphi) - \cos(2\omega t - \gamma - \varphi)]\end{aligned}$$

for coil B,

where k is a constant, β the angular displacement of the coils, and θ the angle between the magnetic axis of the fixed coil and the central plane containing coil B and the pivotal axis.

The mean values of the torques are

$$\mathfrak{T}_A = k\, I\,(E/R)\cos(\theta - \beta)\cos\varphi$$

and
$$\mathfrak{T}_B = -k\, I\,(E/Z)\cos\theta\cos(\gamma - \varphi)$$

Hence, since the moving system is mechanically balanced and is under no restraint from controlling forces, the position of equilibrium, on the assumption of no friction, must be such that the resultant torque is zero, i.e.

$$kI(E/R)\cos(\theta - \beta)\cos\varphi - kI(E/Z)\cos\theta\cos(\gamma - \varphi) = 0$$

Hence, if $R = Z$, and $\beta = \gamma$, we have

$$\cos(\theta - \gamma)\cos\varphi = \cos\theta\cos(\gamma - \varphi)$$

or $\quad \cos\theta\cos\gamma\cos\varphi + \sin\theta\sin\gamma\cos\varphi = \cos\theta\cos\gamma\cos\varphi + \cos\theta\sin\gamma\sin\varphi$

Whence $\quad \sin\theta\cos\varphi = \cos\theta\sin\varphi$

or $\quad \tan\theta = \tan\varphi \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (190)$

i.e. $\quad \theta = \varphi$

Similarly, if the current in the circuit is leading with respect to the line voltage (i.e. $i = I_m \sin(\omega t + \varphi)$, we have

$$-\tan\theta = \tan\varphi$$

or
$$-\theta = \varphi$$

Thus, for the conditions assumed, the angular displacement between the central plane of coil B and the axis of the fixed coil is equal to the phase difference between the voltage and current in the circuit to which the instrument is connected. Observe that the displacement is in one direction (counter-clockwise) when the current is lagging with respect to the voltage, and is in the other direction (clockwise) when the current is leading. The scale of the instrument, however, is usually marked to give the power factor ($\cos\varphi$) directly, and, under the conditions assumed above, the scale divisions are proportional to the cosine of the displacement angle, θ.

If the "supply" and "load" are interchanged (i.e. if the direction of current in the fixed coil is reversed) the position of equilibrium of the moving system will be 180° from the above positions, i.e. the pointer will travel in the lower quadrants of Fig. 237.

Owing to the ligaments, the arc of movement must be restricted to less than 360°, and, in practice, the scale is confined to the upper quadrants, as in Fig. 237.

In connecting the instrument to a circuit, care must be exercised to see that the "supply" and "load" are connected to the correct ends of the fixed coil, as denoted by the markings of these terminals.

The above theory shows that the indications of the instrument are unaffected by variations of current and voltage. But, in practice, the accuracy may be affected if the current and voltage are much below normal, as, under these conditions the frictional torque may be comparable with the resultant torque due to a change in power factor. With high-class instruments the accuracy at normal voltage and frequency is unaffected until the current in the fixed coil is below about one-fifth of the normal current for which the instrument is designed. The instrument readings, however, as already shown, are affected by frequency variations and also wave-form distortion.

Electro-dynamic polyphase power-factor meters for balanced loads. Instruments for *two-phase* and *four-phase systems* closely resemble

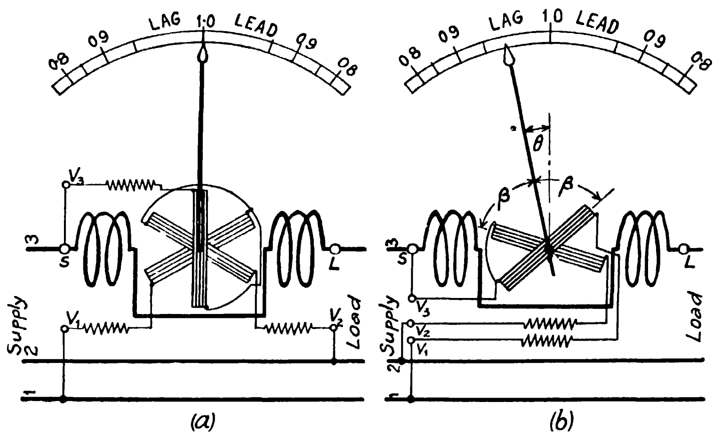


FIG. 238.—Principle of Electro-Dynamic Three-Phase Power-Factor Meters for Balanced Loads. (*a*) Instrument with Three Moving Coils. (*b*) Instrument with Two Moving Coils

the single-phase instrument described above, but the reactance in the "B" moving-coil circuit is replaced by an equivalent resistance. Moreover, if a cosine-law scale is required, the angular displacement of the coils (β) must be exactly 90°, and these coils must be connected across line wires between which the E.M.Fs. have a phase difference of 90°.

Instruments for *three-phase systems* are constructed with a fixed coil and with either two or three moving coils, connected as shown in Fig. 238. When three moving coils are employed they are symmetrically spaced with respect to one another, but when two coils are employed the angular spacing may be chosen according to the type of scale required. For example, with a spacing of 120° a normal cosine-law scale is obtained, as with a single-phase

instrument, but if the angular spacing is increased, certain parts of the scale are opened out (i.e. the range in power factor for a given angular deflection of the pointer is diminished).

Theory of electro-dynamic power-factor meter for balanced three-phase circuits. Let 2β denote the angular spacing between the two moving coils and θ the angle of displacement of the pointer (which is symmetrically placed with respect to the moving coils) from the central plane perpendicular to the axis of the fixed coil (Fig. 238*b*) when the moving system is in equilibrium. Then, if the supply system is symmetrical, and the line voltages are given by

$$v_{1\text{-}2} = V_m \sin(\omega t + \tfrac{1}{6}\pi),\ v_{2\text{-}3} = V_m \sin(\omega t - \tfrac{1}{2}\pi),\ v_{3\text{-}1} = V_m \sin(\omega t - \tfrac{7}{6}\pi)$$

the voltages impressed upon the moving-coil circuits will be given by

$$v_{3\text{-}1} = V_m \sin(\omega t - \tfrac{7}{6}\pi)\,; \qquad v_{3\text{-}2} = V_m \sin(\omega t - \tfrac{1}{2}\pi).$$

If R is the resistance of each of the moving-coil circuits, and self and mutual inductances are negligible, the currents in the coils will be

$$i_A = (V_m/R)\sin(\omega t - \tfrac{7}{6}\pi)\,; \qquad i_B = (V_m/R)\sin(\omega t - \tfrac{1}{2}\pi).$$

The current in the fixed coil is given by $i = I_m \sin(\omega t - \tfrac{4}{3}\pi - \varphi)$, where $\cos\varphi$ is the power factor of the three phase circuit.

Hence the instantaneous torques acting on the moving coils are

$$\begin{aligned}
\mathfrak{T}_{A(inst)} &= k\, i\, i_A \sin(\beta + \theta) \\
&= k\, I_m\, (V_m/R)\sin(\beta + \theta)\sin(\omega t - \tfrac{4}{3}\pi - \varphi)\sin(\omega t - \tfrac{7}{6}\pi) \\
&= k\, I\, (V/R)\sin(\beta + \theta)\,[\cos(\tfrac{1}{6}\pi + \varphi) - \cos(2\omega t - \tfrac{1}{2}\pi - \varphi)] \\
\mathfrak{T}_{B(inst)} &= -k\, i\, i_B \sin(\beta - \theta) \\
&= -k\, I_m\, (V_m/R)\sin(\beta - \theta)\sin(\omega t - \tfrac{4}{3}\pi - \varphi)\sin(\omega t - \tfrac{1}{2}\pi) \\
&= -kI\, (V/R)\sin(\beta - \theta)\,[\cos(\tfrac{1}{6}\pi - \varphi) - \cos(2\omega t - \tfrac{5}{6}\pi - \varphi)]
\end{aligned}$$

and the mean torques are

$$\mathfrak{T}_A = k\, I\, (V/R)\sin(\beta + \theta)\cos(\tfrac{1}{6}\pi + \varphi)$$

$$\mathfrak{T}_B = -k\, I\, (V/R)\sin(\beta - \theta)\cos(\tfrac{1}{6}\pi - \varphi)$$

Since the condition for equilibrium of the moving system is that $\mathfrak{T}_A - \mathfrak{T}_B = 0$, we have

$$\sin(\beta + \theta)\cos(\tfrac{1}{6}\pi + \varphi) = \sin(\beta - \theta)\cos(\tfrac{1}{6}\pi - \varphi)$$

i.e. $\quad \sin(\beta + \theta)\,[\tfrac{\sqrt{3}}{2}\cos\varphi - \tfrac{1}{2}\sin\varphi] = \sin(\beta - \theta)\,[\tfrac{\sqrt{3}}{2}\cos\varphi + \tfrac{1}{2}\sin\varphi]$

or $\quad \sqrt{3}\cos\varphi\,[\sin(\beta + \theta) - \sin(\beta - \theta)] = \sin\varphi\,[\sin(\beta - \theta) + \sin(\beta + \theta)]$

$\therefore \quad \sqrt{3}\cos\varphi\cos\beta\sin\theta = \sin\varphi\sin\beta\cos\theta$

Whence $\quad \tan\theta = (1/\sqrt{3})\tan\beta\tan\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (191)$

If the angular displacement of the coils is 120°, $\beta = 60°$ and $\tan\beta = \sqrt{3}$. Hence

$$\tan\theta = \tan\varphi$$

and $\quad \theta = \varphi$

Thus the angular deflection of the pointer from the plane of reference is equal to the phase difference between the phase E.M.F. and current in the circuit to which the instrument is connected.

When the angular displacement of the coils differs from 120°, the angular deflections of the pointer are no longer equal to φ, and the scale of power factor no longer follows the cosine law. Examples of scales for three values

of β are given in Fig. 239, from which it will be seen that, by making β greater than 60°, the portion of the scale in the vicinity of unity power factor is opened out, but that the range of power factor corresponding to a given full-scale deflection is reduced. Conversely, by making β less than 60°, the range of power factor corresponding to a given full scale deflection is increased, but the openness of the scale is reduced.

Observe that the indications of the instrument are unaffected by variations of frequency: they are also unaffected by wave-form distortion.

Comparison of theoretical and practical scale shapes. The form of scale deduced theoretically in the preceding sections refers to an ideal instrument in which: (1) The magnetic field produced by

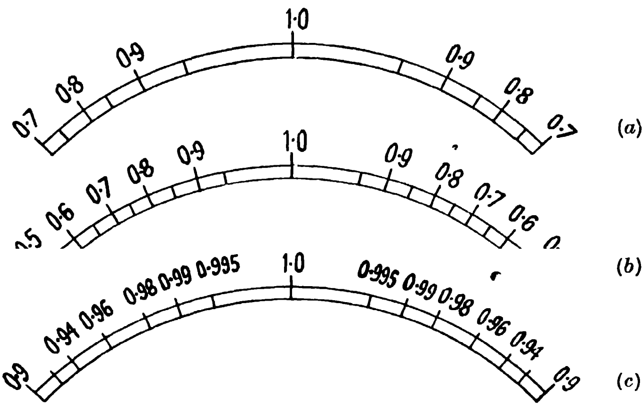


FIG. 239.—Forms of Scale for Electro-Dynamic Three Phase Power-Factor Meter with Two Moving Coils. Angular Spacing Between Coils is 120° for Scale (*a*), 90° for Scale (*b*), and 150° for Scale (*c*)

the fixed coil is (*a*) in phase with the current (i.e. there are no eddy currents in the conductors or coil supports), (*b*) uniform over the space traversed by the moving coils when the latter are deflected over the full scale; (2) the moving coils are identical in dimensions and number of turns, and their self and mutual inductances are negligible in comparison with the resistances of the circuits; (3) the friction is zero; (4) the reactance employed in the single-phase instrument has no losses and produces no distortion of the wave-form of the current.

The agreement between the form of scale of a commercial instrument with the theoretical form of scale will depend upon the closeness with which the actual instrument approaches the ideal. For example, in a typical high-class instrument (such as that

manufactured by the Weston Electrical Instrument Co.), the several layers of the moving coils are interlaced at the diametrical crossing points, so that each coil has the same mean diameter. The coils are extremely short and of light weight, and the moving system is pivoted in jewelled bearings. Moreover, in the polyphase instruments the values of the series resistances in the moving-coil circuits are such that the self and mutual inductances of these circuits are negligible. Again, the supports for the fixed coils are constructed of a material having a high specific resistance, and the amount of metal located in the magnetic field is reduced to a minimum, so that the eddy-current losses are very small. These supports are of similar design to those of the polyphase wattmeter illustrated in Fig. 232.

In consequence of these features, the scale is almost identical with that for an ideal instrument, as is shown by the following data—

Deflection $\theta°$	0	5	10	15	20	25	30	35	40	43
Actual scale marking ($\cos\varphi$)	1·0	0·985	0·942	0·88	0·808	0·735	0·665	0·599	0·537	0·5
Theoretical scale marking* ($\cos\varphi'$)	1·0	0·987	0·9505	0·895	0·8275	0·77	0·682	0·605	0·54	0·5

Calibration. Since the indications of the polyphase electrodynamic power-factor meter are unaffected by frequency variations, the instrument may be calibrated on a direct-current circuit. The normal current is passed through the fixed coil, and such voltages are applied to the moving-coil circuits that the torques corresponding to a given position of the moving system are the same as the mean torques under normal operating conditions. The mean torques under normal operating conditions are given by the equations on p. 421, thus

$$\mathfrak{T}_A = kI(V/R)\sin(\beta + \theta)\cos(\tfrac{1}{6}\pi + \varphi)$$

$$\mathfrak{T}_B = -kI(V/R)\sin(\beta - \theta)\cos(\tfrac{1}{6}\pi - \varphi)$$

and the torques with direct current passing through the coils are

$$\mathfrak{T}_A' = kI'I_A'\sin(\beta + \theta) = kI'(V_A'/R)\sin(\beta + \theta)$$

$$\mathfrak{T}_B' = -kI'I_A'\sin(\beta - \theta) = kI'(V_B'/R)\sin(\beta - \theta)$$

where I' is the current in the fixed coil and I_A', I_B', the currents in the moving coils, and V_A', V_B', the voltages applied to these circuits. The constant k has the same value for both direct- and alternating-current operation.

* Calculated from $\tan\varphi' = \sqrt{3}\tan\theta/\tan 43°$

Hence, with normal current in the fixed coil, we have

$$(V_A'/R)\sin(\beta + \theta) = (V/R)\sin(\beta + \theta)\cos(\tfrac{1}{6}\pi + \varphi)$$

or $$V_A' = V\cos(\tfrac{1}{6}\pi + \varphi) \quad . \quad . \quad . \quad (192)$$

and similarly $$V_B' = V\cos(\tfrac{1}{6}\pi - \varphi) \quad . \quad . \quad . \quad (193)$$

where V is the normal (alternating) operating voltage.

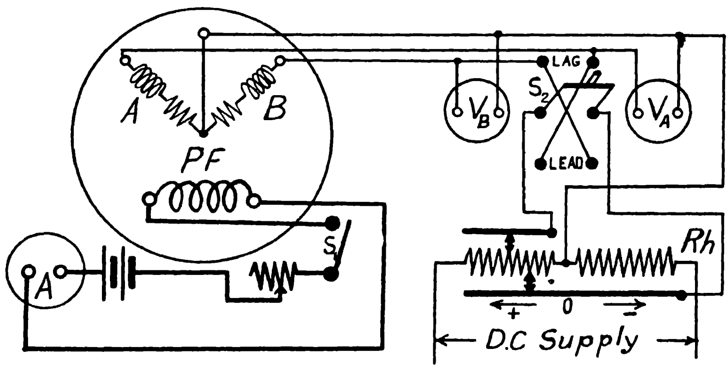


Fig. 240.—Connections for Calibrating Electro-Dynamic Three-Phase Power-Factor Meter

Thus, if the normal operating voltage is 100, the values of V_A' and V_B' to be applied to the moving-coil circuits to obtain the deflection corresponding to a given power factor (lagging) are

$\cos\varphi$.	1·0	0·95	0·9	0·8	0·7	0·6	0·5	0·4	0·3	0·2	0·1
V_A' .	86·6	81·16	56·13	39·33	24·61	11·9	0	−11·13	−21·74	−31·63	−41
V_B' .	86·6	91·17	99·74	99·29	96·35	91·93	86·6	80·4	73·65	66·34	58·43

To obtain the deflections corresponding to leading power factors, the moving-coil circuits must be interchanged with respect to V_A', V_B'. The connections are arranged as shown in Fig. 240; the moving-coil circuits are supplied—*via* a change-over switch and suitable rheostats—from a source having a voltage equal to twice the normal operating voltage of the instrument, and the fixed coil is supplied with current from a low-voltage battery.*

Moving-iron power-factor meters. These instruments may be divided into two sub-classes, according to whether the operation depends upon a rotating magnetic field or a number of alternating

* If calibration at power factors below 0·5 is not required the right-hand rheostat, *Rh*, Fig. 240 (which is necessary for obtaining the reversal of the voltage V_A'), may be dispensed with, in which case the voltage of the direct-current supply should be equal to the normal operating voltage of the instrument.

fields. Since, in both forms of instruments, all coils are stationary, the moving system may be given complete freedom of movement, and a scale extending over the full 360° of arc may, therefore, be employed as shown in Fig. 241.

The *essential features of a* **rotating field instrument** *for balanced polyphase loads* are shown in Fig. 242. The electrical portion of the instrument consists of (1) a set of coils, A, capable of producing a uniformly rotating magnetic field when supplied with suitable polyphase currents; and (2) a single coil, B, which is fixed coaxially inside A and is excited from one of the phases of the (polyphase) system. The outer coils, A, are surrounded by a laminated iron ring and are usually supplied, by means of current transformers, with currents proportional to the line currents in the polyphase system. The inner coil, B, together with a series resistance, is connected across two line wires of the system.

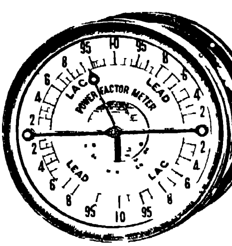


FIG 241 – Moving-Iron Form of Power Factor Meter with Scale Extending Over 360°
[Nalder Bros and Thompson]

The moving system consists of a pair of light iron vanes, C, C', fixed to the ends of an iron core and projecting perpendicularly from the latter in opposite directions, as shown in Fig. 242. The core and spindle are coaxial with the coils A, B and the vanes project over the ends of coil B. The iron core is fixed to a spindle which is pivoted in jewelled bearings and carries the pointer P and the light mica damping vanes D.

The moving system is perfectly balanced and is free from controlling forces. Hence, when the coils A, B, are excited, the iron vanes C, C, set themselves along the direction of the resultant M.M.F. due to these coils.

The *theory* of the instrument may be developed in a similar manner to that of the electro-dynamic instrument by considering the moving core and vanes to be magnetized, by the inner coil B, with current which is proportional to, and in phase with, the line voltage of the system. Then, if the effects of hysteresis and eddy currents are ignored, the core, vanes, and inner coil are equivalent electromagnetically to a rectangular moving coil pivoted within the outer coils, A, the centre line of the moving coil being coincident with the axis of the iron vanes.

Expressions for the torque acting upon such a coil are readily obtained, and it is easy to show that the angular deflection of the

coil from a given position is equal to the phase difference between the voltage and current in the polyphase system.

In the actual instrument, however, the effects of hysteresis and eddy currents in the moving core and iron vanes, and the inductance of the inner magnetizing coil, B, cause the indications to be affected by variations of frequency and wave-form. Hence the instruments

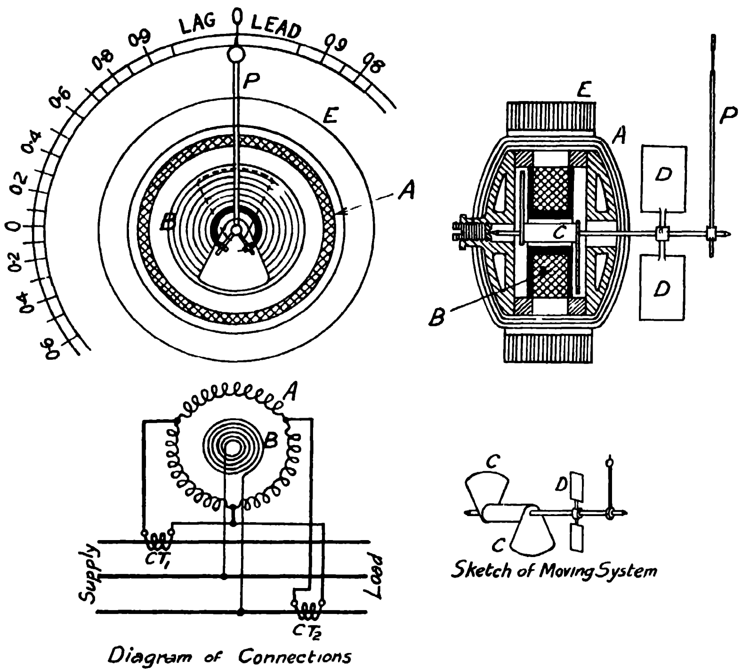


FIG. 242.—Moving-Iron Form of Power-Factor Meter in which a Rotating Magnetic Field is Employed

must be calibrated at the particular frequency at which they will be used.

The *essential features of* **alternating-field** *instruments* (due to Lipman, and manufactured by Messrs. Nalder Bros. and Thompson) for three-phase balanced and unbalanced loads are shown in Figs. 243, 244.*

In the instrument for *balanced* loads (Fig. 243) the moving system comprises three pairs of iron vanes and cores, C_1, C_2, C_3,

* See also "Power-Factor Meters," by F. E. J. Ockenden, *Electrical Review*, vol. 93, p. 164.

which are fixed to a common spindle pivoted in jewelled bearings, the spindle also carrying the pointer, P, and mica damping vanes, D.

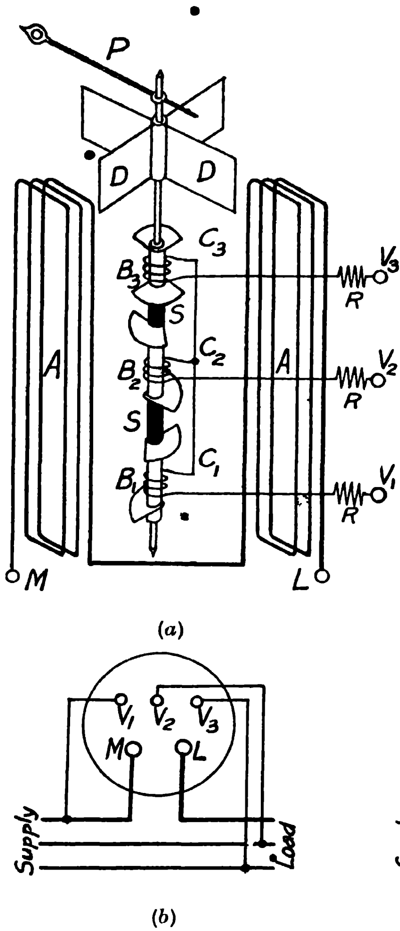


FIG 243

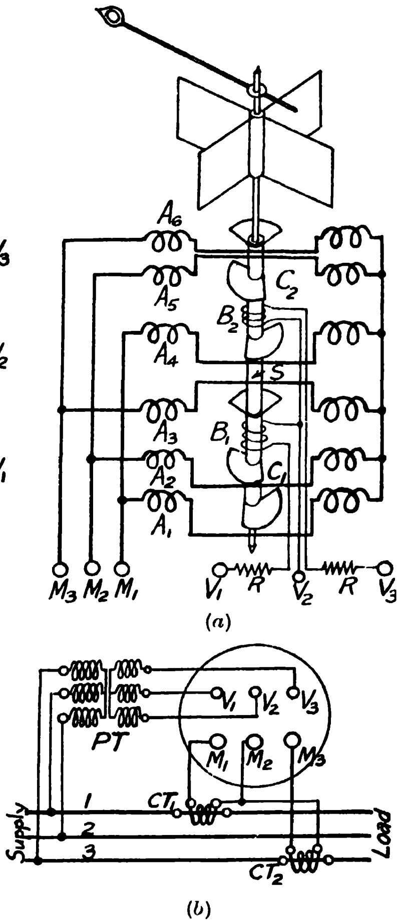


FIG 244

(*a*) Principle of Moving-Iron, Alternating Field, Three Phase Power-Factor Meters for Balanced and Unbalanced Loads.
(*b*) Diagrams of External Connections

The vanes are sector-shaped—the arc subtending an angle of 120°—and the vanes forming each pair (which are magnetically connected together by the core) are fixed 180° apart (as in the rotating-field instrument, Fig. 242). The cores are separated on the spindle by

distance pieces, S, of non-magnetic material, and the axes of symmetry of the three pairs of vanes are displaced 120° with respect to one another.

The cores and vanes are magnetized by the fixed coaxial pressure coils B_1, B_2, B_3, which are mounted coaxially with the spindle and are excited with currents proportional to the phase voltages of the three-phase system.

The current coil, A, is wound in two equal sections, which are mounted parallel to each other on opposite sides of the spindle, with their magnetic axes lying in a plane passing through the pivotal axis of the moving system. The two sections are connected in series or parallel, according to the current range of the instrument, and are supplied with current proportional to the current in one of the line wires of the three-phase system.

In the position of equilibrium of the moving system the resultant mean torque is zero, i.e. the mean torque acting on one pair of vanes is balanced by the mean torques due to the other pair of vanes. Under these conditions, and if the effects of hysteresis, eddy-currents, and inductance of the pressure-coil circuits are ignored, it can be shown that, for the pair of vanes which are magnetized from the same phase as the fixed coil, the angle of displacement of the axis of symmetry of these vanes is equal to the phase difference between the phase voltage and current of the three-phase system.

The instrument for *unbalanced* loads operates on the same principle as, but the construction differs in a number of features from, the instrument for balanced loads, since the moving system must be acted upon by the currents in all line wires. The essential features are shown in Fig. 244.

The moving system consists of two sets of vanes, C_1, C_2, each set comprising three 120° sectors fixed with their axes of symmetry 120° apart and magnetically connected together by iron cores; the two sets of cores being separated on the spindle by a distance piece of non-magnetic material, S. The cores and vanes are magnetized by the two coaxial pressure coils B_1, B_2, which, together with suitable non-inductive series resistances, are connected across the line wires of the three-phase system.

Six current coils, A_1–A_6 are employed, three coils being provided for each set of vanes. Each group of (three) coils is star-connected, and the two groups are connected in parallel and supplied, by means of current transformers, with currents proportional to the currents in the line wires of the three-phase system. Alternatively, for low-voltage circuits and currents not exceeding 30 A., the star

connections may be omitted, in which case the corresponding coils of each group are connected in parallel with each other and inserted directly in the line wires of the three-phase system.

The current coils are of the flat type; they are mounted parallel to one another and adjacent to the vanes, the magnetic axes of the coils being perpendicular to the spindle and coplanar with the vanes.

The mean resultant torque acting on one set of vanes is due to the combined effects of the currents in all line wires and the voltage across one pair of line wires, while the mean resultant torque acting on the other set of vanes is due to the combined effect of the currents in all line wires and the voltage across the other pair of line wires. These torques balance each other in the position of equilibrium of the moving system, and this position, therefore, corresponds to the average phase difference between the phase voltages and currents for the three-phase system.

The terminals are marked in accordance with the B.E.S.A. standard scheme (p. 416), and a knowledge of the phase rotation of the supply system is necessary when the instrument is connected in circuit.

The instrument for *two-phase unbalanced circuits* is similar in general construction to the instrument for three-phase circuits, but each set of vanes now comprises two, instead of three, vanes, which take the form of quadrants and are fixed with their axes of symmetry mutually at right angles, as shown in Fig. 252, which refers to a deflectional frequency meter. Two magnetizing, or pressure, coils, and four current coils are employed.

In the *single-phase instrument* the moving system is identical with that of the two-phase instrument. The two magnetizing coils are supplied, from the single-phase system, with currents having a phase difference of 90°, a reactance being connected in series with one coil and a non-inductive resistance in the other coil. A single current coil, wound in two sections, is employed, and is mounted in the same manner as the current coil of the three-phase balanced-load instrument (Fig. 243).

SYNCHROSCOPES

A synchroscope, or synchronism indicator, is a special form of phase meter for indicating the phase coincidence of the E.M.Fs. of two alternators, or two alternating-current supply systems, which are to be operated in parallel.

Requirements. When an alternator is to be operated in parallel with other alternators already in service, the switch connecting the incoming machine to the bus-bars must be closed at the instant

when the E.M.F. of this machine is equal to, and is in phase-opposition with, the E.M.F. at the bus-bars, and the frequency of the two E.M.Fs. has the same value. With a polyphase machine the direction of phase rotation must agree with that of the bus-bars.

The equality in the magnitudes of the E.M.Fs. is indicated either by two voltmeters or a differential voltmeter (called a "paralleling" voltmeter). The latter consists of two moving systems fitted to a common spindle and acted upon differentially by operating coils excited from the bus-bars and the incoming machine respectively.

The coincidence of phase* may be indicated in a number of ways, such as by incandescent lamps suitably connected; by a combination of vibrating reeds; and by a special form of phase meter. Two important requirements—especially in large generating stations where the consequences resulting from incorrect closing of the switch are very serious—are that the coincidence of phase shall be indicated accurately, and that accurate indications shall be given of small phase differences in the E.M.Fs. of the incoming machine and bus-bars. These requirements are satisfied only with the phase-meter type of indicator, in which precautions have been taken to secure sensitiveness and low inertia of the moving system. Moreover, this type of indicator also indicates whether the frequency of the incoming machine is lower than, equal to, or higher than, the frequency at the bus-bars.

With polyphase alternators and systems, the phase coincidence is indicated by a single-phase instrument connected across corresponding line wires of the bus-bars and incoming machine. The direction of phase rotation of the incoming machine must agree with that of the bus-bars, and this must be tested before any attempt is made to parallel the machine. Methods of testing the phase rotation are given in Chapter XV.

Principles of operation. The electromagnetic principle of operation is employed in the phase-meter type of synchroscope. Both the moving-iron and the electro-dynamic forms of construction are employed in practice, but the moving-iron form of instrument is in more general use, as the moving system can be given complete freedom of movement, and the pointer can travel over the full 360° of arc, as shown in Fig. 245. To obtain accurate indications from a moving-iron instrument, however, the effects

* Although the E.M.Fs. are actually in phase-opposition relative to each other, it is convenient to consider that they are in phase but that one is reversed relatively to the other. The term "phase coincidence," as employed in connection with synchroscopes, is to be interpreted in this manner.

of hysteresis and eddy currents in the moving-iron elements must be reduced to a minimum, and the inertia and friction of the moving system must be as small as practicable.

The moving-iron instrument may operate with either a rotating magnetic field or an alternating magnetic field. When a rotating field is employed, the moving-iron vanes are liable to be acted upon by the rotating field, as in an induction instrument, and precautions must be taken to reduce to a negligible amount the torque due to this effect. Instruments which operate with an alternating field are free from this defect

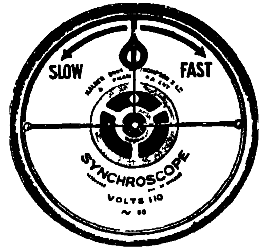


FIG 245—External View of Moving-Iron Synchroscope [Nalder Bros. and Thompson]

Construction. Two examples of modern instruments will be considered—a moving-iron instrument and an electro-dynamic instrument.*

A diagram showing the connections and essential parts of the moving-iron synchroscope is given in Fig 246. This instrument is of the Lipman alternating-field type. The moving system is identical with that of a single-phase power factor meter, and the magnetizing coils B_1, B_2 and the method of obtaining the required phase difference in their exciting currents are also identical in the two cases. The magnetizing coils are excited from the incoming machine.

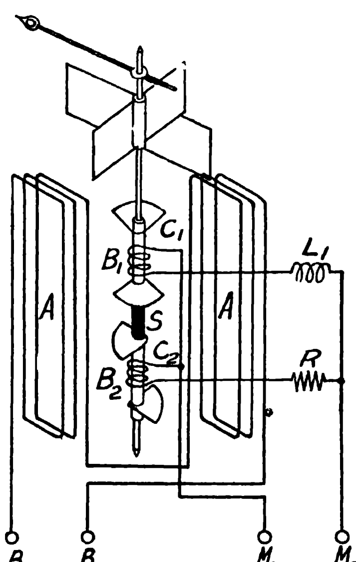


FIG. 246.—Principle of Moving-Iron Alternating-Field Synchroscope

The field coil, A, however, is wound with fine wire and, together with a non-inductive series resistance, is connected to the bus-bars.

A phantom view of the Weston electro-dynamic synchroscope is given in Fig. 247, and a diagram of the internal connections, together with the connections of the auxiliary apparatus, is given in Fig. 248.

* For other examples, including lamp synchroscopes, see *Power Wiring Diagrams*, pp. 124–128.

The mechanism of the synchroscope is identical with that of a single-phase wattmeter, in which the fixed coils are designed for a very small current, but the control springs are so adjusted that, with no current in the instrument, the axis of the moving coil is perpendicular to that of the fixed coils, i.e. the pointer is at the mid-scale position. The instrument is not provided with the usual scale and dial, but an opal dial is fitted in *front* of the pointer. The dial is marked with a central mark and also the words *fast*, *slow*, as shown in Fig 247. A small lamp, capable of illuminating the dial is fitted behind the pointer. This lamp is supplied from the secondary of a special transformer, *Tr*, Fig 248, which has two primary windings. One of the latter is excited from the bus-bars and the other from the incoming machine, and the connections are so arranged that the E.M.F. induced in the secondary winding is

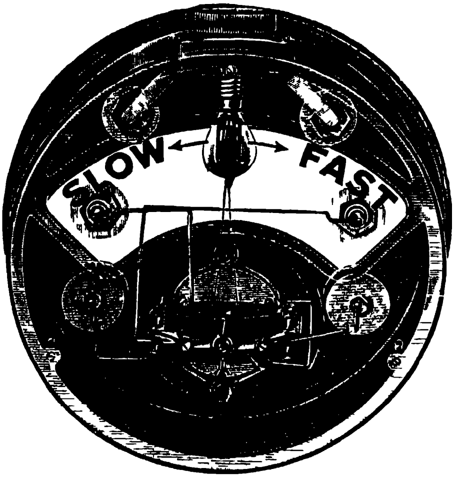


FIG. 247.—Phantom View of Weston Synchroscope showing Fixed and Movable Coils, Pointer and Lamp

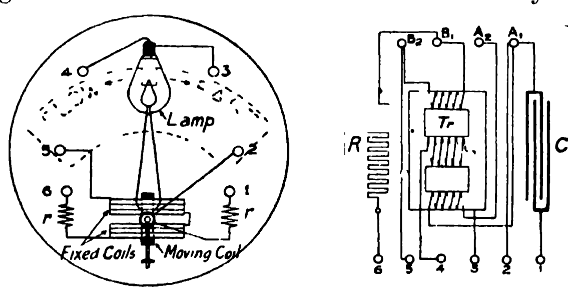


FIG. 248.—Internal Connections of Weston Synchroscope and Auxiliary Box

proportional to the vector sum of the E.M.Fs. applied to the primary windings.

The fixed coil, together with a non-inductive series resistance, *R*, Fig. 248, is connected to the bus-bars. The moving coil, together

with a condenser, C, in series, is connected to the incoming machine.

[NOTE. Both types of synchroscopes are usually supplied from transformers.]

Theory of action. (*a*) *Moving-iron instrument.* When the frequency of the incoming machine is equal to that at the bus-bars, the action of the moving-iron synchroscope is identical with that of the corresponding form of power-factor meter, the angular displacement of the pointer from the central position being proportional to the phase difference between the two E.M.Fs. When, however, the E.M.Fs. have different frequencies, the moving system rotates in one direction or the other, according to whether the frequency of the incoming machine is higher or lower than that at the bus bars. The speed of rotation, in revolutions per second, is equal to the difference between the two frequencies.

The theory of action can be developed mathematically by considering the equivalent ideal electro-dynamic instrument, in which moving system is free to rotate, the axes of the moving coils are perpendicular to each other, the phase difference between the currents in these coils is 90°, and the frequency of these currents is equal to that of the incoming machine. The fixed coil is excited with current proportional to, in phase with, and of the same frequency as, the E.M.F. at the bus-bars.

Hence if E_1, E_2 denote the E.M.Fs. applied to the fixed and moving-coil circuits, f the frequency at the bus-bars, $f \pm f'$ the frequency of the incoming machine; then, from Fig. 237, and the deductions on p. 419, we have, for the instantaneous torques,

$$\mathfrak{T}_{A(inst)} = k_1 E_{1m} E_{2m} \cos(\theta - 90^\circ) \sin[2\pi(f + f')t \pm \alpha] \sin 2\pi ft$$
$$= k_1 E_1 E_2 \cos(\theta - 90^\circ)\,[\cos(\pm 2\pi f't \pm \alpha) - \cos(2\pi(2f \pm f')t \pm \alpha)]$$
$$\mathfrak{T}_{B(inst)} = -k_1 E_{1m} E_{2m} \cos\theta \sin[2\pi(f + f')t - \tfrac{1}{2}\pi \pm \alpha] \sin 2\pi ft$$
$$= -k_1 E_1 E_2 \cos\theta\,[\cos(\pm 2\pi f't \pm \alpha - \tfrac{1}{2}\pi) - \cos(2\pi(2f \pm f')t \pm \alpha - \tfrac{1}{2}\pi)]$$

where α is the phase difference between the two E.M.Fs.

If $f' = 0$, i.e. the frequency of the incoming machine is equal to that at the bus-bars, the above expressions reduce to

$$\mathfrak{T}_{A(inst)} = k_1 E_1 E_2 \sin\theta\,[\cos \pm \alpha - \cos(4\pi ft \pm \alpha)]$$

and $$\mathfrak{T}_{B(inst)} = -k_1 E_1 E_2 \cos\theta\,[\sin \pm \alpha - \sin(4\pi ft \pm \alpha)]$$

Hence the mean torques are

$$\mathfrak{T}_A = k_1 E_1 E_2 \sin\theta \cos \pm \alpha$$

and $$\mathfrak{T}_B = -k_1 E_1 E_2 \cos\theta \sin \pm \alpha$$

The condition for equilibrium is that $\mathfrak{T}_A + \mathfrak{T}_B = 0$,

i.e. $$\sin\theta \cos \pm \alpha = \cos\theta \sin \pm \alpha \quad . \quad . \quad . \quad . \quad . \quad (194)$$

or $$\theta = \pm \alpha$$

Thus the pointer is stationary, and its deflection from the "zero" position (which is vertically upwards) represents the phase difference between the E.M.Fs.

If the frequency of the incoming machine differs slightly from that at the bus-bars, the torques tending to deflect the moving system will be given by

$$\mathfrak{T}_A = k_1 E_1 E_2 \sin\theta \cos(\pm 2\pi f't \pm \alpha)$$

and $$\mathfrak{T}_B = -k_1 E_1 E_2 \cos\theta \sin(\pm 2\pi f't \pm \alpha)$$

since the component torques having a frequency of $2f \pm f'$ will have no effect upon the moving system.

Hence, for equilibrium $\mathfrak{T}_A + \mathfrak{T}_B = 0$, or

$$\sin\theta \cos(\pm 2\pi f't \pm \alpha) = \cos\theta \sin(\pm 2\pi f't \pm \alpha) \quad . \quad . \quad (195)$$

i.e. $$\theta = \pm 2\pi f't \pm \alpha$$

Thus the moving system rotates with an angular velocity equal to that corresponding to the difference in the two frequencies (i.e. one revolution corresponds to a difference of one cycle in these frequencies), and the direction of rotation depends upon whether the frequency of the incoming machine is higher (f' positive) or lower (f' negative) than the frequency at the bus-bars.

(*b*) *Electro-dynamic* (*Weston*) *instrument. Case I. Frequency of incoming machine equal to that at bus bars.* Since the voltage impressed upon the illuminating lamp is proportional to the vector sum of the E.M.Fs. of the incoming machine and the bus-bars, the brilliancy of the dial will vary with the phase difference between these E.M.Fs.; the brilliancy being a maximum when the E.M Fs. are in phase and zero when the phase difference is 180°. Moreover, since the currents in the fixed and moving coils are in time-quadrature, the torque will be zero when the impressed E.M.Fs. are in phase and in phase opposition. The pointer and dial, however, are only illuminated when the E.M.Fs. are in phase*; and under these conditions the pointer will be seen stationary and vertical. If there is a phase difference of less than 90° between the E.M.Fs., and there is no difference in frequency, the pointer will be deflected to a position to the left or right of the central position and the dial will be illuminated. But if the phase difference is greater than about 120° the dial will be dark and the pointer will not be seen.

The theory of the action for the above conditions may be developed quite easily by deriving the expressions for the torque and the voltage impressed upon the lamp which illuminates the dial. Thus, considering an ideal instrument, the currents in the fixed and moving coils may be expressed as $i_1 = k_1(E_{1m}/R)\sin 2\pi ft$, and $i_2 = k_2fCE_{2m}\sin(2\pi ft \pm \alpha + \frac{1}{2}\pi)$, respectively where R denotes the resistance of the fixed-coil circuit (the reactance of which is assumed to be zero), and C the capacity connected in series with the moving coil, the resistance and inductance of this circuit being assumed to be zero. The instantaneous value of the deflecting torque corresponding to a deflection θ from the central position of the moving system is proportional to $i_1 i_2 \cos\theta$, and is given by

$$\mathfrak{T}_{D(inst)} = k_1k_2(E_{1m}E_{2m}fC/R)\cos\theta\sin 2\pi ft\sin(2\pi ft \pm \alpha + \tfrac{1}{2}\pi)$$
$$= k_1k_2(E_1E_2fC/R)\cos\theta[\cos(\pm\alpha + \tfrac{1}{2}\pi) - \cos(4\pi ft \pm \alpha + \tfrac{1}{2}\pi)]$$

and the mean torque by

$$\mathfrak{T}_D = k_1k_2(E_1E_2fC/R)\cos\theta\cos(\pm\alpha + \tfrac{1}{2}\pi)$$

Since the controlling torque ($\mathfrak{T}_C$) is due to a spring, we have, for equilibrium, $\mathfrak{T}_C = \mathfrak{T}_D$, i.e.

$$k'\theta = k_1k_2(E_1E_2fC/R)\cos\theta\cos(\pm\alpha + \tfrac{1}{2}\pi)$$

or $\theta/\cos\theta = K\sin\pm\alpha$ . . . . (196)

Hence, the deflection (θ) is zero when $\alpha = 0°$ or $\pm 180°$, and is a maximum when $\alpha = \pm 90°$ or $\pm 270°$. The following values show the relationship between θ and α for a given case, e.g. when the maximum deflection is 45° ($K = 1{\cdot}11$).

$\theta°$.	0	5	10	15	20	25	30	40	45
$\alpha°$.	0	4·5	9·2	14·1	19·6	25·7	33	56	90
	180	175·5	170·8	165·9	160·4	154·3	147	124	

Thus, for small deflections, the angular displacement of the pointer from the central position is practically equal to the phase difference between the E.M.Fs.

Consider now the illuminating lamp and assume that the E.M.F. of the incoming machine is equal to the voltage at the bus-bars. Then, if the

* The remarks concerning phase difference are to be interpreted in accordance with the footnote on p. 430

phase difference due to the transformer is ignored, the voltage impressed upon the lamp is

$$e = k[E_{1m} \sin 2\pi ft + E_{1m} \sin(2\pi ft \pm a)]$$
$$= kE_{1m} \sin 2\pi ft(1 + \cos a) \pm kE_{1m} \cos 2\pi ft \sin a \quad . \quad . \quad (197)$$

When $a = 0$, $e = 2kE_{1m} \sin 2\pi ft$; when $a = \pm 180°$, $e = 0$;

when $a = \pm 45°$, $e = kE_{1m}(1{\cdot}707 \sin 2\pi ft + 0{\cdot}707 \cos 2\pi ft)$
$= 1{\cdot}85\, kE_{1m} \sin(2\pi ft + 22{\cdot}6°)$;

and when $a = \pm 135°$
$$e = kE_{1m}(0{\cdot}293 \sin 2\pi ft + 0{\cdot}707 \cos 2\pi ft)$$
$$= 0{\cdot}764\, kE_{1m} \sin(2\pi ft + 60°)$$

The R.M.S. values of these voltages are $a = 0$, $E = 1{\cdot}41\, kE_1$; $a = \pm 180°$ $E = 0$; $a = \pm 45°$, $E = 1{\cdot}31\, kE_1$; $a = \pm 135°$, $E = 0{\cdot}54\, kE_1$.

Hence the dial will not be illuminated, and the pointer will not be visible when the phase difference between the E.M.Fs. exceeds about 135°.

Case II. Frequency of incoming machine not equal to that at bus-bars. The equation for equilibrium of the moving system is now

$$\theta/\cos\theta = K \sin(\pm 2\pi f't \pm a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (198)$$

where f' is the difference between the two frequencies. Hence the pointer will oscillate backwards and forwards over the dial, one complete oscillation corresponding to one cycle difference in the frequencies.

The voltage impressed on the illuminating lamp is given by

$$e = kE_{1m}[\sin 2\pi ft + \sin(2\pi(f \pm f')t \pm a)] \quad . \quad . \quad . \quad (199)$$

It can be shown that the illumination pulsates at a frequency f'; that if f' is positive (i.e. the frequency of the incoming machine is greater than that at the bus-bars) the dial is illuminated only when the pointer swings in one direction (clockwise); and that if f' is negative the dial is illuminated only when the pointer swings in the opposite (counter-clockwise) direction. Thus, when viewed at a distance, the pointer appears to be rotating in one direction or the other, according to whether the frequency of the incoming machine is higher or lower than that at the bus-bars.

In an actual instrument the condition for the currents in the fixed and moving coils to be in quadrature is that $\tan^{-1} \omega L/R = \tan^{-1} \omega Cr$, i.e. $Rr = L/C$ where L is the inductance of the fixed coil and r the resistance of the moving coil, the inductive reactance of the latter being negligible in comparison with the capacity reactance. Observe that this relation is independent of frequency. The adjustment is effected at the factory by connecting both circuits to a common supply and adjusting the resistances (r, Fig. 248) so that the pointer stands at the central position.

FREQUENCY METERS

Principles of operation. A number of principles may be employed such as (1) mechanical resonance, (2) electrical resonance, (3) variation of electrical characteristics with frequency of two parallel circuits containing resistance and inductance. Instruments operating on electrical principles form interesting examples of the application of the elementary principles discussed in the earlier chapters.

Mechanical resonance or vibrating-reed frequency meters. In these instruments a number of steel reeds, which are so tuned that

the natural frequencies of successive reeds differ slightly from one another, are acted upon by an electromagnet excited from the supply system. Only those reeds having a natural frequency approximately equal to that of the exciting current will show visible vibration, and, if this frequency coincides with the natural frequency of any particular reed, that reed will have a large amplitude of vibration. If the frequency of the exciting current lies between the natural frequencies of two adjacent reeds, these reeds will vibrate with approximately equal amplitudes, which, however, will be less than the maximum. Thus the accuracy with which the frequency can be measured depends upon the interval between the natural frequencies to which the successive reeds are tuned. This interval may be one alternation, one cycle, or two cycles, according to the range of the instrument and the accuracy required.

Theoretically, this type of instrument is unaffected by variations in the magnitude and wave-form of the exciting current or voltage, but, in practice, the amplitude of vibration of the reeds diminishes (owing to damping) as the exciting current, and voltage impressed on the instrument, is reduced below normal, and the indications become unreliable at voltages below about 70 per cent of normal. When instruments are required for use over a range of voltages, a special winding, with tappings connected to a multi-contact switch, is employed.

The frequency range of a given set of reeds may be doubled by magnetically polarizing the reeds by means of either a permanent magnet or an electromagnet excited with direct current. When the reeds are unpolarized they vibrate at a frequency equal to that of the flux alternations of the operating electromagnet, and, therefore, the reed which has a natural frequency *twice* that of the exciting current will have the maximum amplitude of vibration. But when the reeds, or the alternating-current electromagnet, are polarized magnetically, the magnetic effect of the operating magnet is neutralized at each alternate half-cycle by the polarizing magnet, and the reeds vibrate at the same frequency as the exciting current. Thus, in this case, the maximum amplitude is obtained on the reed for which the natural frequency is the same as that of the exciting current. Hence, if the instrument is direct reading with unpolarized reeds, its range will be doubled when the reeds are polarized.

Construction. Figs. 249, 250 show two forms of construction; the former (Fig. 249) is representative of an instrument having a double row of non-polarized reeds, and the latter (Fig. 250) is representative of an instrument in which the reeds are permanently

polarized by a permanent magnet. The methods of vibrating the reeds differ in the two cases. For example, in Fig. 249 the reeds are rigidly clamped to the base of the instrument and are attracted

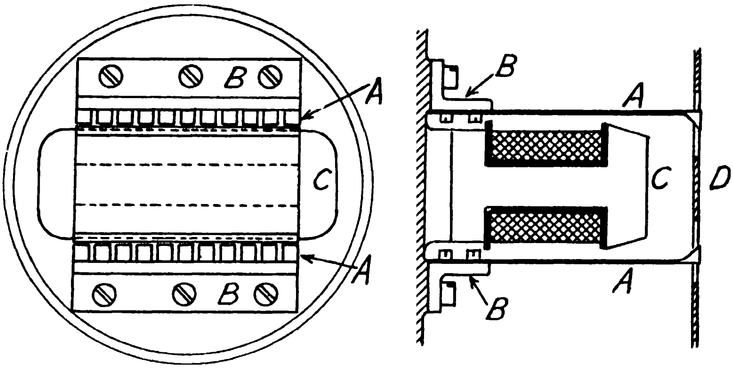


FIG 249 —Vibrating-Reed Frequency Meter with Non-Polarized Reeds. *A*, Reeds ; *B*, Supports ; *C*, Laminated Pole Face ; *D*, Dial

directly by the alternating-current electromagnet, the pole pieces of which extend the whole length of the double row of reeds. In Fig 250 the reeds are fixed to a bar, *B*, which is spring-supported

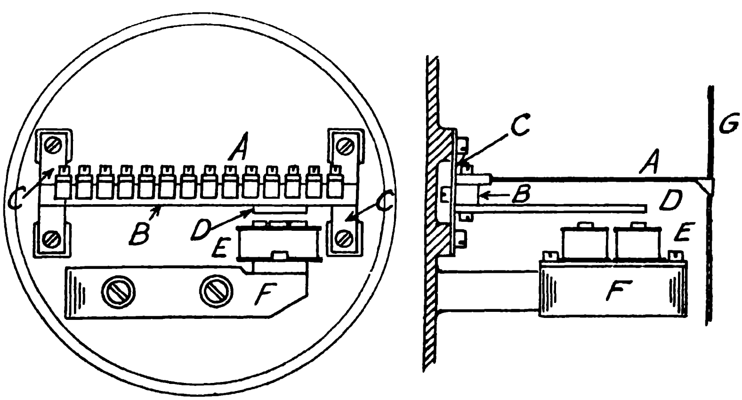


FIG. 250 —Vibrating-Reed Frequency Meter with Polarized Permanently Reeds. *A*, Reeds ; *B*, Reed-Supporting Bar ; *C*, Spring Supports ; *D*, Armature ; *E*, Exciting Coils ; *F*, Permanent Magnet ; *G*, dial

from the base of the instrument by transverse flat springs *C*, *C*. A soft-iron armature, *D*, is rigidly fixed to the bar *B* and projects towards the alternating-current electromagnet *E*, the poles of which are fixed to soft iron extensions of the poles of the permanent

horse-shoe magnet, F, which is supported from the base of the instrument. The exciting ampere-turns of the electromagnet are so adjusted that the magnetizing force due to a steady current in one direction is approximately neutralized by that due to the permanent magnet. Hence, when the electromagnet is excited with alternating current, the poles are effective only at alternate half-cycles of the current, and therefore the armature will be attracted once for each cycle of the current.

With both forms of instrument the reeds consist of spring steel about 4 mm. wide and 0·4 to 0·5 mm. thick, the lengths depending upon the frequency range. The outer end of each reed is bent at right angles to form a flag, the outer face of which is enamelled white, in order to render the vibrations visible from a distance. Unless the dimensions of the reeds are determined accurately to give the required frequency of vibration, the final adjustment is effected by weighting the free ends of the reeds by means of a little solder at the back of the flags.

Electrical resonance frequency meters. The operation of these instruments depends upon the principle that both the power and factor of, and the current in, a series resonant circuit supplied at constant voltage change with the frequency. At resonance frequency the power factor is unity and the current is a maximum; when the frequency is decreased or increased, the power factor becomes lagging or leading, respectively, and the current diminishes rapidly (see Fig. 51, p. 96).

Construction. The instruments are similar in construction to the moving-coil and moving-iron forms of single-phase power-factor meters. The operation of the moving-coil form of instrument may be made to depend upon either the variation of power factor, with frequency, of a single resonant circuit, or the variation of current, with frequency, in two resonant circuits which have different resonance frequencies and are connected in parallel. With moving-iron instruments the operation depends upon the variation of power factor, with frequency, of a single resonant circuit.

Moving-coil instrument. A diagram showing the connections of an instrument having two resonant circuits is given in Fig. 251. The principal resonance circuits are represented by $R_1\ L_1\ C_1$ $R_2\ L_2\ C_2$, the resonance frequencies of which are slightly below and slightly above the lower and upper limits of the scale of the instrument. These circuits, each of which includes a moving coil, are connected in parallel, and the fixed coil, A, is connected in series with them. The moving system is entirely free from controlling forces, as in a power factor meter.

When the frequency of the circuit to which the instrument is connected corresponds to that of the lower limit of the scale, the current in the circuit $R_2 L_2 C_2$, which includes the moving coil, B, will be much greater than that in the other circuit $R_1 L_1 C_1$, which includes the moving coil, C. Moreover, the current in B will be nearly in phase with the current in the fixed coil, A, while the current in C will have a large phase difference with respect to the currents in B and A. Hence the position of equilibrium of the moving system will be such that the plane of coil B is nearly perpendicular to that of the fixed coil. The conditions are reversed when the frequency corresponds to the upper limit of the scale.

At the mid-scale frequency the currents in the moving coils will be approximately equal, and will be lagging and leading by approximately equal phase angles with respect to the current in the fixed coil. Hence the moving coils will occupy symmetrical positions with respect to the fixed coil.

At abnormally low frequencies the currents in the moving coils will become approximately equal, and the moving system would tend, unless otherwise prevented, to occupy the symmetrical, or mid-scale, position, thereby giving a false indication. An auxiliary resonant circuit, $R_3 L_3 C_3$, having a resonance frequency much lower than that of the principal circuits (e g. 25 to 30 cycles for a 45/55 cycle instrument) is therefore connected in parallel with the circuit $R_2 L_2 C_2$, and the pointer is prevented from appearing on the scale.

The inductances L_1, L_2, L_3 of the resonant circuits are, for economical reasons, constructed with iron cores, but, in order that the resonance frequencies of these circuits may be sharply defined, the cores must be provided with air gaps, the iron must be worked at low flux densities, and the hysteresis and eddy-current losses must be reduced to a minimum.

Moving-iron, resonant circuit, frequency meter. The circuits of an instrument of the Lipman type are shown in Fig. 252. The moving system is practically identical with that of the synchroscope (Fig. 246), and the two magnetizing and field coils are also identical in the two cases.

The resonant circuit consists of the field coil, A, and a condenser of suitable capacity connected in series, the field coil itself providing the required inductance. This circuit is adjusted so that its resonance frequency corresponds to the mid-point of the frequency range of the instrument.

The deflection of the moving system from the central, or mid-scale, position is proportional to the phase difference between the

voltage impressed upon, and the current in, the resonant circuit, and the action of the instrument is similar to that of a single-phase power-factor meter.

An inductance, L_2, is connected in series with the instrument for the purpose of rendering its indications practically independent of the wave-form of the supply circuit.

Frequency meters with parallel circuits having dissimilar electrical characteristics. Frequency meters depending for their operation upon the variation of current, with change of frequency, in parallel-

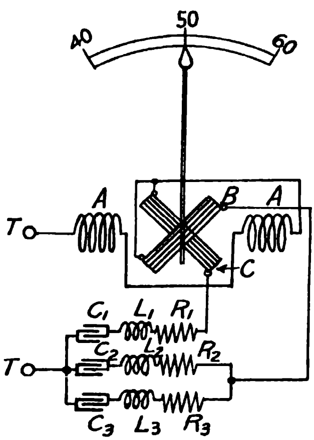


FIG. 251.—Principle of Moving-Coil Resonant-Circuit Frequency Meter

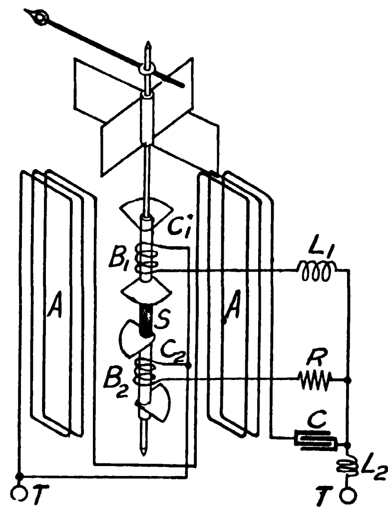


FIG. 252.—Principle of Moving-Iron, Alternating-Field, Resonant-Circuit Frequency Meter

connected circuits containing resistance and inductance, are constructed in both the induction and moving-iron forms.

In the **induction instrument,** Fig. 253, two similar shielded-pole electromagnets A, B (similar to those employed with induction-type voltmeters) act upon a pivoted aluminium disc, C, which is free from controlling forces. The magnets are so arranged that the torques produced upon the disc oppose each other. The winding of one magnet, B, has a non-inductive resistance connected in series with it, and the circuit of the other, A, is rendered highly inductive by means of an iron-cored reactance. Both circuits are connected in parallel, and an inductance is sometimes connected in series with the combined circuits to reduce the effect of harmonics on the exciting currents of the electromagnets.

Hence if the voltage of the circuit is constant and the frequency changes, the current in the magnet B will remain almost constant, while that in the other magnet, A, will vary inversely as the frequency. The disc will therefore tend to rotate, but it is shaped eccentrically and is arranged to present a diminishing surface to the magnet causing rotation, and an increasing surface to the other magnet. Thus a definite position of equilibrium is obtained for all frequencies within the range for which the instrument is designed.

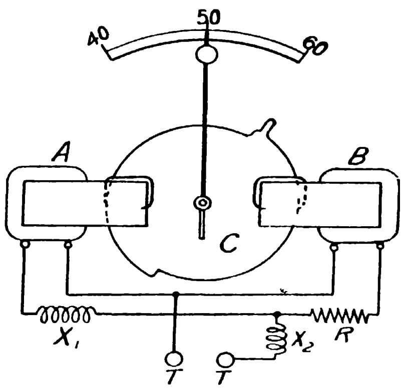


FIG. 253 —Principle of Induction Frequency Meter

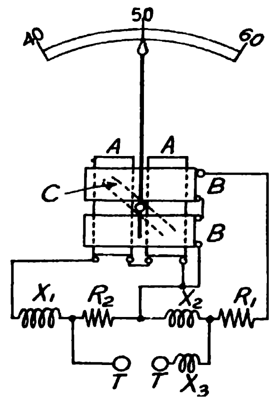


FIG. 254.—Principle of Moving-Iron Rotating (Elliptical) Field Frequency Meter

The indications are unaffected by moderate variations of voltage, since both magnets are affected equally. But variations of wave-form affect the currents in the magnets and impair the accuracy. With a suitable series inductance, however, the errors due to variation of wave-form may be considerably reduced.

In the **moving iron (Weston) instrument,** Fig. 254, two flat intersecting coils A, B, are fixed with their magnetic axes at right angles to each other. The moving system consists of a pointer, damping vanes, and a relatively long, thin, soft-iron needle, C, which is pivoted in the magnetic field of the coils A, B, and is free from controlling forces. The coils A, B, together with auxiliary resistances R_1, R_2, and reactances X_1, X_2, are connected to form a Wheatstone network, and a reactance, X_3, is connected in series with the network to render the indications of the instrument practically independent of the wave-form of the supply voltage.

By suitable adjustment of the resistances and reactances of the branch circuits, the ratio of the currents in the coils A, B is affected only by variations of frequency; the current in one coil increasing, and that in the other coil decreasing, as the frequency increases. The combined action of the currents in the coils sets up an elliptical rotating field, the position of the major axis of which depends upon the ratio of the currents in the coils, which again depends upon the frequency. The moving-iron needle sets itself along this major axis, and its angular position, therefore, changes as the frequency changes.

GALVANOMETERS

Two distinct classes of galvanometers for alternating-current measurements have been developed: (1) deflectional galvanometers, which are suitable for the direct measurement (by the angular deflection of the moving system) of the R.M.S. values of small currents and potential differences; (2) vibration galvanometers, which are employed for detecting purposes in connection with alternating-current bridges and potentiometers, these instruments being more sensitive for this purpose than deflectional instruments. Electromagnetic oscillographs are special forms of the vibration galvanometer and are considered in a separate section.

Deflectional galvanometers. These instruments operate on the electro-dynamic and electro-thermic principles. The electro-thermic instrument has already been considered, and only electro-dynamic instruments will be considered here. In general, these instruments consist of a fixed coil, or pair of coils, and a moving coil, which is suspended so as to move in the field produced by the fixed coil. The moving system carries a mirror to enable an optical method to be employed for measuring the deflection. The controlling force is provided by the suspension, and air damping, by means of mica vanes, is usually employed. The instruments may be constructed either with an air-cored or an iron-cored magnetic circuit, but with the former an electrically astatic moving system should be employed, in order to eliminate interference from stray magnetic fields.

Vibration galvanometers. These instruments are special forms of moving-coil (D'Arsonval) and moving-magnet (Thomson) galvanometers, in which the natural frequency of the moving system is tuned to the frequency of the circuit to which the instrument is connected. Under these conditions the resulting resonance vibration of the moving system may be very much greater than the forced vibrations which would occur if the natural frequency were

not the same as that of the current passing through the instrument. For example, with a typical galvanometer, tuned to a resonance frequency of 100 cycles per second, the current sensivity for a frequency of 99·5 cycles is about 30 per cent of that at resonance frequency, while for frequencies of 300 and 500 cycles the sensitivities are only about $\frac{1}{4000}$ and $\frac{1}{12000}$, respectively.

The vibration galvanometer may, therefore, be rendered extremely sensitive to currents having a particular frequency, and relatively insensitive to currents of other frequencies. Such an instrument is of great value as a detector in null methods of measurement (as employed in the alternating-current potentiometer and bridge networks) as it enables a balance to be obtained at one particular frequency, even if the supply E.M.F. is non-sinusoidal.*

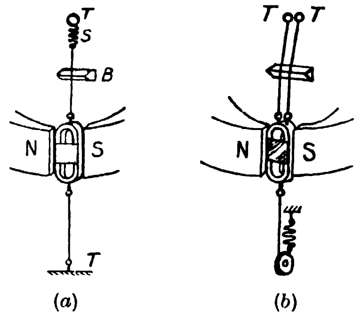


FIG. 255.—Principle of Unifilar (a) and Bifilar (b) Forms of Moving-Coil Vibration Galvanometer

The **moving-coil pattern** of vibration galvanometer may be constructed with either a bifilar or a unifilar suspension, the former being suitable for a greater range of frequency than the latter. The moving coil must have low inertia and the damping must be small. The coil is suspended in the magnetic field of a permanent magnet by phosphor-bronze wire or strip suspensions, as indicated in the sketches of Fig. 255. In both the bifilar and unifilar instruments the effective length of the upper suspension is adjustable by means of a movable bridge-piece, and the tension in the suspension as a whole is also adjustable. These two adjustments, viz., the length of, and the tension in, the suspension, enable the natural frequency of the instrument to be adjusted to any desired value within the working range. The moving coil carries a mirror for use with a lamp and scale.

The **method of tuning** a vibration galvanometer is as follows: The instrument having been set up on a support free from mechanical vibration, the lamp and scale are adjusted to give a bright and sharp reflected "spot" on the scale when no current is passing through the instrument. A small current, having a frequency to

* The theory and practice of bridge measurements is dealt with exhaustively in Hague's *Alternating-Current Bridge Measurements* (Pitman).

which the instrument is to be tuned, is passed through the galvanometer, the effect of which is to cause the "spot" to open out into a wide band of light across the scale. The length and tension of the suspension, or other tuning adjustment, is then varied until the band of reflected light on the scale attains its maximum breadth.

OSCILLOGRAPHS

An oscillograph is an instrument for showing, in either rectangular or polar co-ordinates, the wave-form of rapidly-recurring and transient phenomena. The outfit comprises, (1) a sensitive vibration galvanometer, which is operated by the current or potential difference to be determined and has a very high natural frequency of vibration; (2) an optical system, whereby the deflections of this instrument at successive instants are shown, with respect to a time axis, upon either a screen or a moving photographic plate.

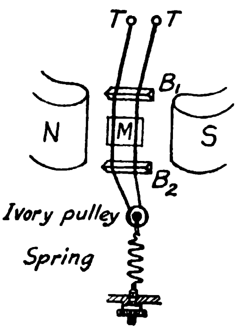


FIG. 256.—Principle of Duddell Oscillograph

It is essential that the deflections of the galvanometer shall be strictly proportional to the instantaneous values of the alternating quantity causing the deflection. Hence the inertia of the moving system must be very small, the natural frequency must be much higher than that of the phenomena to be investigated, the damping must be aperiodic, the self-induction and capacity must be as small as possible, and hysteresis and eddy-current effects must be negligible.

Principles of operation. The principles of operation employed in commercial oscillographs include the electromagnetic, electro-thermic, electrostatic, and thermionic principles. Of these, the electromagnetic and electro-thermic principles are suitable for low frequencies and low voltages, the electrostatic principle is suitable for low frequencies and high voltages, and the thermionic (cathode-ray) principle is suitable for the whole range of frequencies occurring in practice, this principle of operation possessing especial advantages for very high frequencies.

Electromagnetic oscillograph. The galvanometer may be of either the moving-magnet or the moving-coil patterns, the former being due to Blondel and the latter to Duddell. The moving-coil instrument possesses the advantage of practically no self-induction or capacity and is entirely free from hysteresis errors.

The moving coil, called the *vibrator*, consists of a single loop of thin and narrow phosphor-bronze strip: it is stretched between the pole faces of a powerful magnet as represented diagrammatically in Fig 256, the ivory bridge-pieces, B_1, B_2 being for the purpose of limiting the length of the vibrating portion of the strip to the portion actually in the magnetic field A very small and light mirror, M, is fixed to both strips A current passing through the loop will cause one side to advance and the other side to recede from the normal position. The mirror is therefore turned through a small angle, which, for the small movements which are employed in any actual instrument, is proportional to the current.

Construction of Duddell oscillograph. In the actual instrument —Fig. 257—each strip is stretched in a separate air gap, and a uniform tension is maintained on the strips by means of the ivory pulley and tension spring, the tension being adjustable. The clearances between the edges of the strips and pole faces are very small, being from 0·04 to 0 15 mm., according to the type and sensitiveness. Hence, when this space is filled with oil, very efficient damping is obtained.

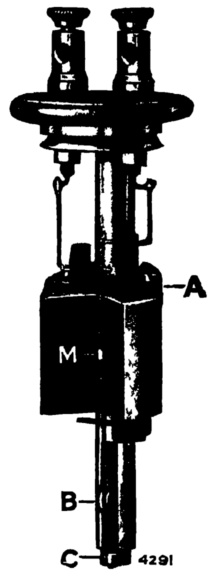


FIG. 257. -Vibrator of Duddell Oscillograph
[Cambridge Instrument Co.]

With the high-frequency* type of vibrator the natural period of vibration of the undamped loop is $\frac{1}{12000}$ second, and the damping effect of the oil is such that accurate results are obtained for sinusoidal currents of frequencies up to 2000 cycles per second. Two vibrators are usually mounted side by side between the pole faces of an electromagnet, with a fixed mirror between the vibrating mirrors. Thus, two wave-forms (e.g. current and voltage), together with a common zero line, may be obtained simultaneously. In an alternative form of construction each vibrator is mounted separately between the pole faces of a permanent magnet, so as to form a self-contained unit.

* The term "high-frequency" here refers to frequencies (sine wave) of the order of 2000 cycles per second.

Data of these vibrators (as manufactured by the Cambridge Instrument Co.) are—

Type.*	I.	II.	III.
Undamped free period of vibrator (sec.)	0 00008	0·0003	0·0005
Normal scale distance (cm.)	50	50	300
Scale deflection per ampere (direct current) with damping oil and normal tension on strips (cm.)	30	30	50
Safe working R M S current (amp.)	0·1	0·1	0·5
Resistance of vibrator, without fuse (ohms)	4 to 5	5 to 6	0 7 to 0·9

Optical system. The optical arrangements for an oscillograph vary with the nature of the record required. In general, a steady beam of light of high intrinsic brilliancy and having a fixed focus is necessary for illuminating the mirrors

If a photographic record is required, the beams of light reflected from the mirrors of the vibrators are focused on to a moving photographic plate or film, the direction of motion of which is perpendicular to the direction of movement of the reflected beams. The speed at which the plate or film moves across the reflected beams depends upon the nature of the phenomena under investigation.

For visual observations the beams of light reflected from the mirrors are intercepted by a vibrating or a rotating mirror, and are thereby given a uniform motion, proportional to time, about an axis which is in the plane of vibration of the beams and is at right angles to the zero positions of the beams. The doubly-reflected beams of light, if received upon a stationary screen, will then trace out the time curves for the variation of the currents in the vibrators.

If the variations of current are periodic, the vibrating, or rotating, mirror is operated in synchronism with the currents in the vibrators; e.g. by driving it by a small synchronous motor, and the reflected beams of light will trace out the same wave-form over and over again, which, therefore, remains stationary on the screen. When a vibrating mirror is employed, a shutter must be synchronized with the mirror to intercept the beam of light during the return movement of the mirror.

Method of using electromagnetic (Duddell) oscillograph. Since the resistance of a vibrator is about 5 ohms, and the safe working

* I. High-frequency vibrator and electromagnet. II. Low-frequency vibrator and permanent magnet. III. Projection type for demonstration purposes.

current of the strips is about 0·1 ampere, series resistances and shunts are necessary when the instrument is to be used on commercial circuits. The series resistance must be non-inductive and should be capable of carrying a current of 0·1 A. without overheating. The shunts must also be non-inductive and the pressure drop at their rated currents should be about 0·5 V.

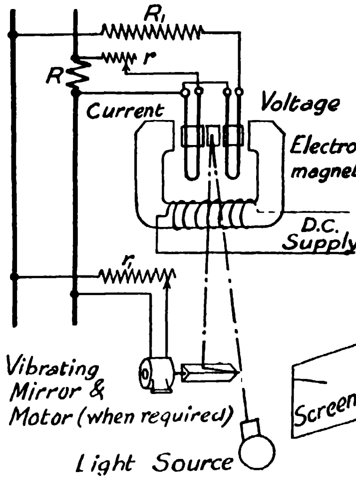


FIG. 258.—Elementary Diagram showing Connections of Duddell Oscillograph for Obtaining Current and Voltage Wave-forms

Fig. 258 shows the connections for obtaining simultaneous wave-forms of current and voltage, and Fig. 259 is a reproduction of an actual photographic record.

In order to determine the scales of these wave-forms, the vibrators must be calibrated on a direct-current circuit by means of a standard ammeter and a voltmeter. The calibration is effected by observing on the screen the displacements of the reflected beam of light from the zero line when a known current is passed through the shunt, or a known voltage is applied to the pressure circuit, the values of the shunt and series resistances being unaltered.

FIG. 259.—Photographic Record obtained from Three-Element Oscillograph, having Two Electromagnetic Vibrators and One Electrostatic Vibrator (*A*, *C*, Records from Electromagnetic Vibrators; *B*, Record from Electrostatic Vibrator)

[Cambridge Instrument Co.]

Theory of electromagnetic (moving-coil) oscillograph. This instrument is essentially a vibration galvanometer in which the natural frequency of vibration of the moving system is high and the damping is critical. Since

the angular deflections of the moving system are always extremely small, the deflecting torque at any instant may be assumed to be directly proportional to the instantaneous value of the current, i.e $\mathfrak{T}_d = ki$.

The deflecting torque is expended in three ways: (1) in accelerating the moving system; (2) in doing work against the restoring or controlling forces; (3) in supplying the friction losses due to damping.

The torque required for acceleration is proportional to the product of the moment of inertia and the angular acceleration of the moving system, and may be expressed as $\mathfrak{T}_a = ad^2\theta/dt^2$.

The restoring or controlling torque—which is due to the elasticity of the strips—is proportional to the angular deflection, and may, accordingly, be expressed as $\mathfrak{T}_c = c\theta$.

The torque due to damping friction is proportional to the angular velocity of the moving system, and may be expressed as $\mathfrak{T}_f = b\, d\theta/dt$.

Hence the equation of motion is

$$\mathfrak{T}_d = \mathfrak{T}_a + \mathfrak{T}_c + \mathfrak{T}_f$$

or

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = ki \quad . \quad . \quad . \quad . \quad . \quad (200)$$

Since the current is given by the general equation

$$i = I_{1m}\sin\omega t \pm I_{2m}\sin(2\omega t \pm \varphi_2) \pm I_{3m}\sin(3\omega t \pm \varphi_3) \pm \ldots$$

$$= \sum_{n=1}^{n=\infty} I_{nm}\sin(n\omega t \pm \varphi_n),$$

where n is the order of any harmonic (i.e. $n = 1$ for the fundamental, $n = 3$ for the third harmonic, etc.), and φ the phase displacement between the corresponding zero values of fundamental and harmonic, the equation of motion reduces to

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = k\sum_{n=1}^{n=\infty} I_{nm}\sin(n\omega t \pm \varphi_n) \quad . \quad . \quad (201)$$

Observe that if the inertia and damping of the moving system are both zero, the deflection, θ, is directly proportional to the instantaneous value of the current. In practice, however, the effects of inertia and damping cannot be ignored, and, therefore, a solution must be obtained for the general equation (201).

The solution to this equation consists of two parts—one representing the "steady" conditions, corresponding to the motion of the moving system after the initial conditions due to switching have subsided; the other representing the "transient" conditions and involving the initial conditions at the instant of closing the circuit.

The portion of the solution corresponding to the steady conditions is called the "particular solution," or "particular integral," of equation (201), and is given by

$$\theta = k\sum_{n=1}^{n=\infty} \frac{I_{nm}}{\sqrt{[b^2n^2\omega^2 + (c - an^2\omega^2)^2]}}\sin(n\omega t \pm \varphi_n - \beta) \quad . \quad (202)$$

where $\beta = \tan^{-1}[bn\omega/(c - an^2\omega^2)]$.

The transient portion of the general solution (which must be added to the particular solution to obtain the general solution) is obtained by solving equation (200) for the condition $i = 0$, since the current is zero at the instant of closing the circuit. The solution is of the form

$$\theta_1 = A\varepsilon^{m_1 t} + B\varepsilon^{m_2 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (203)$$

where m_1, m_2 are the roots of the auxiliary equation $am^2 + bm + c = 0$; and A and B are constants, the values of which are determined from the initial

conditions (e.g. the initial position of the moving system and the initial value of the impressed E.M.F).

In the present case

$$m_1 = -\frac{b}{2a} + \frac{\sqrt{(b^2-4ac)}}{2a}\,;\quad m_2 = -\frac{b}{2a} - \frac{\sqrt{(b^2-4ac)}}{2a}$$

Hence, three expressions will be obtained for θ_1, according to whether (b^2-4ac) is positive, zero, or negative.

If $b^2 > 4ac$

$$\theta_1 = \varepsilon^{-(b/2a)t}\left(A\varepsilon^{t\sqrt{(b^2-4ac)}/2a} + B\varepsilon^{-t\sqrt{(b^2-4ac)}/2a}\right) \quad . \quad . \quad . \quad (204)$$

if $b^2 = 4ac$

$$\theta_1 = \varepsilon^{-(b/2a)t}(A + Bt) \quad . \quad . \quad . \quad . \quad . \quad (205)$$

and if $b^2 < 4ac$

$$\theta_1 = \varepsilon^{-(b/2a)t}\sqrt{(A_1^2 + B_1^2)}\sin[(\sqrt{(4ac-b^2)}/2a)t + \gamma] \quad . \quad (206)$$

where $A_1 = A + B$, $B_1 = j(A - B)$, $\gamma = \tan^{-1} B_1/A_1$

The first and second expressions show that the transient conditions cause a non-oscillatory transient motion of the moving system—which may be considered to be superimposed upon the motion due to the steady-state conditions (equation (202)) - and if θ_1 and t are plotted in rectangular co-ordinates, logarithmic curves are obtained. The third expression (206) shows that the motion is oscillatory, but that the oscillations diminish in amplitude and become ultimately zero.

With the oscillograph, the second expression (205)—representing the conditions when $b^2 = 4ac$—is of special significance, as the damping is then just sufficient to cause the motion to be non-oscillatory. This degree of damping is called *critical damping.*

It is important in the oscillograph that the damping should be critical, as the transient effects on the moving system, corresponding to given initial conditions, are then of the shortest duration and the moving system quickly reaches its steady-state condition of motion. Moreover, if transient conditions occurring in a circuit are to be investigated, it is important that the transient term (205) in the equation of motion of the oscillograph should be of infinitesimal duration. This condition is approached if the damping is critical and the moment of inertia of the moving system is reduced to a practical minimum.

With critical damping, equation (202), representing the steady-state conditions of motion of the moving system, becomes

$$\theta = k\sum_{n=1}^{n=\infty} \frac{I_{nm}}{c + an^2\omega^2}\sin(n\omega t + \varphi_n - \beta') \quad . \quad . \quad . \quad . \quad (207)$$

where $\beta' = \tan^{-1}[2n\omega\sqrt{ac}/(c - an^2\omega^2)]$.

This equation shows that (1) the various harmonics do not have the same proportional effect on the deflection, the lower harmonics having a greater effect than the higher harmonics due to the term $n^2\omega^2$ in the denominator; (2) the relative phase displacements between the harmonic components of the deflection is not the same as that between the harmonic components of the operating current.

The amplitude (θ_{nm}) of the deflection due to the nth harmonic of the operating current is readily obtained from equation (207), and is

$$\theta_{nm} = kI_{nm}/(c + an^2\omega^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

In an ideal instrument, without inertia or damping, the amplitude would be

$$\theta'_{nm} = k\,I_{nm}/c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

Hence

$$\frac{\theta_{nm}}{\theta'_{nm}} = \frac{c}{c + an^2\omega^2} = \frac{1}{1 + n^2\omega^2(a/c)}$$

Now from equation (206), the frequency of the damped natural oscillations of the moving system is $f_d = \frac{1}{2\pi}\left(\frac{\sqrt{(4ac - b^2)}}{2a}\right)$, and if there were no damping (i e $b = 0$), the natural frequency of the undamped oscillations would be $f_0 = \frac{1}{2\pi}\sqrt{\frac{c}{a}}$ Hence, if T_0 is the natural, or free, period of the moving system, $T_0 = 1/f_0 = 2\pi\sqrt{(a/c)}$, or $a/c = T_0^2/4\pi^2$

Whence

$$\frac{\theta_{nm}}{\theta'_{nm}} = \frac{1}{1 + n^2\omega^2 T_0^2/4\pi^2}$$

Moreover, if f is the frequency of the fundamental of the operating current and T is the period, $\omega^2 = 4\pi^2/T^2$ Therefore

$$\frac{\theta_{nm}}{\theta'_{nm}} = \frac{1}{1 + n^2(T_0/T)^2} \tag{210}$$

The phase displacement (β_n) due to the nth harmonic in the operating current is, from equation (207), given by

$$\tan\beta_n = \frac{2n\omega\sqrt{ac}}{c - a_n^2\omega^2} = \frac{2n\omega\sqrt{(a/c)}}{1 - n^2\omega^2(a/c)} = \frac{2nT_0/T}{1 - n^2(T_0/T)^2}$$
$$= \frac{2n(f_0/f)}{(f_0/f)^2 - n^2} \tag{211}$$

Example. In the low frequency type Duddell oscillograph (which is intended for use on circuits of supply frequencies) the undamped natural period of the moving system is 0 0003 sec and the length, on the screen, corresponding to one cycle of 50 frequency, is 6 cm The relative accuracies for, say, the fundamental, seventh and nineteenth harmonics, when the instrument is used on a 50 cycle circuit, are calculated from equations (210), (211), with the results given in the accompanying table In the present case $T_0 = 0\ 0003$, $T = 0\ 02$ $T_0/T = 0\ 015$, $f_0/f = 66\ 7$

Order of Harmonic	Relative Amplitude	Phase Displacement	Linear Displacement on Screen (cm)
Fundamental	$\theta_{1m}/\theta'_{1m} = 0\ 9999$	$\beta_1 = 1\ 7°$	0 0283
7th	$\theta_{7m}/\theta'_{7m} = 0\ 9891$	$\beta_7 = 12°$ (with respect to 7th harmonic) 1,713° (with respect to fundamental)	0 0285
19th	$\theta_{19m}/\theta'_{19m} = 0\ 925$	$\beta_{19} = 31\ 8°$ (with respect to 19th harmonic) − 1 675° (with respect to fundamental)	0 0279

Thus the harmonic components are displaced from their correct positions on the screen by practically the same amount, and no error of importance occurs in practice Moreover, since, for the conditions under which this

(low frequency) type of vibrator would be used in practice, the amplitude of the 19th harmonic would rarely exceed 1 or 2 per cent of the fundamental, the $7\frac{1}{2}$ per cent error in the amplitude, as shown in the table, is insignificant.

Electro-thermic oscillograph. An oscillograph operating on the hot-wire principle has been devised by Irwin, and its theory is given in the *Journal of the Institution of Electrical Engineers*, vol. 39, p. 617. The instrument, however, has limitations which are not present in

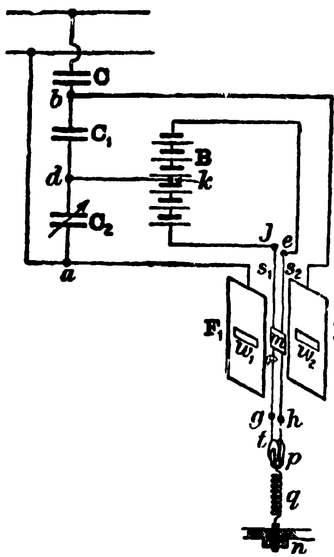


FIG. 260.—Diagram showing Connections and Principle of Electrostatic Oscillograph

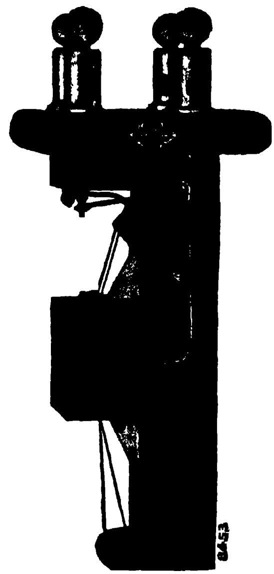


FIG. 261.—Vibrator of Electrostatic Oscillograph
[Cambridge Instrument Co.]

the electro-magnetic type, and, in consequence, its manufacture has been discontinued.*

Electrostatic oscillograph. The electrostatic oscillograph (due to Ho and Koto) is a special form of double electrometer in which the natural frequency of vibration is very high and the damping is critical.

The essential features of the instrument are shown diagrammatically in Fig. 260, and a vibrator is shown in Fig. 261. The "quadrants" of the electrometer take the form of two parallel

* The theory of this instrument, together with that of other oscillographs, is given in Irwin's book on *Oscillographs* (Pitman).

plates, F_1, F_2, which are mounted a short distance apart and are connected to the source of E.M.F. under investigation through the condenser-multiplier C, C_1, C_2; the condensers C_1, C_2 (which are of approximately equal capacity) being for the purpose of obtaining a point (d), having a potential midway between that of the "quadrants."

The "needles" of the electrometer—which form the vibrator—consist of two parallel phosphor bronze strips S_1, S_2, which are insulated from each other and are connected to a source of constant E.M.F. (about 300 V.) so that the strips are oppositely charged. The strips are stretched approximately symmetrically between the "quadrants," with the plane containing the strips parallel

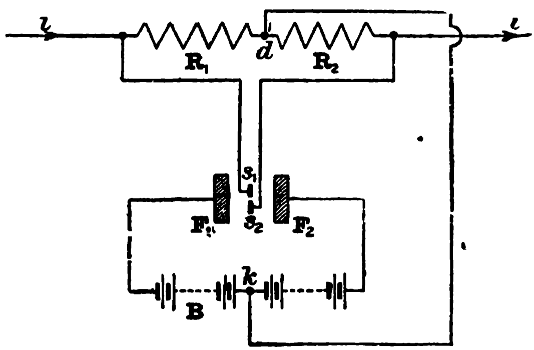


FIG. 262.—Connections for Obtaining Record of Current Wave-Form on Electrostatic Oscillograph

to the plates, and the strips and quadrants are rendered electrically symmetrical by adjusting the capacity of one of the condensers, C_2, of the condenser-multiplier.

The upper ends, e, j, of the strips are fixed and the lower ends, g, h, are joined by a silk thread, t, which passes round an ivory pulley, p, to which the tension spring, q, is attached. The strips are damped by being immersed in an oil bath, which, in the instrument manufactured by the Cambridge Instrument Co., is of similar construction, and interchangeable with, the oil bath of the electromagnetic vibrator.

A small mirror, m, is cemented to both strips, and two windows, w_1, w_2, are cut in the "quadrants"; one being for the purpose of illuminating the mirror, the other being for the purpose of maintaining electrical symmetry between "quadrants" and "needle." A "zero" mirror (not shown in Fig. 260) is fitted to the damping chamber, as in the electromagnetic vibrator.

Applications. The electrostatic oscillograph is specially suitable for high-voltage circuits (for voltages above 2000 volts) and possesses the important advantages that the energy consumption is *nil* and that the operating current is exceedingly small, being only that necessary to charge the "quadrants" and their condenser multipliers. Another important feature is that the vibrators are small enough to be interchangeable with those of the electromagnetic (Duddell) oscillograph, and, therefore, a common optical and photographic system may be employed for both the electrostatic and electromagnetic vibrators.

The electrostatic oscillograph may be adapted to measure small currents by arranging the connections as shown in Fig. 262. The current to be investigated is passed through two equal non-inductive resistances, R_1, R_2; the strips are connected to the terminals of these resistances and the mid-point, d, is connected to the mid-point, k, of a high-tension battery, B, which is employed for charging the "quadrants." Instead of the battery, an electrostatic influence machine may be employed, and in this case the point k is obtained by connecting two similar condensers across the "quadrants," as in Fig. 260.

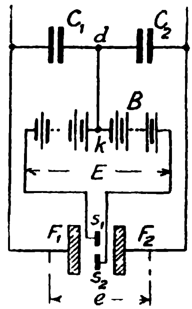


FIG. 263.—Elementary Diagram of Electrostatic Oscillograph

Theory of electrostatic oscillograph. The instrument may be considered as equivalent to two electrometers, in which the "needles" are mechanically coupled together and the "quadrants" are common to both. Thus, in Fig. 263, the plates F_1, F_2, and the strip s_1 form one electrometer, while these plates and the strip s_2 form the other electrometer.

Assuming the needles to be perfectly symmetrical with respect to the quadrants, and to have polarities as indicated in the diagram, then if E is the potential difference between the needles and e is the potential difference between the quadrants at any instant, the force tending to move strip s_1 towards the plate F_2 is

$$f_1 = k_1[2e(-\tfrac{1}{2}e + \tfrac{1}{2}E) + e^2] = k_1Ee$$

and that tending to move strip s_2 towards the plate F_1 is

$$f_2 = k_1[2e(-\tfrac{1}{2}e - \tfrac{1}{2}E) + e^2] = -k_1Ee$$

Hence the couple, or torque, acting upon the moving system is proportional to Ee, and is given by $\mathfrak{T}_d = kEe$.

Therefore, the equation of motion is

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = kEe$$

where the constants a, b, c have the same significance as those in equation (200), p. 448.

In the present case

$$e = E_{1m} \sin \omega t + E_{2m} \sin(2\omega t \pm \varphi_2) + \ldots$$

$$= \sum_{n=1}^{n=\infty} E_{nm} \sin(\omega t \pm \varphi_n).$$

Hence if the conditions relating to damping, natural free period, and moment of inertia of the moving system are satisfied, the deflection, θ, will be proportional to e.

The above theory assumes that there are no dielectric losses in the damping oil and condensers, and that the electrometers are perfectly symmetrical. If the symmetry is not perfect the strips will vibrate when they are both connected to the point which has a potential midway between that of the quadrants.

In practice, an electrical adjustment is necessary to compensate for any mechanical dissymmetry in construction. The adjustment is effected by disconnecting the strips from the battery and connecting them to the mid-point, d, Fig. 260, of the condensers C_1, C_2. The normal alternating E.M.F. is then applied to the plates and the condenser C_3 is adjusted until there is no vibration of the strips.

Cathode-ray oscillograph. The electronic discharge between the electrodes of a highly exhausted vacuum, or Crookes' tube, consists of a stream of electrons issuing perpendicularly from the surface of the cathode. This electronic stream (which was discovered by Crookes and called "cathode rays") possesses a number of special properties: thus, it is without inertia; it may be deflected by either electric or magnetic forces applied transversely to the stream; it will cause the glass upon which it impinges to fluoresce slightly, and other more fluorescent materials to fluoresce strongly; it may be intercepted by a metallic screen or target.

The properties were applied to an oscillograph by Braun, who modified the Crookes' tube by: (1) interposing in the electronic stream a metal screen, pierced with a small hole, so that only a small pencil of rays reached the fluorescent screen, thereby giving a sharp bright "spot" on the screen; (2) deflecting this pencil of rays either magnetically or electrostatically by the alternating current or E.M.F. to be investigated.

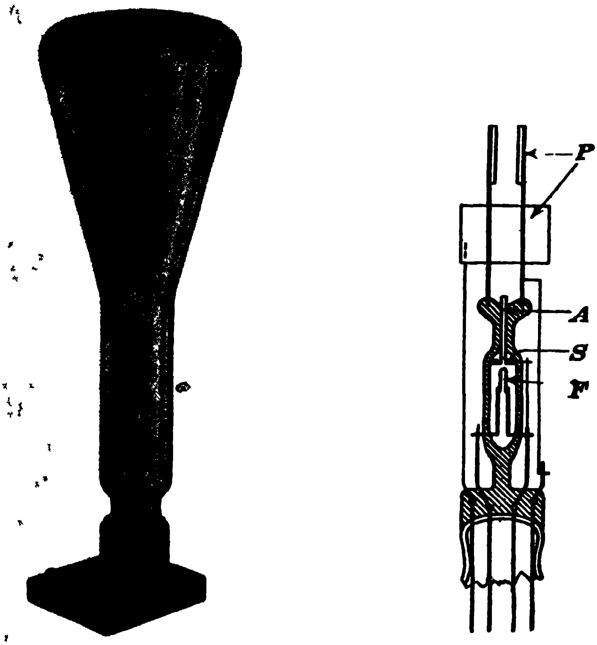
The principal disadvantage of the Braun tube is the high continuous voltage (from 10,000 to 20,000 volts) which must be maintained between the electrodes to produce the electronic stream.

The exciting voltage, however, may be reduced to a relatively low value if a *heated cathode* be employed, as in the thermionic tubes, or valves, which are now so largely employed for radiotelegraphy and telephony.

The development of the thermionic principle for oscillographic purposes is due to the Western Electric Co.,* and a commercial oscillograph of this type is shown in Fig. 264.

* Now Standard Telephones and Cables.

Construction of Western-Electric thermionic (cathode-ray) oscillograph. The tube, shown at (*a*), Fig. 264, is about 30 cm. long, about 4 cm. diameter at the small end, and about 10 cm. diameter at the large end, which forms a fluorescent screen. The small end of the tube is fitted with the electrodes, filament, intercepting screen, and deflecting plates, the arrangement of these being shown



(*a*) (*b*)

FIG. 264.—Western-Electric Cathode-Ray Oscillograph and Electrodes

at (*b*), Fig. 264, and also in the diagram of connections, Fig. 265. The tube contains a small amount of gas, which serves the double purpose of focusing the pencil of electron rays and preventing the accumulation of electronic charges on the walls of the tube.

The filament, or cathode, *f*, is of oxide-coated tungsten or molybdenum, and supplies the requisite electronic emission at a dull red heat.

The anode, *a*, is a small tube of platinum fixed about 1 mm. in front of the filament. A potential difference of from 250 to 400 volts is maintained between anode and cathode.

The intercepting screen, s, is a small metal disc located between the filament and the anode, and is pierced with a hole just smaller than the filament (which is bent to form nearly a complete circle).

The two pairs of deflecting plates P_x, P_y (of non-magnetic, high resistance material) are mounted at right angles to each other in front of the anode, so that the pencil of rays emerging from the tubular anode passes centrally between both pairs of plates. One plate of each pair is connected to the anode, and the E.M.Fs. to

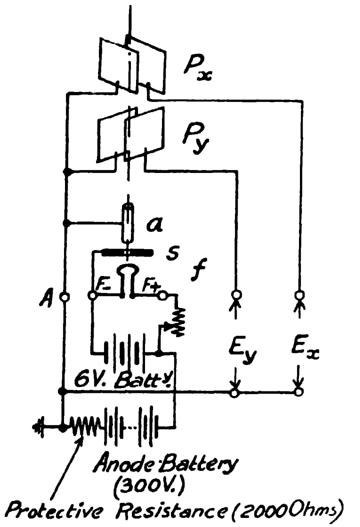


FIG. 265

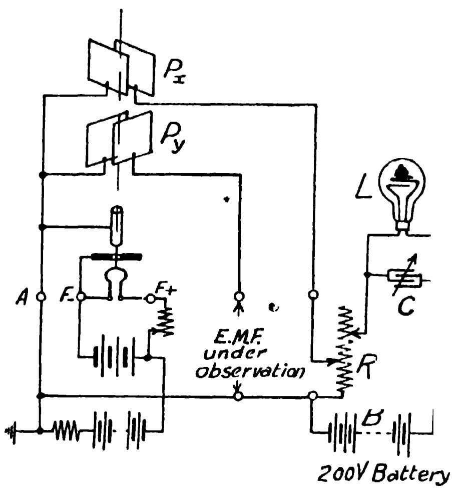


FIG. 266

Connection Diagrams for Cathode-Ray Oscillograph

be investigated are applied between the anode and the other plate of each pair.

Method of using thermionic (cathode-ray) oscillograph. (1) *Adjustments.* The filament requires a direct current of from 1·2 to 1·7 A. at a voltage of about 2 V.

The normal potential difference between anode and filament is 300 V., and the anode current is about 0·5 mA. This should preferably be supplied by a battery of small storage cells.

The sensitiveness of the tube varies inversely as the anode voltage, but the brightness of the fluorescent spot (which is of a bright-green colour) increases rapidly with increase of anode voltage. The sharpness of the spot, however, depends upon the filament current, and for each anode voltage there is a particular value of filament current, which gives the sharpest spot; this value increasing as the anode voltage increases. Therefore, the life of the tube will be prolonged if the spot is worked below its maximum brightness. As the cathode rays are deflected by the earth's magnetic field, the centering of the spot on the screen is effected by means of a permanent magnet placed near the tube.

(2) *Visual observations.* If a steady E.M.F. be applied to one pair of plates the spot will be deflected along a straight line, and the deflection from the zero position will be directly proportional to the E.M.F.* If, however, the E.M.F. is alternating, the spot will trace a straight line across the screen. If now another E.M.F., varying approximately linearly with respect to time, be applied to the other pair of deflecting plates, the spot will trace out on the screen, in rectangular co-ordinates, the variation of the alternating E.M.F. with respect to time. Alternatively, a current varying approximately linearly with respect to time may be passed through magnetic deflecting coils arranged externally to the tube.

An interesting method (due to the Western Electric Co.†) of obtaining the linear variation of the auxiliary E.M.F. depends upon the property possessed by the neon-filled (glow-discharge) lamp of a relatively large difference between the voltages at which the discharge "strikes" and "fails." The connections are shown in Fig. 266. The neon lamp, L, is shunted across an adjustable condenser C, which is connected in series with a high resistance R, and the circuit is supplied from a direct-current source of constant E.M.F (about 200 V.).

At the instant of closing the circuit the lamp is equivalent to an open circuit, and the voltage at the terminals of the condenser and lamp therefore increases according to the exponential equation : $e = E(1 - \varepsilon^{-t/RC})$, where e is the voltage at the terminals of the condenser after an interval of t seconds from the instant of closing the circuit, E is the voltage of the source of direct-current, and R, C, are the values of the resistance and capacity, respectively.

The voltage at the lamp then builds up to the critical striking value, the discharge through the lamp commences, and the increase of current through the resistance, R, so reduces the voltage at the lamp that the discharge ceases. The cycle is then repeated and the lamp is maintained in a state of blinking. The frequency of the blink can be calculated when the "striking" and "failing" voltages of the lamp, together with the values of R and C, are known.

A suitable proportion of the voltage drop across the resistance R is applied to one pair of the deflecting plates, P_x, of the oscillograph, and, therefore, the potential difference between these plates varies according to the rate at which the condenser is charged and discharged. Since the combination of lamp, condenser, and resistance results in the condenser being never fully charged or discharged, only portions of the charge-discharge-time curves are utilized, but these approximate to straight lines. Thus the potential difference applied to the deflecting plates, P_x, varies approximately as a linear function of time. Owing to this irregularity, the time-axis is not precisely even, but the irregularity causes only a slight distortion in the wave-form traced on the screen. The actual effect which occurs is a slight shortening of the time scale in one direction, resulting in a slight crowding of the waves in this direction.

(3) *Photographic record.* A photographic record of recurring phenomena (giving a stationary wave-form on the screen) is obtained by photographing the trace of the spot on the screen by an ordinary camera.

Applications of thermionic (cathode-ray) oscillograph. In addition to the investigation of wave-forms and alternating-current phenomena of any frequency, the thermionic oscillograph may be utilized for other purposes, such as for showing the relationship between two interdependent electromagnetic quantities. The necessity for a time-axis then ceases, and the two variables are made to cause deflections at right angles to each other.

* With the normal anode voltage a deflection of 1 cm. is obtained with a potential difference of about 10 V. applied to the deflecting plates.

† A full description of this method, together with examples of wave-forms, is given in *Electrical Communication*, vol. 3, p. 69 ; July, 1924. (Western Electric Co's. publication.) The characteristics of the neon-filled lamp on direct-current circuits are given in *Electrician*, vol. 89, p. 626.

The tube is then essentially a wattmeter, and the diagram traced upon the screen usually represents the power diagram for the circuit under investigation The tube has been employed in this manner for measuring dielectric and iron losses *

The volt ampere characteristics of X ray tubes and thermionic valves may also be determined by arranging that the two deflections are proportional to the current and voltage, respectively † Then, if the supply frequency is constant, the spot traces a curve which is the volt ampere characteristic of the circuit

The tube has also been largely employed in the investigation of phenomena of radio frequency

* "An investigation of dielectric losses with the cathode ray tube, J P Minton, *Transactions of American Institute of Electrical Engineers*, vol 34, p 1627

† *Electrician*, vol 89, p 611

CHAPTER XIV

INSTRUMENT TRANSFORMERS

Use of instrument transformers in practice. The range of alternating current switchboard instruments is usually extended by means of transformers—in preference to series resistances and shunts—as, in addition to extending the range, the transformers also insulate the instruments from the main circuit, and therefore enable low-voltage instruments (supplied through transformers) to be used with safety on high-voltage circuits. In such cases the current circuits of the instruments are designed for a maximum current of 5 A., and the pressure circuits are designed for a maximum pressure of about 100 V. Again, with transformers, the range of an instrument may be extended almost indefinitely with only a small increase in the power consumption, as the losses in the transformer are extremely small.

The use of transformers with instruments, however, introduces slight errors into the instrument readings, and these errors must be taken into consideration when accurate results are required Usually the errors are only important in connection with wattmeter readings at low power factors, and the method of applying the corrections is discussed later

Construction. *Current transformers.* The primary winding of a current, or series, transformer is connected in series with the main circuit, and the secondary winding is connected to the current coil of the measuring instrument, the function of the transformer being to supply the instrument with current proportional to that in the main circuit and in phase opposition thereto.* The current range of the instrument connected to the secondary winding is usually 5 A.

The number of turns in the primary winding vary with the type of magnetic circuit and the magnitude of the current in the main circuit, and when this current is large the primary winding consists of a single conductor.

The insulation of the primary winding is of extreme importance in transformers for high-voltage circuits. After being wound and

* The phase relationship of the currents is important only in connection with power measurements.

insulated, the windings are impregnated, in vacuo, with insulating compound, and the entire transformer may be either immersed in oil or fitted into a case and filled with insulating compound applied in vacuo.

The magnetic circuit is constructed of high quality laminations and may take the form of a ring, a rectangle, or a double rectangle, as shown in Fig. 267. For the highest accuracy the laminations must be of good quality alloyed iron, and the magnetic circuit must be without joints, thereby necessitating hand-wound coils. In commercial transformers, considerations of cost require the use of

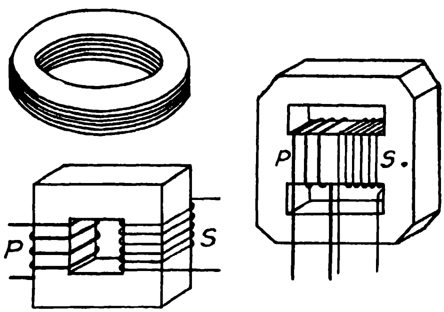


FIG 267.—Forms of Core Construction for Current Transformers

machine wound coils, and therefore a jointless magnetic circuit cannot be employed. But the laminations at the joints must be interleaved so as to reduce the reluctance.

Examples of transformers for low-voltage and high-voltage circuits are shown in Fig. 268. In the low-voltage transformer the coils are placed side by side and the core is constructed with interleaved joints, the laminations being assembled after the coils are in position. In the high-voltage transformer (Fig. 268*b*) the primary winding is insulated from the secondary winding, core, and supports by porcelain tubes. The core is of the ring type and is wound with the secondary winding. After being impregnated, the wound core is slipped over one of the porcelain tubes, the other porcelain tube is placed in position, and the primary winding is threaded through the tubes.

Potential transformers. In these transformers the primary winding is connected across the supply system and the secondary winding is connected to the pressure circuit of the measuring instrument, the function of the transformer being to supply the latter with a

potential difference proportional to the voltage of the supply system and in phase opposition thereto.*

The primary winding must therefore consist of a large number of turns and must be adequately insulated. Moreover, the terminals

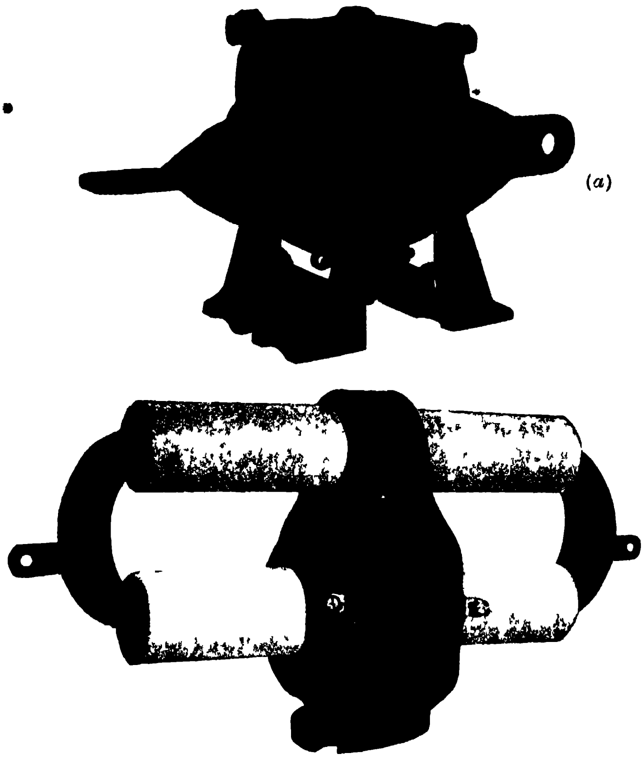


FIG. 268 – Current Transformers for (*a*) Low-Voltage, (*b*) High-Voltage Circuits
[British Thomson Houston Co.]

must be separated from one another and must be adequately insulated from the frame by porcelain or other bushings. Fuses are necessary as a protection against breakdown of the insulation, internal short-circuits, etc., and these are usually mounted on bushings adjacent to the terminal bushings.

The magnetic circuit is constructed of high-quality laminations

* The phase relationship of the voltages is important only in connection with power measurements.

and takes the form of a single rectangle (Fig. 269*a*) for single-phase transformers, and a double rectangle (Fig. 269*b*) for three-phase transformers. As former wound coils are only permissible in the present case, the laminations forming the cores and yokes must be assembled after the coils are in position and the joints must be interleaved.

The coils forming the secondary and primary windings are located symmetrically one over the other on each core of the magnetic circuit, the primary coils being outside the secondary coils. For high-voltage circuits the primary coils are wound in sections, the several sections being separately insulated and connected in series. The windings are impregnated in vacuo in the same manner as those of current transformers.

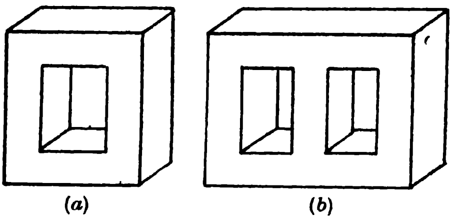


FIG 269.—Forms of Core Construction for Potential Transformers

Connections of transformers and instruments. The diagrams of Fig. 270 show the connections of an ammeter, a voltmeter, and a single-phase wattmeter, or power-factor meter, when used with instrument transformers.

If the ammeter, voltmeter, and wattmeter are scaled as straight-through (i.e. low voltage) instruments, the current in the main

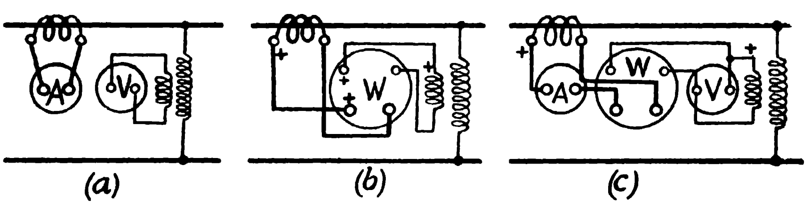


FIG. 270 —Connection Diagrams for Instruments Used with Instrument Transformers

circuit will be given by : Ratio of current transformer × ammeter reading ; the voltage by : Ratio of potential transformer × voltmeter reading ; and the power by : Wattmeter reading × ratio of current transformer × ratio of potential transformer.

When extreme accuracy is required in connection with power measurements, the correction factors of the transformers must be known.

In the connections for the wattmeter and power-factor meter, a

knowledge of the **relative polarities** of the secondary terminals of the transformers with respect to the primary terminals is important, otherwise the instruments may be so connected as to read improperly.

The relative polarities of the terminals of the primary and secondary windings should be marked by the manufacturer, either by conventional signs or letters. In the event of a transformer having no markings the relative polarities of the terminals may be determined very simply by passing a small direct current through one winding and connecting a low reading permanent-magnet moving-coil voltmeter across the other winding. Then, if the voltmeter deflects *up* the scale on *closing* the direct-current circuit, the terminal of the transformer which is connected to the positive terminal of the voltmeter will have the same polarity as that which is connected to the positive terminal of the direct-current supply. Precautions should be taken to avoid leaving the magnetic circuit of the transformer in a highly magnetized condition, otherwise its accuracy will be impaired.

Precautions to be observed with current transformers. When using current transformers it is important that the secondary circuit be *always closed* when current is passing through the primary winding, otherwise the magnetic circuit may become highly saturated. The iron losses and the heating of the core will then become excessive, and a relatively high voltage will be induced in the secondary winding which may cause a breakdown of the insulation between adjacent turns and a burning-out of the transformer. Even if the transformer is capable of successfully withstanding these abnormal conditions, it is highly probable that, when the primary current is switched off, or the secondary circuit is again closed, the magnetic circuit will be left in a highly-magnetized condition, which will impair the accuracy for future work. To restore the core to its normal magnetic state, it must be demagnetized by passing an alternating current through the primary winding (with the secondary winding open circuited) and gradually decreasing this current to zero.

When current transformers are used with portable instruments, precautions are necessary to avoid errors due to (1) the relatively large stray magnetic field which may exist with open-type transformers for high-voltage circuits owing to the separation of the primary and secondary coils ; (2) the inadvertent substitution of instruments having a range (e.g. 1, 2, or 3 A.) lower than the standard range (5 A.), as the former have a much higher impedance than the latter and may involve operating conditions in the

transformer which may be quite different from the normal conditions.

Theory of current transformer. In an ideal transformer the magnetizing ampere-turns and the losses are zero, and therefore, (1) the ratio of primary and secondary currents is constant for all loads; and (2) the phase difference between these currents is 180°.

The vector diagram representing these conditions is shown in Fig. 271*a*, in which $O\Phi$ represents the flux; OE_1, OE_2, the E.M.Fs. induced in the primary and secondary windings (which are directly proportional to the

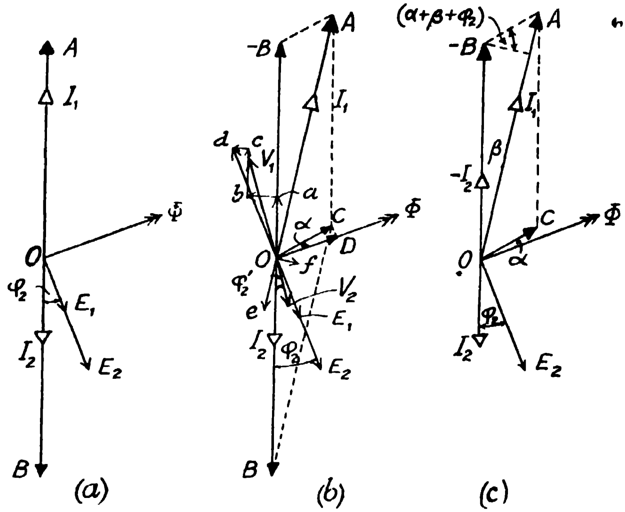


FIG. 271.—Vector Diagrams for Current Transformer

numbers of turns in these windings); $OV_1(=-OE_1)$, the potential difference at the terminals of the primary winding; OI_2, the current in the secondary circuit lagging $\varphi°$ with respect to OE_2; OB, the ampere-turns due to the secondary winding; $OA(=-OB)$, the ampere-turns due to the primary winding; and OI_1, the current in the primary circuit. Observe that, since the resultant ampere-turns are zero, the ratio of currents is equal to the inverse ratio of turns, i.e. $I_1/I_2 = N_2/N_1$, where I_1, I_2, denote the primary and secondary currents, respectively, and N_1, N_2, the number of turns in the primary and secondary windings, respectively.

In a practical transformer a definite number of ampere-turns are required for the magnetic circuit, there are losses in the core and windings, and magnetic leakage occurs between the windings. The vector diagram representing these conditions is shown in Fig. 271*b*, some of the vectors being exaggerated to obtain legibility. The diagram is drawn with the secondary current as the vector of reference, since when this current, together with data of the transformer, are known, the induced E.M.Fs., the flux, and the primary current may be readily determined.

The E.M.F. induced in the secondary winding is represented by OE_2, and balances the vector sum (Od) of the internal E.M.Fs. due to the resistance and reactance of the secondary circuit, of which Oa, bc—in phase opposition with respect to the secondary current—represent the E.M.Fs. due to the resistance$_s$

of the secondary winding and load (i.e. ammeter or other instrument), respectively, and ab, cd—lagging 90° with respect to the current—represent the E.M.Fs. due to the reactance (inductive) of the load and the leakage reactance of the secondary winding, respectively.

The E.M.F. induced in the primary winding is represented by OE_1, the ratio OE_2/OE_1 being equal to the ratio of the numbers of secondary and primary turns.

The flux is represented by $O\Phi$, which leads the induced E.M.Fs. by 90°. The magnetizing ampere-turns are represented by OD, and the exciting ampere-turns by OC, the latter leading the flux by the angle a. The exciting ampere-turns are the resultant of the ampere-turns due to the primary and secondary windings. Hence, if the ampere-turns of the secondary winding are represented by OB, then the vector sum (OA) of OB reversed and OC will represent the primary ampere-turns. The primary current is, therefore, represented by OI.

The voltage at the terminals of the primary winding is represented by OV_1, and is equal to the vector sum of the induced E.M.F. (OE_1) and the E.M.Fs. due to the resistance and leakage reactance of the primary winding, these E.M.Fs. being represented by Oe and Of, respectively.

If this vector diagram, Fig. 271*b*, is compared with that, Fig. 271*a*, for an ideal transformer, we observe that, in the commercial transformer—

1. The ratio of primary and secondary currents is not strictly proportional to the ratio of the numbers of turns.
2. The phase difference between these currents is less than 180°.
3. The ratio and phase difference are not constant, but vary with the magnitudes of the currents and the impedance of the secondary circuit.

To approach ideal conditions in a commercial transformer, the exciting ampere-turns must be small in comparison with the primary, or secondary, ampere-turns, and the resistance and reactance of the secondary circuit must be kept as low as practicable.

Expressions for ratio and phase angle of current transformer. The ratio of the primary and secondary currents (called in practice the "ratio" of the transformer) may be calculated from the vector diagram, the vectors concerned being drawn separately in Fig. 271*c*, in which the secondary ampere-turn vector is reversed from its normal position. From this diagram we obtain the relation

$$OA = OB\cos\beta + OC\sin(a + \varphi_2 + \beta)$$

where β is the angle between the primary ampere-turns vector and the reversed secondary ampere-turn vector. This angle (β) is called the "phase angle" of the transformer, and is usually very small (from about 0·5° to 3·0°).

Substituting ampere-turns in this expression, we have

$$I_1N_1 = I_2N_2\cos\beta + I_0N_1\sin(a + \varphi_2 + \beta)$$

where I_0 is the fictitious exciting current (i.e. I_0 = exciting ampere-turns$/N_1$).

Whence $$\frac{I_1}{I_2} = \frac{N_2}{N_1}\cos\beta + \frac{I_0}{I_2}\sin(a + \varphi_2 + \beta). \qquad (212)$$

or, since β is a small angle, we have, to a very close approximation,

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} + \frac{I_0}{I_2}\sin(a + \varphi_2) \qquad (212a)$$

Hence, on account of magnetizing current, losses, and reactance in the secondary circuit, the ratio of turns (i.e. N_2/N_1) must be smaller than the ratio required for the currents. For example, a transformer designed for a ratio of 10 : 1 would have N_2/N_1 equal to between 9·8 to 9·9.

Observe that both the exciting current (I_0) and a depend upon the flux density and the magnetic qualities of the core. With a given transformer, the flux density is almost directly proportional to the E.M.F. induced in the secondary winding, and with a given secondary current, this E.M.F. is proportional to the impedance of the secondary circuit.

The phase angle, β, is readily determined from the diagram, Fig. 271c. Thus

$$\tan\beta = \frac{OC\ \cos(\alpha + \varphi_2 + \beta)}{OB\cos\beta} = \frac{I_0 N_1 \cos(\alpha + \varphi_2 + \beta)}{I_2 N_2 \cos\beta} \quad . \quad (213)$$

Since β is a small angle, we have, to a close approximation,

$$\beta_{\text{ radians}} = \frac{N_1}{N_2} \cdot \frac{I_0}{I_2}\cos(\alpha + \varphi_2) \quad . \quad . \quad . \quad . \quad . \quad (213a)$$

Observe that the vector OI_2 of the secondary current is leading with respect to the reversed vector of the primary current.

Data of current transformer. A commercial current transformer, intended for a primary current of 50 A., a secondary current of 5 A., and a secondary load of 40 V.A., has a ring core wound with 30 primary turns and 294 secondary turns. The resistances (at 20° C.) of the windings are 0·018 ohm

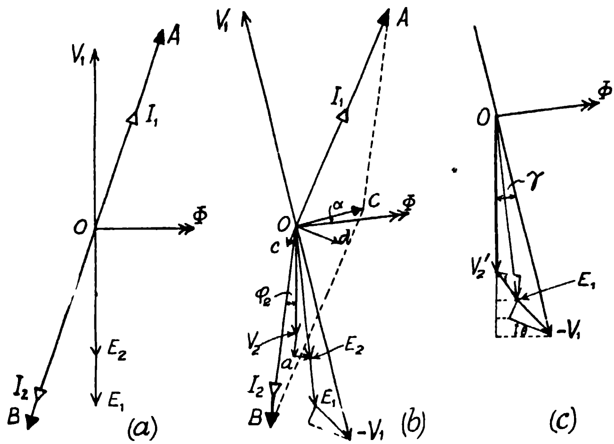


FIG. 272.—Vector Diagrams for Potential Transformer

(primary) and 0·65 ohm (secondary). The mean diameter of the core is 10 cm. and the magnetic cross-section is 5 cm.²

Theory of potential transformer. In an ideal potential transformer without losses, the voltages at the terminals of the primary and secondary windings are equal to the E.M.Fs. induced in these windings, and their ratio is constant and equal to the ratio of the numbers of turns in the windings. Thus, $V_1/V_2 = E_1/E_2 = N_1/N_2$. *Moreover, with constant frequency the flux in the core is proportional to the voltage applied to the primary winding.*

The vector diagram is shown in Fig. 272*a*, in which $O\Phi$ represents the flux in the core; OE_1, OE_2, the E.M.Fs. induced in the primary and secondary windings, respectively; $OV_1(= -OE_1)$, the impressed E.M.F.; OI_2, the current in the secondary circuit; OB, the secondary ampere-turns; $OA(= -OB)$, the primary ampere-turns; and OI_1, the primary current.

In the diagram for the practical transformer, Fig. 272*b*, the secondary terminal voltage is taken as the vector of reference, OV_2. The current in the secondary circuit is represented by OI_2, which, in practice, lags by a small angle with respect to OV_2. The E.M.F. induced in the secondary winding is represented by OE_2, and is obtained by compounding with OV_2 the pressure drops due to the resistance and leaking reactance of the secondary winding, these pressure drops being represented by V_2a and aE_2, respectively.

The flux leads OE_2 by 90°, and is represented by $O\Phi$. The exciting ampere-turns are represented by OC, and from this vector and the vector OB, representing the secondary ampere-turns, the vector OA, representing the primary ampere-turns, is obtained. The primary current is therefore represented by OI_1.

The E.M.F. induced in the primary winding is represented by OE_1, the ratio OE_1/OE_2 being equal to the ratio of the numbers of turns in the windings. The external voltage at the terminals of the primary winding is represented by OV_1, and balances the induced E.M.F., OE_1, together with the internal E.M.Fs. due to the resistance and reactance of the primary winding, these E.M.Fs. being represented by Oc and Od, respectively.

Expressions for ratio and phase angle of potential transformer. The ratio of the primary and secondary voltages (called in practice the "ratio" of the transformer) is easily calculated from the vector diagram if the diagram is re-drawn with all secondary voltage vectors increased in the ratio N_1/N_2, and the secondary current vector diminished in the ratio N_2/N_1; i.e. the scale for secondary voltages is N_1/N_2 times that for the primary voltages, and the scale for secondary currents is N_2/N_1 times that for primary currents. Thus the vector representing the E.M.F. induced in the secondary is now of equal length to that representing the E.M.F. induced in the primary winding. The new vector diagram for the quantities concerned is shown in Fig. 272c.

If the various voltage vectors are projected upon OV_2 produced, we have

$$V_1 \cos \gamma = V_2 \frac{N_1}{N_2} + \frac{N_1}{N_2}(I_2R_2 \cos \varphi_2 + I_2X_2 \sin \varphi_2) + I_1(R_1 \cos \theta + X_1 \sin \theta)$$

where γ is the angle between OV_2 and OV_1 reversed; θ is the angle between OV_2 reversed and I_1; R_2, X_2 are the resistance and leakage reactance, respectively, of the secondary winding; R_1, X_1, the resistance and leakage reactance, respectively, of the primary winding.

Now γ is usually much less than 1°, so that, to a very close approximation,

$$\cos \gamma = 1{\cdot}0;$$

$$I_1 \cos \theta = (I_2N_2/N_1) \cos \varphi_2 + I_0 \sin a;$$

$$I_1 \sin \theta = (I_2N_2/N_1) \sin \varphi_2 + I_0 \cos a$$

Hence, substituting in the preceding expression, we have

$$\begin{aligned} V_1 &= V_2 \frac{N_1}{N_2} + I_2 \frac{N_1}{N_2}(R_2 \cos \varphi_2 + X_2 \sin \varphi_2) + R_1\left(I_2 \frac{N_2}{N_1} \cos \varphi_2 + I_0 \sin a\right) \\ &\qquad + X_1\left(I_2 \frac{N_2}{N_1} \sin \varphi_2 + I_0 \cos a\right) \\ &= V_2 \frac{N_1}{N_2} + I_2 \cos \varphi_2 \left(\frac{N_1}{N_2} R_2 + \frac{N_2}{N_1} R_1\right) + I_2 \sin \varphi_2 \left(\frac{N_1}{N_2} X_2 + \frac{N_2}{N_1} X_1\right) \\ &\qquad + I_0(R_1 \sin a + X_1 \cos a) \\ &= V_2 \frac{N_1}{N_2} + I_2 \frac{N_2}{N_1}\left[\left(R_1 + \left(\frac{N_1}{N_2}\right)^2 R_2\right) \cos \varphi_2 + \left(X_1 + \left(\frac{N_1}{N_2}\right)^2 X_2\right) \sin \varphi_2\right] \\ &\qquad + I_0(R_1 \sin a + X_1 \cos a) \end{aligned}$$

Now I_2N_2/N_1 is the component of the primary current which balances the secondary current I_2, and is called the equivalent secondary current referred to the primary circuit. Similarly, $R_2(N_1/N_2)^2$ and $X_2(N_1/N_2)^2$ are called the equivalent resistance and reactance, respectively, of the secondary winding

referred to the primary circuit.* These equivalent quantities will be denoted by I_2', R_2', X_2'. Hence the preceding expression becomes

$$V_1 = V_2N_1/N_2 + I'_2[(R_1 + R_2')\cos\varphi_2 + (X_1 + X'_2)\sin\varphi_2] + I_0(R_1\sin\alpha + X_1\cos\alpha)$$

and the ratio (V_1/V_2) is given by

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} + \frac{I_2'[(R_1 + R_2')\cos\varphi_2 + (X_1 + X_2')\sin\varphi_2] + I_0(R_1\sin\alpha + X_1\cos\alpha)}{V_2} \quad (214)$$

or, since φ_2 and α are usually small angles, we have, approximately,

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} + \frac{I_2'(R_1 + R_2') + I_0X_1}{V_2} \quad . \quad . \quad . \quad . \quad . \quad (214a)$$

Observe that, on account of losses, magnetic leakage between the primary and secondary windings (to which the leakage reactances X_1, X_2, are due), and magnetizing current, the ratio of turns (N_1/N_2) must be slightly smaller than the ratio required for the terminal voltages. Also, since I'_2 is usually small in comparison with I_0, and X_1 is greater than $(R_1 + R_2')$, the term I_0X_1 in equation (214a) is more important than the term $I_2'(R_1 + R_2')$. Thus the ratio is given very approximately by

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} + \frac{I_0X_1}{V_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (214b)$$

The **phase angle**, γ, of the transformer is readily calculated from the vector diagram. Thus

$$\sin\gamma = \frac{1}{V_1}\left[\frac{N_1}{N_2}(I_2X_2\cos\varphi_2 - I_2R_2\sin\varphi_2) + I_1(X_1\cos\theta - R_1\sin\theta)\right]$$

or, since γ is a very small angle, $\sin\gamma = \gamma$ (in radians),

$$\gamma\ \text{radians} = \frac{1}{V_1}[I_2'\{(X_1 + X_2')\cos\varphi_2 - (R_1 + R_2')\sin\varphi_2\} + I_0(X_1\sin\alpha - R_1\cos\alpha)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

Application of correction factors for ratio and phase angle. The correction factors for ratio and phase angle of instrument transformers vary with the magnitude and nature of the load connected to the secondary winding. The correction factors for a given transformer, are therefore expressed in the form of curves, typical examples being given in Figs. 273, 274.

With measurements of current and voltage using instrument transformers, the instrument readings are multiplied by the ratio correction factors, the application of which in this case is quite straightforward.

With **power measurements in single-phase circuits** by the wattmeter method, correction factors for both ratio and phase angle must be applied to the instrument readings, together with the correction factors for the wattmeter itself.

In applying the correction factors for phase angle, we observe, from

* The quantities are called equivalent quantities because, if a resistance or reactance of this (equivalent) value is connected in the primary circuit and carries the equivalent secondary current (I_2N_2/N_1) the voltage drop, as measured in the primary circuit, is N_1/N_2 times that due to the passage of the secondary current, I_2, through the actual secondary resistance, R_2, or reactance, X_2. Thus

$$I_2\frac{N_2}{N_1} \times R_2\left(\frac{N_1}{N_2}\right)^2 = \frac{N_1}{N_2}(I_2R_2)$$

$$I_2\frac{N_2}{N_1} \times X_2\left(\frac{N_1}{N_2}\right)^2 = \frac{N_1}{N_2}(I_2X_2)$$

Fig. 271*b*, that, for the current transformer, the secondary current leads the reversed primary current, but that, for the potential transformer and the conditions represented in Fig 272*b*, the secondary terminal voltage is lagging with respect to the reversed primary terminal voltage With potential transformers, however, the secondary terminal voltage may be in phase with, or lagging, or leading the reversed primary voltage, according to the design of the transformer and the nature of the load

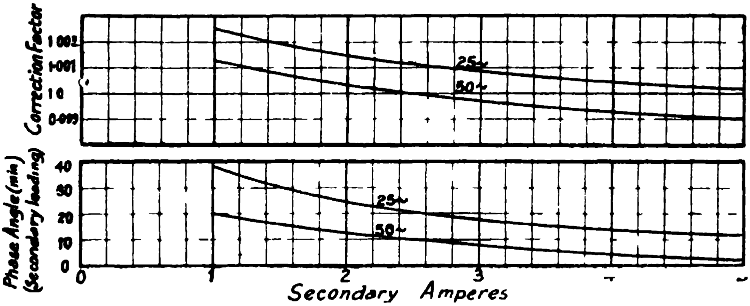


FIG. 273 — Correction Factor and Phase Angle Curves for High Class Current Transformer with Ammeter and Wattmeter Connected in Secondary

Hence, if φ is the phase difference between line voltage and line current, the phase difference between the secondary terminal voltage of the potential transformer and the current in the secondary circuit of the current transformer is $\varphi - (\beta \pm \gamma)$ when φ is lagging, and $\varphi + \beta \pm \gamma$ when φ is leading, the *plus* sign being used in connection with γ when the secondary terminal voltage is lagging, and the *minus* sign when this voltage is leading, with respect to the reversed primary terminal voltage

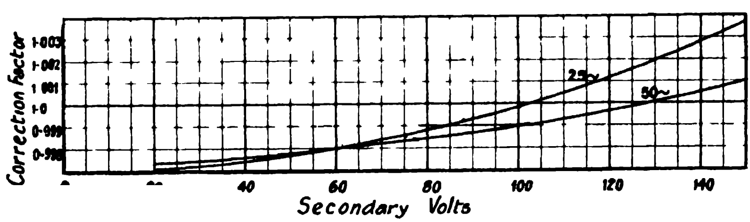


FIG 274 —Correction Factor Curves for Potential Transformer

If δ is the phase difference (lagging) between the voltage applied to the potential circuit of the wattmeter and the current in this circuit, the phase difference between the currents in the fixed and moving coils is $\varphi - (\beta \pm \gamma + \delta)$ when φ is lagging, and $\varphi + \beta \pm \gamma + \delta$ when φ is leading

Hence, if $\cos \varphi_a$ is the apparent power factor determined from the ratio of the uncorrected wattmeter reading and the readings of the ammeter and voltmeter, the power factor $\cos \varphi$ of the primary circuit is given by $\cos [\varphi_a \pm (\beta \pm \gamma + \delta)]$, the *plus* sign to be taken for lagging power factors and the *minus* sign for leading power factors

Therefore the power in the primary circuit is given by

$$P = [\cos \varphi / \cos \{\varphi_a \pm (\beta \pm \gamma + \delta)\}]$$

$\times$ wattmeter reading $\times$ corrected ratio of current transformer $\times$ corrected

ratio of potential transformer, when the power loss in the instruments is ignored.

Example. The following readings (corrected for calibration of instruments) were taken on a voltmeter, an ammeter, and a wattmeter connected (as in Fig. 270c) to instrument transformers on a high-voltage circuit, the load having a lagging power factor.

Volts 100·2; Amperes 3·4; Watts 280.

Data of the instrument transformers are as follow—

Current transformer—

Nominal ratio	15 : 1
Ratio correction factor for a secondary load of 3·4 A.	0·992
Phase angle at this load (β)	0·75°

Potential transformer—

Nominal ratio . .	20 : 1
Ratio correction factor .	1·001
Phase angle (γ) . .	−0·25°

Hence,

Current in main circuit $= 3\cdot4 \times 15 \times 0\cdot992 = 50\cdot6$ A.

Voltage of main circuit $= 100\cdot2 \times 20 \times 1\cdot001 = 2006$ V.

Apparent power factor (from instrument readings)

$$= \cos\varphi_a = \frac{280}{100\cdot2 \times 3\cdot4} = 0\cdot822$$

Phase difference (φ_a) between voltage and current in instrument circuits $= \cos^{-1} 0\cdot822 = 34\cdot7°$

Actual phase difference (φ) between voltage and current in main circuit $= \varphi_a - (\beta - \gamma) = 34\cdot7 - 0\cdot75 + 0\cdot25$
$= 34\cdot2°$

Power factor of main circuit

$$= \cos\varphi = 0\cdot827$$

Power in main circuit $= \dfrac{0\cdot827}{0\cdot822} \times 280 \times 15 \times 0\cdot992$
$\times 20 \times 1\cdot001 = 83{,}900$ W.

With **power measurements in three-phase circuits** by the two-wattmeter method the effect of the phase angles of the instrument transformers is to reduce the phase differences of the currents in the fixed and moving coils of the wattmeters by the angle $\beta \pm \gamma + \delta$. Hence, with balanced loads, the phase difference between the currents in the coils of wattmeter No. 1 is $[30° + \{\varphi - (\beta \pm \gamma + \delta)\}] = (30° + \varphi_a)$, and that between the currents in the coils of wattmeter No. 2 is $[30° - \{\varphi - (\beta \pm \gamma + \delta)\}] = (30° - \varphi_a)$, assuming that similar transformers are used in each phase.

If V_2, I_2 denote the voltage and current for the secondary circuits of the transformers, the readings of the wattmeters represent the quantities $V_2I_2\cos(30° + \varphi_a)$, and $V_2I_2\cos(30° - \varphi_a)$. Whence the algebraic sum of the readings represents the quantity $\sqrt{3}\,.\,V_2I_2\cos\varphi_a$.

Therefore the power in the primary circuit is given by

$$P = [\cos\varphi/\cos\{\varphi_a \pm (\beta \pm \gamma + \delta)\}]$$

× algebraic sum of wattmeter readings × corrected ratio of current transformer × corrected ratio of potential transformer.

CHAPTER XV

ALTERNATING-CURRENT MEASUREMENTS

In this chapter we shall discuss a few of the methods available for determining the principal constants of electric circuits and apparatus as well as methods of measuring current, potential difference, and power. The methods of determining the direction of the phase rotation of a polyphase system will also be considered.

Simple measurement of inductance and capacity by the "impedance" method. This method possesses the advantage of simplicity. It is very convenient for the test-room and laboratory when a high degree of accuracy is not required, and when the value of the impedance under test is within the range of the instruments available.

The general procedure is to measure the impedance of the apparatus (using the ammeter and voltmeter method) at a known frequency, employing preferably a source of sinusoidal E.M.F., and, if possible, an electrostatic voltmeter

Then, for the inductive circuit, if I is the current, V the applied voltage, f the frequency, and R the resistance, the inductance is given by

$$L = \sqrt{(V^2 - R^2I^2)}/2\pi fI \text{ henries,}$$

or, if RI is negligible in comparison with V,

$$L = V/2\pi fI \text{ henries.}$$

For the capacitive circuit

$$C = I/2\pi fV \text{ farads.}$$

The **mutual inductance** of two circuits may be determined by measuring the induced E.M.F. in the secondary circuit when a known current, at a known frequency, is passing in the primary circuit. Then, if E_2 is the value of the induced E.M.F., I_1 the primary current, and f the frequency, we have $E_2 = 2\pi fMI_1$, whence

$$M = E_2/2\pi fI_1$$

It is to be observed that in all these measurements the **accuracy** of the results will be affected by any distortion of the wave-form from the sine wave, the greatest errors occurring in the measurement of capacity. Examples, showing the magnitude of these errors for a particular case, are given in Chapter X.

The error due to non-sinusoidal wave-form may be calculated and allowed for when the equation to the E.M.F. wave is known. The method of doing so is explained in Chapter X.

Again, if an electromagnetic voltmeter is employed instead of an electrostatic voltmeter, the instrument should be so connected that its operating current passes through the ammeter, and the reading of the ammeter should be corrected to allow for this current

Measurement of power and power factor in single-phase circuits. We shall only consider the case in which the power to be measured is small and the measurements are to be made by means of a wattmeter, ammeter, and voltmeter. The connections should be made according to Fig. 275. The wattmeter reading then includes the losses in the pressure circuit of this instrument and the voltmeter; the ammeter reading includes the currents in the pressure circuit of the wattmeter and the voltmeter.

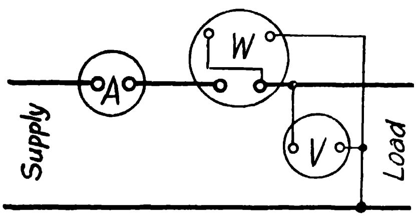


Fig. 275.—Connections of Instruments for Measuring a Small Amount of Power

If P, I, denote the true power and current, respectively, in the load; P', I', the readings of the wattmeter and ammeter, respectively; V the voltage at the load; R_v the resistance of the voltmeter; and R_w the resistance of the pressure circuit of the wattmeter then

$$P = P' - V^2/R_v - V^2/R_w$$

and $$I = I' - V/R_v - V/R_w$$

The power factor is given by

$$\cos \varphi = P/VI$$

[Note. The corrections for the inductance of the pressure circuits are assumed to be negligible.]

Example. The following readings were taken on instruments connected as in Fig. 275—

Volts 100; Amperes 0·4; Watts 30.

The resistance of the voltmeter was 1200 ohms, and that of the pressure circuit of the wattmeter 4,000 ohms. Calculate the power supplied to the load and also the power factor.

Employing the expressions previously obtained, we have

$$P = 30 - 100^2/1200 - 100^2/4000 = 30 - 8{\cdot}33 - 2{\cdot}5 = 19{\cdot}17 \text{ W.}$$

$$I = 0{\cdot}4 - 100/1200 - 100/4000 = 0{\cdot}4 - 0{\cdot}0833 - 0{\cdot}025$$
$$= 0{\cdot}2917 \text{ A.}$$

$$\cos \varphi = 19{\cdot}17/100 \times 0{\cdot}2917 = 0{\cdot}657$$

Note that if the power factor is calculated from the uncorrected readings of the instruments, we obtain

$$\cos\psi' = 30/100 \times 0{\cdot}4 = 0{\cdot}75$$

Measurement of power in single-phase, high-voltage circuit, using a wattmeter without instrument transformers. When the power in a high-voltage circuit is to be measured by a "straight-through" wattmeter, a series resistance, of suitable value, must be connected in series with the moving coil of the instrument. If the moving-coil circuit is designed for a normal voltage V and the resistance

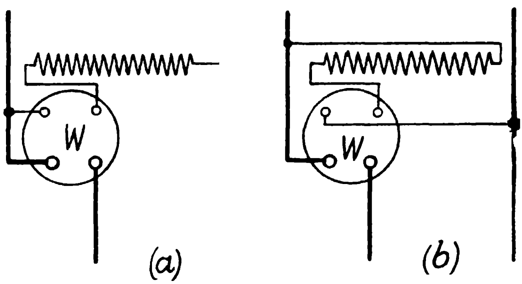


FIG. 276.—Correct and Incorrect Connections of Wattmeter on High-Voltage Circuit

of this circuit is R, the additional resistance, R_1, to be connected in this circuit when the instrument is used on a circuit of voltage V_1 is $R_1 = R(V_1/V - 1)$. The readings of the wattmeter must then be multiplied by the quantity $(1 + R_1/R)$ to obtain the power in the main circuit, and, if necessary, correction factors must be applied (as explained in Chapter XII) for the power expended in the instrument, and for the inductance (and distributed capacity, if any) of this circuit.

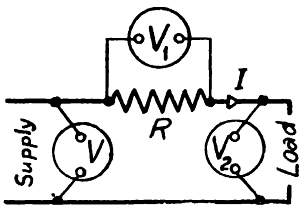
Precautions must be taken when connecting the instrument to the circuit to (1) support the instrument upon an insulating stand (or, if one side of the system is earthed, to connect the instrument to this side of the circuit), (2) make a common connection between the fixed and moving coils as shown in Fig. 276*a*, to ensure that a high voltage cannot exist between these coils.

It is of the utmost importance that the connections shown in Fig. 276*b* be not inadvertently made, as in this case the full line voltage would exist between the coils, thereby causing a breakdown of the insulation and a burning-out of the instrument.

Measurement of power in single-phase circuits without using a wattmeter. The wattmeter method of measuring power is described

in Chapter V, and the corrections which have to be applied to the wattmeter readings under certain circumstances are discussed in Chapter XII. In cases where a wattmeter is not available, and the conditions are favourable, the power (and power-factor) may be measured by alternative methods, known as the "three-voltmeter" and "three-ammeter" methods.

Three-voltmeter method of measuring power and power factor. A non-inductive resistance is connected in series with the apparatus under test, and the voltages across the apparatus, non-inductive resistance, and supply are measured, preferably by electrostatic instruments. The diagram of connections is shown in Fig. 277,



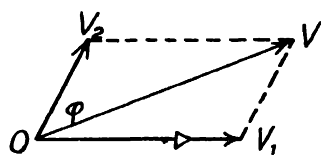


FIG. 277 FIG. 278

Circuit and Vector Diagrams for the "Three-Voltmeter" Method of Measuring Power

and the vector diagram for the circuit is given in Fig. 278. From the vector diagram we have

$$V^2 = V_1^2 + V_2^2 + 2V_1V_2 \cos\varphi$$

where V is the line voltage, V_1 the voltage across the non-inductive series resistance, and V_2 the voltage across the apparatus under test.

The power supplied to the apparatus is

$$P = V_2 I \cos\varphi = (V_2V_1/R)\cos\varphi,$$

where R is the value of the non-inductive resistance.

Hence, substituting in the preceding expression and re-arranging, we obtain

$$P = \tfrac{1}{2}(V^2 - V_1^2 - V_2^2)/R \quad . \quad . \quad . \quad . \quad (216)$$

Also $$\cos\varphi = (V^2 - V_1^2 - V_2^2)/2V_1V \quad (217)$$

These expressions are valid when the wave-form of the supply voltage is sinusoidal, and (in cases of non-sinusoidal wave-form) when the wave-form of the current is identical with that of the impressed E.M.F.

NOTE. Equation (217) may also be employed to calculate the phase difference between two E.M.Fs. which are acting in series.

Thus if V_1, V_2 are the R.M.S. values of the separate E.M.F's., and V their resultant, or vector sum, then if θ is the phase difference between V_1 and V_2 we have

$$\theta = \cos^{-1} [(V^2 - V_1^2 - V_2)/2V_1V_2].$$

Three-ammeter method of measuring power and power factor. In this method a non-inductive resistance is connected in parallel with the apparatus under test, and the currents in this resistance, the apparatus, and the line are measured. The diagram of connections is shown in Fig. 279, and the vector diagram for the circuit is given in Fig. 280. From the vector diagram we have

$$I^2 = I_1^2 + I_2^2 + 2I_1I_2 \cos \varphi$$

where I is the line current, I_1 the current in the non-inductive resistance, and I_2 the current in the apparatus under test

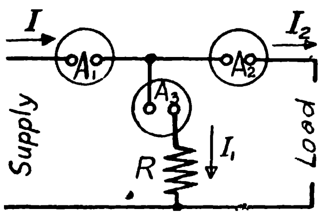


FIG. 279

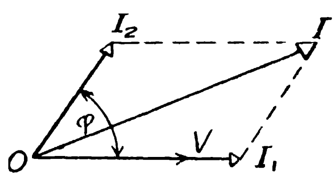


FIG. 280

Circuit and Vector Diagram for the "Three-Ammeter" Method of Measuring Power

The power supplied to the apparatus is

$$P = VI_2 \cos \varphi = I_1R \, . \, I_2 \cos \varphi,$$

where R is the value of the non-inductive resistance. Hence, substituting in the preceding expression and re-arranging, we obtain

$$P = \tfrac{1}{2}R(I^2 - I_1^2 - I_2^2) \quad . \quad . \quad . \quad . \quad . \quad (218)$$

Also $$\cos \varphi = (I^2 - I_1^2 - I_2^2)/2I_1I_2 \quad . \quad . \quad . \quad . \quad . \quad (219)$$

The validity of each of these expressions is governed by the same conditions as apply to the three-voltmeter method.

Electrostatic method of measuring power. In this method a quadrant electrometer is employed in conjunction with a standard non-inductive resistance and a potential divider. The method is particularly suitable for research and standardizing laboratories, and has been perfected by the National Physical Laboratory for measurements of high accuracy.* It is the standard method of

* "The use of the electrostatic method for the measurement of power." *Journal of the Institution of Electrical Engineers*, vol. 51, p. 294.

measuring power in the testing of watt-hour meters and the calibration of wattmeters at this laboratory. The method also possesses advantages over other methods for the measurements of small amounts of power at high voltages and low power factors, such as losses in dielectrics.

To measure power by the quadrant electrometer, a potential difference proportional to, and in phase with, the current in the circuit is applied to the quadrants, and the voltage of the circuit, or a definite fraction thereof, is applied between the needle and quadrants. The deflection is then proportional to the power expended between the point to which the needle is connected and

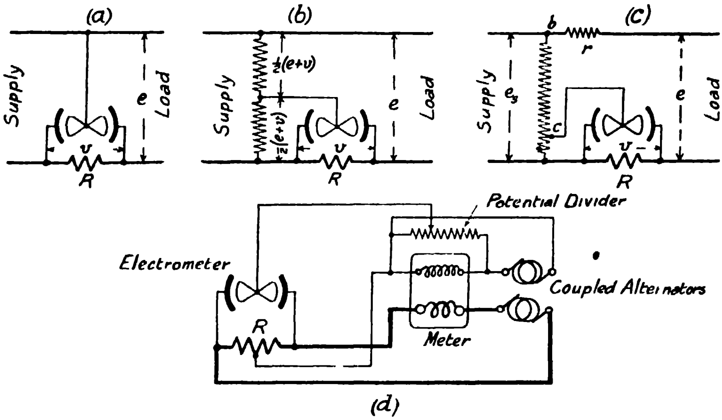


FIG. 281.—Connections for Measurement of Power by Quadrant Electrometer

the point which has a potential midway between the potentials of the quadrants, as proved later.

Thus if the connections are arranged as shown in Fig. 281*a*—which refers to the case where the load is connected directly to the supply system—the deflection of the electrometer is proportional to the power expended in the load *plus* one-half of the power expended in the non-inductive series resistance, or shunt, to which the quadrants are connected.

The correction for the power expended in one-half of the series resistance, *R*, may be eliminated either by connecting the needle to the mid-point of a non-inductive resistance (or a potential divider) connected across the supply system as shown in Fig. 281*b*. or, if this is impracticable, by inserting a second non-inductive resistance, *r*, in the main circuit, as shown in Fig 281*c*, and making

the value of this resistance equal to $\frac{1}{2}R(n-2)$, where R is the value of the resistance across which the quadrants are connected and n is the ratio (supply voltage/voltage between needle and "line" quadrants).

If, however, the mid-point of the resistance R can be utilized, the wattmeter reading is directly proportional to power expended in the load. This is the case when a fictitious load is employed as shown in Fig. 281d, which refers to the test circuit for the calibration of a wattmeter or a watt-hour meter. The current coils of the meter are supplied through a transformer from an alternator, and the pressure coils are supplied from another alternator. The latter is direct coupled to the first machine, and is so arranged that its frame may be given angular displacements with reference to this machine, in order to obtain any desired phase difference between the current and voltage supplied to the meter.

In cases where no correction is necessary for the power expended in the series resistance, R, the power is given by

$$P = kn\,\theta/2R \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

where θ is the deflection, k the "constant" of the instrument (which is determined in the manner described later), and n, R refer to the quantities mentioned previously.

Theory of electrostatic method of measuring power. Let e denote the instantaneous value of the potential difference between the needle and one pair of quadrants, and v the instantaneous value of the potential dfference between the quadrants. Then, from the law of the quadrant electrometer, the deflection, θ, is given by

$$k\theta = \frac{1}{T}\int_0^T (2ev + v^2)dt$$

Now, with the connections arranged as in Fig 281a, $v = iR$, where i is the instantaneous value of the current in the circuit. Hence

$$k\theta = \frac{1}{T}\int_0^T (2Rei + i^2R^2)dt$$

$$= \left[2R \cdot \frac{1}{T}\int_0^T ei\,dt + R \cdot \frac{1}{T}\int_0^T Ri^2\,dt\right]$$

But $\frac{1}{T}\int_0^T ei\,dt$ represents the power (P) supplied to the circuit, and $\frac{1}{T}\int_0^T Ri^2\,dt$ represents the power (P_R) expended in the series resistance, R.

Therefore $\quad k\theta = 2R(P + \frac{1}{2}P_R)$

whence $\quad P = k\theta/2R - \frac{1}{2}P_R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221)$

With the connections arranged as in Fig. 281b, the potential difference

between the needle and the pair of quadrants connected to the load is $[\frac{1}{2}(e + v) - v]$. Hence

$$k\theta = \frac{1}{T}\int_0^T (2v[\tfrac{1}{2}(e + v) - v] + v^2)dt = \frac{1}{T}\int_0^T ev\,dt$$

$$= R \,.\, \frac{1}{T}\int_0^T ei\,dt = RP$$

Whence $\qquad P = k\theta/R \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (222)$

With the connections arranged as in Fig. 281*c*, the potential difference between the needle and the pair of quadrants connected to the load is $(e + ir + iR)/n - iR$, where n is the ratio of the voltage across the tapped

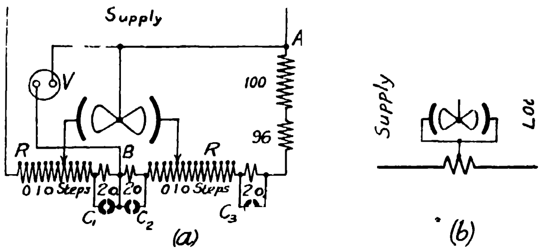


FIG. 282.—Connections for Calibrating Quadrant Electrometer for Power Measurements

portion of the potential divider to the supply voltage, i.e. n = voltage across points $a\,c$/voltage across points $a\,b$. Hence

$$k\theta = \frac{1}{T}\int_0^T (2v[(e + ir + iR)/n + iR] + v^2)dt$$

$$= \frac{2R}{n}\cdot\frac{1}{T}\int_0^T ei\,dt + \frac{1}{T}\int_0^T \frac{i^2R}{n}(2r - R(n-2))dt$$

The first integral represents the power (P) supplied to the load. The second integral becomes zero when $2r = R(n-2)$, or when $r = \frac{1}{2}R(n-2)$. Under these conditions, we have

$$P = nk\theta/2R \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (220)$$

When the instrument is used with a fictitious load, as in Fig. 281*d*, the potential difference between the needle and the pair of quadrants connected to the load is $(e/n - \frac{1}{2}v)$. Hence

$$k\theta = \frac{1}{T}\int_0^T [2v(e/n - \tfrac{1}{2}v) + v^2]dt$$

$$= \frac{2R}{n}\cdot\frac{1}{T}\int_0^T ei\,dt = \frac{2R}{n}\,.\,P$$

whence $\qquad P = nk\theta/2R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (220)$

Method of determining the "constant" of the electrometer. The method employed by the National Physical Laboratory requires the use of standard non-inductive resistances and an electrostatic voltmeter, the calibration of which is accurately known. The connections are shown in Fig. 282.

The alternating supply voltage is adjusted to give the normal voltage (100 V.) between the needle and the neutral point of the quadrants (i.e. 100 V. between points A and B, Fig. 282), and the potential difference between the quadrants is adjusted to any desired value by means of selector switches

connected to standard non-inductive resistances. The resistance in circuit between the points A and B is maintained constant at 200 ohms, and, since, a constant potential difference of 100 V. is maintained between these points the current in the circuit is 0·5 A. Hence when the 2-ohm coils, C_1, C_2, are short-circuited the potential difference between the quadrants can be varied from 0·1 V. to 2 V. in 0·1 V. steps.

[NOTE. The positions of the selector switches must always be such that equal resistances are included between each quadrant and the neutral point A.]

This potential difference can be extended up to 4 V., in 0·1 V. steps by unplugging the 2-ohm coils, C_1, C_2, but when C_2 is inserted an equal resistance (C_3) must be cut out between the points A, B, in order to maintain the total resistance between these points constant at 200 ohms.

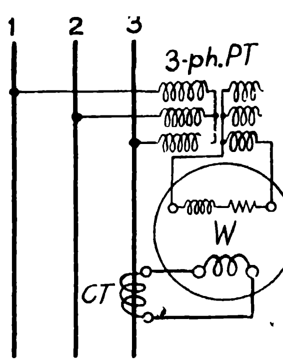


FIG. 283

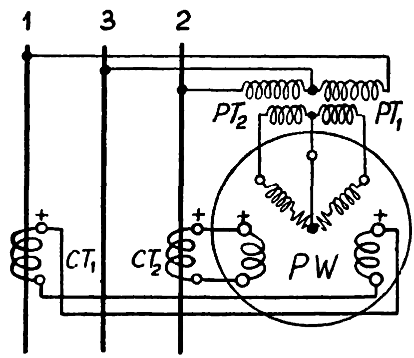


FIG. 284

Connections of Wattmeter and Instrument Transformers for Measuring Power in Balanced and Unbalanced Loads

If θ_1 is the deflection from the "electrical zero," E the R.M.S. voltage between the points A, B; I the current in the circuit; R the resistance connected across the quadrants; and k the constant of the instrument corresponding to the deflection θ_1, then, from the law of the electrometer,

$$k\theta_1 = 2IR(E - \tfrac{1}{2}IR) + I^2R^2 - 2EIR$$

whence $\qquad k = 2EIR/\theta_1$

The "electrical zero" is obtained by connecting both quadrants to the point A (as shown in the diagram b, Fig. 282), with normal voltage between the points A, B.

Measurement of power in three-phase circuits. The principles of the methods of measuring power in three-phase circuits have been discussed in Chapters VIII and IX, and diagrams of connections for low-voltage circuits have been given in these chapters

With high-voltage circuits it is customary to use instrument transformers with the wattmeters, and connection diagrams for balanced and unbalanced loads are given in Figs. 283, 284.

In cases where it is desired to use straight-through wattmeters the connections must be made in accordance with Fig. 285, and two separate wattmeters (*not* a polyphase wattmeter) must be employed, both of which must be supported upon insulating stands.

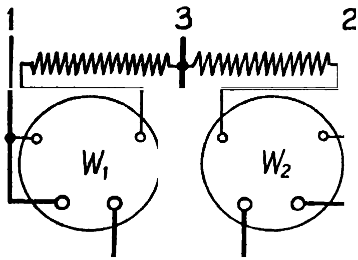


FIG. 285.—Connections of Wattmeters for Measuring Power in High Voltage Circuit

The precautions already mentioned in connection with the measurement of power in single-phase, high-voltage circuits apply with equal force in the present case.

Measurement of iron losses by wattmeter. The commercial method of measuring the losses in sheet iron and steel employed in the manufacture of electrical machinery and transformers is by means of a wattmeter and alternating current of sine wave-form The test samples are in the form of strips 25 cm. long by 5 cm. wide and are assembled to form a closed magnetic circuit, the weight of material required being about 3½ lb. The test strips are carefully insulated from one another by strips of insulating material having the same width and thickness as the test strips, but a length 2 cm. shorter than the latter. The insulated strips are then built into four equal bundles, each about 1 cm. thick, and are inserted into the magnetizing solenoids, each of which is about 23 cm. long and has internal dimensions of 5 cm. by 1 cm. The solenoids are arranged in the form of a square, as shown in

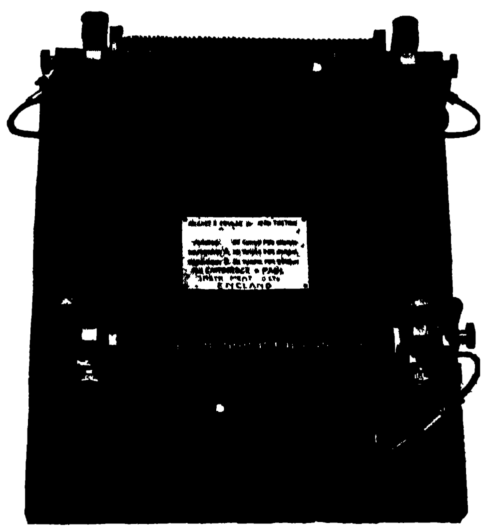


FIG. 286.—Epstein Iron Testing Square
[Cambridge Instrument Co.]

Fig. 286,* and each is wound with two equal, and uniformly distributed, secondary windings and a primary, or magnetizing, winding, the latter being placed outside the former. The magnetic circuit at the corners is completed by small angle pieces of the material under test, these pieces being interleaved with the test strips and secured by non-metallic clamps, as shown in Fig. 287.

One of the secondary windings is connected to an accurate voltmeter, and is used for determining the flux and flux density in the test sample, and the other secondary winding is connected

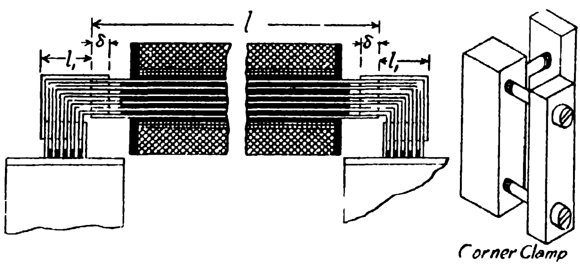


FIG. 287 —Arrangement of Corner-pieces in Epstein Iron Testing Square

to the moving-coil circuit of a sensitive electro-dynamic wattmeter, the fixed, or current, coil of which is connected in series with the primary winding. This (primary) winding is supplied with current from an alternator giving a sine wave-form, and the frequency is measured by a frequency meter.

The reading of the wattmeter, when corrected for the losses in the secondary circuits (i.e. the loss in the pressure circuit of the wattmeter, the loss in the voltmeter, and the small I^2R losses in the secondary windings), gives the iron loss (hysteresis and eddy currents) in the test sample.

It is customary to measure this loss at two frequencies (e.g. 25 and 50 cycles) and at two flux densities (e.g 5000 and 10,000 lines per cm.^{2}), as from these readings the hysteresis loss can be separated from the eddy current loss.

Precautions are necessary when making the observations to ensure that the normal cycle of magnetization is obtained. For example, the material must first be demagnetized by applying the

* This method of arranging the samples and the method of testing is due to Epstein. The apparatus (Fig. 286) is called the "Epstein iron-testing square," and its construction has been standardized in Europe, this country, and the United States of America.

exciting current and reducing this gradually to zero At subsequent applications of the exciting current the circuit must be closed through a high resistance, which must be cut out gradually until the desired magnetization is obtained. Again, when a change is made from a higher to a lower flux density, the exciting current must be reduced gradually to the lower value.

Theory of wattmeter method of measuring iron losses. The instantaneous value of the E.M.F. induced in each secondary winding is

$$e_2 = -10^{-8} \times N_2\, d\Phi/dt$$

and the R.M.S. value is

$$E_2 = 4k_f N_2 f \Phi_m \times 10^{-8} = 4k_f N_2 f B_m A \times 10^{-8},$$

where k_f is the form-factor of the flux wave-form, N_2 the number of turns in each secondary winding, Φ the flux linked with these turns, f the frequency of the magnetizing current, B_m the crest value of the flux density in the test sample, and A the magnetic cross-section of each bundle of test samples.

The cross-section, A, is determined indirectly from the mass, length, and density of the test strips. Thus, if W is the mass in grammes, $A = 7{\cdot}77\, W/4l$, where 7·77 represents the density in grammes per cm.^{3}, and l is the length, in cm., of the test strips.

Whence $B_m = E_2 \times 10^8/(4k_f N_2 f A)$

Therefore the flux density in the test samples may be determined from the E.M.F. induced in the secondary winding, together with a knowledge of the frequency and form-factor of the flux wave-form. Since, for the reasons given in Chapter XI, the tests should be carried out with sinusoidal flux wave-form, the value of $k_f = 1{\cdot}11$.

The instantaneous power supplied to the primary winding is given by

$$e_1 i_1 = i_1^2 R_1 + 10^{-8} \times N_1 i_1\, d\Phi/dt + Li_1\, di_1/dt$$

where e_1 is the instantaneous value of the impressed E.M.F., i_1 the primary current, R_1 the resistance of the primary winding, N_1 the number of turns in the primary winding, Φ the flux linked with these turns, and L the self-inductance due to any flux not included in Φ.

The mean power (i.e. the power expended during a period) is given by

$$\frac{1}{T}\int_0^T e_1 i_1 dt = i_1^2 R_1 + \frac{1}{T}\int_0^T i_1 \frac{N_1}{10^8}\frac{d\Phi}{dt}\,dt + \frac{1}{T}\int_0^T i_1 L\frac{di_1}{dt}\,dt$$

$$= i_1^2 R_1 + \frac{1}{T}\int_0^T i_1 \frac{N_1}{10^8}\frac{d\Phi}{dt}\,dt$$

since $\frac{1}{T}\int^T i_1\, L\frac{di_1}{dt}\,dt = 0.$

Hence the quantity $\frac{1}{T}\int_0^T i_1 \frac{N_1}{10^8}\frac{d\Phi}{dt}$ represents the power expended in iron losses, together with the power supplied to the secondary circuits. This quantity may be expressed in the form

$$\frac{N_1}{N_2}\left(\frac{1}{T}\int_0^T i_1 \frac{N_2}{10^8}\frac{d\Phi}{dt}\,dt\right) = \frac{N_1}{N_2}\left(\frac{1}{T}\int_0^T i_1 e_2\, dt\right)$$

Now if P is the reading (in watts) of the wattmeter, R the resistance of the pressure circuit of the wattmeter, and R_2 the resistance of the secondary winding to which this circuit is connected, then, from the theory of the electro-dynamic wattmeter given in Chapter XII, we have

$$P = R\frac{1}{T}\int_0^T i_1 \frac{e_2}{R + R_2}\,dt = \frac{1}{1 + R_2/R}\left(\frac{1}{T}\int_0^T i_1 e_2\, dt\right)$$

or $$\frac{1}{T}\int_o^T i_1 e_2 \, dt = (1 + R_2/R)P$$

and if R_2 is very small in comparison with R, we have

$$P = \frac{1}{T}\int_o^T i_1 \, e_2 \, dt$$

Therefore the reading of the wattmeter, multiplied by the ratio of the numbers of turns in primary and secondary windings, gives the iron losses and the losses in the instruments (pressure circuits).

Whence $$P_i = P - (E_2{}^2/R + E_2{}^2/R_v)$$

where P_i denotes the iron losses, E_2 the reading of the voltmeter, and R, R_v the resistances of the pressure circuits of wattmeter and voltmeter respectively.

To obtain the iron losses in watts per kg. of the test samples, corrections must be applied for the mass of the corner pieces and the decreased flux density at the lapped joints. Let W denote the mass, in grammes, of the test strips, w_1 the mass of the corner pieces, B_m the maximum flux density in the strips (as determined from the voltmeter reading), l the length of the test strips, l_1 the maximum projection of the corner pieces beyond the test strips (Fig. 287), and δ the width of the overlap at the corner pieces. Then, since the corner pieces are of the same material and thickness as the test strips and the cross-section, A, is uniform throughout the magnetic circuit, the flux density in the overlapping portions of the test strips and corner pieces is $\frac{1}{2}B_m$.*

Hence if $2w_2$ is the mass of the strips and corner pieces in which the flux density is $\frac{1}{2}B_m$, and $P_{i(kg)}$ is the specific iron loss in watts per kg., then the total iron loss (P_i) is given by

$$\begin{aligned} P_i &= P_{i(kg)} \, [(W + w_1 - 2w_2) + 2w_2(\tfrac{1}{2}B_m/B_m)^x] \\ &= P_{i(kg)} \, W \left[1 + \frac{w_1}{W} - \frac{2w_2}{W}(1 - (\tfrac{1}{2})^x)\right] \end{aligned}$$

where x is the index of the flux density to which the iron loss is proportional. The value of x lies between 1·6 and 2.

If $x = 1{\cdot}6$, the term $(1 - (\frac{1}{2})^x)$ becomes equal to 0·67, and if $x = 2$, this term becomes equal to 0·75. If we take the value 0·7 for the term $(1 - (\frac{1}{2})^x)$, then

$$P_{i(kg)} = \frac{P_i}{W}\left(\frac{1}{1 + w_1/W - 1{\cdot}4w_2/W}\right)$$

Observe that P_i/W is the apparent value of the specific iron loss which is obtained by dividing the corrected wattmeter reading by the mass of the test strips, and the term in parenthesis is the correction factor for the corner pieces and the lapped joints.

Instead of determining w_2 directly, it may be expressed in terms of the quantities W, w_1, l, l_1. Thus

$$\frac{w_2}{W} = \frac{\delta}{l} = \frac{w_1}{W} - \frac{l_1}{l}$$

since $$\frac{w_1}{W} = \frac{l_1}{l} + \frac{\delta}{l}$$

Whence $$P_{i(kg)} = \frac{P_i}{W}\left(\frac{1}{1 - 0{\cdot}4w_1/W + 1{\cdot}4l_1/l}\right) \quad . \quad . \quad . \quad . \quad (223)$$

* Actually the flux density is about 1 per cent lower on account of magnetic leakage. The effect of this leakage and the slight non-uniformity in the flux density (due to the length of the magnetic path being slightly different for each corner piece) has been investigated by the Bureau of Standards and the maximum error due to these causes is only a fraction of 1 per cent.

If the quantity $W(1 - 0{\cdot}4w_1/W + 1{\cdot}4l_1/l)$ is considered as the "equivalent mass," W_e, of the test sample, then $P_{i(kg)} = P_i/W_e$.

Separation of iron loss into hysteresis and eddy-current components. If the specific iron loss follows the equation

$$P_{i(kg)} = \eta f B_m^x + \xi f^2 B_m^y,$$

where the first term represents the hysteresis loss and the second term the eddy-current loss, these components may be separated by determining $P_{i(kg)}$ for two different frequencies at a given flux density. The exponents x, y, may be evaluated by determining $P_{i(kg)}$ for two different flux densities at a given frequency.

Thus, if observations are made at frequencies f_1, f_2, and flux density B_m, we have

$$\frac{P_{i(kg)1}}{f_1} = \eta B_m^x + \xi f_1 B_m^y = a + bf_1$$

$$\frac{P_{i(kg)2}}{f_2} = \eta B_m^x + \xi f_2 B_m^y = a + bf_2$$

where a, b, denote the hysteresis and eddy-current losses, respectively, per kg. per cycle at the flux density B_m.

Whence
$$a = \frac{(P_{i(kg)2}/f_2)f_1 - (P_{i(kg)1}/f_1)f_2}{f_1 - f_2}$$

$$b = \frac{P_{i(kg)1}/f_1 - P_{i(kg)2}/f_2}{f_1 - f_2}$$

If $f_1 = 2f_2$ the computation is greatly simplified, for then

$$a = 2P_{i(kg)2} - P_{i(kg)1}$$

$$b = \frac{1}{f_2}\left(\frac{P_{i(kg)1}}{f_1} - \frac{P_{i(kg)2}}{f_2}\right)$$

Example. The following data refer to a typical set of observations and results:—

Mass of test strips, 1327 grammes.
Mass of corner pieces, 80·2 grammes
Length of test strips, 25 cm.
Projection of corner pieces beyond test strips (dimension l_1, Fig. 267), 0·8 cm.
Correction factor

$$= \frac{1}{1 - 0{\cdot}4(80{\cdot}2/1327) + 1{\cdot}4(0{\cdot}8/25)} = \frac{1}{1{\cdot}0206}$$

Equivalent mass of test strips $= 1327 \times 1{\cdot}0206 = 1355$ grammes.

Frequency	Flux density (B_m).	Wattmeter reading (watts).	Losses in instruments (watts).	Iron losses (watts).	Specific iron loss (W/kg.).	Hysteresis loss.	Eddy-current loss.
						(ergs per cycle per gramme).	
50	10,000	3·58	0·08	3·5	2·583	394	124
25	10,000	1·58	0·04	1·54	1·136	394	62

Alternating-current potentiometer methods of measuring E.M.F. The potentiometer principle of comparing E.M.Fs. enables measurements of high accuracy to be performed, as the balance is obtained

by a null method. In the direct-current potentiometer the balance is obtained between the magnitudes of the "unknown" E.M.F. and the potential difference between certain points in the potentiometer wire, but in the alternating-current potentiometer the balance must be obtained between the phases of these quantities as well as between their magnitudes.

The theory of the alternating-current potentiometer is quite simple. Thus, if the "unknown" E.M F. is represented by

$$\dot{E} = E\underline{/\varphi_e} = \pm \boldsymbol{e}_1 \pm j\boldsymbol{e}_2,$$

and the potential difference against which it is balanced is represented by

$$\dot{V} = V\underline{/\varphi_v} = \pm \boldsymbol{v}_1 \pm j\boldsymbol{v}_2,$$

then the condition of balance is

$$\dot{V} = \dot{E}.$$

Hence $V = E$, and $\varphi_v = \varphi_e$.

Or, alternatively, $\pm \boldsymbol{v}_1 \pm j\boldsymbol{v}_2 = \pm \boldsymbol{e}_1 \pm j\boldsymbol{e}_2$

which requires $\pm \boldsymbol{v}_1 = \pm \boldsymbol{e}_1$

and $\pm j\boldsymbol{v}_2 = \pm j\boldsymbol{e}_2$

Hence, two adjustments, which must be carried out in succession. are necessary in obtaining the balance with an alternating-current potentiometer.

Two methods of effecting the double adjustment have been devised: In one method (due to Drysdale) a phase-shifting transformer is employed ; in the other method (due to Gall) the in-phase and quadrature components of the two quantities are balanced separately. The application of these methods will now be considered.

Drysdale-Tinsley alternating-current potentiometer. This instrument consists of a direct-current potentiometer with the addition of a phase-shifting transformer and a precision electro-dynamic ammeter. A selector switch (to enable four "unknown" E.M.Fs. to be connected to the instrument and to be transferred to the potential slides without changing the external connections) and a change-over switch (to enable the potentiometer to be used on either a direct-current, or an alternating-current, circuit) also form part of the instrument. The resistance coils forming part of the potentiometer wire circuit are, of course, wound non-inductively.

When the instrument is used for alternating-current measurements, an alternating-current, of constant magnitude, is maintained in the potentiometer-wire circuit, the correct value* of the current (50 mA.) being indicated by the electro-dynamic ammeter. This current is obtained from the secondary (or rotor) of a phase-shifting transformer, the primary (or stator) of which

* I.e. the "standard" value of the current in the potentiometer wire when the instrument is used with direct current

has a two-phase winding and is excited from a supply circuit of constant voltage and frequency. Usually, the exciting current is derived from a single-phase supply circuit: in this case one winding is supplied directly, and the other through a phase splitting condenser and resistance, as shown in Fig. 288. [The theory of this method of "phase-splitting" is given on p. 126.] Special features are introduced to ensure that the E.M.F. induced in the rotor is of sine wave-form and is of constant magnitude for all relative positions of rotor and stator.

Since the constancy of the rotor current for different relative positions of rotor and stator depends upon the production of a perfectly uniform rotating field in the stator, i.e. upon the accuracy with which the phase splitting is effected, the phase splitting operation is performed in the following manner—

The resistance and capacity are first roughly adjusted to their appropriate values with the rotor index pointer at zero and the rheostats in the potentiometer-wire circuit are adjusted to give the standard current of 50 mA. The rotor index pointer is then set at 45° *leading*, and if the current has altered in

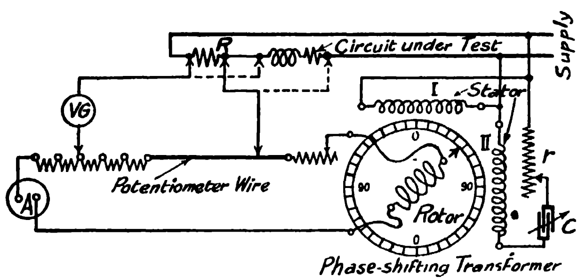


FIG. 288.—Principle of Alternating-Current Potentiometer

value the *capacity* is readjusted to restore the current to its standard value. Next the pointer is set at 45° *lagging*, and if the current has altered the *resistance* is readjusted to restore the current to its standard value. The pointer is then returned to zero, the rheostats in the potentiometer-wire circuit are readjusted, if necessary, and the above cycle of operations is repeated.

The rotor is fitted with an index pointer (marked AXIS) representing the position of the magnetic axis of the rotor, a scale divided into quadrants, and a worm and worm wheel for adjusting the position of the rotor relatively to the stator.

The E.M.F. to be determined is applied to the potentiometer wire *via* potential slides, and a vibration galvanometer, connected in this circuit, is employed for indicating the condition of the balance.

The balance (i.e. zero deflection of the galvanometer) is obtained by adjusting successively the positions of the potential slides and the rotor. For example, firstly, the positions of the potential slides are adjusted to give the minimum deflection of the galvanometer; secondly, the rotor is adjusted to obtain a lower minimum; thirdly, the adjustments are repeated until zero deflection is obtained. The magnitude and phase of the E.M.F., when the balance is obtained, are read off from the potential slides and the rotor pointer and scale.

Gall-Tinsley co-ordinate potentiometer. This instrument (Fig. 289) consists of two potentiometers (which are called the "in-phase" and "quadrature" potentiometers) fitted into a common case. The potentiometers are supplied with equal currents (50 mA.) having a mutual phase difference of 90°, the

currents being obtained from a single-phase supply system through a step-down isolating transformer, the latter being for the double purpose of isolating the potentiometer circuits from the supply system and for obtaining a suitable voltage (6 V.) for these circuits.

The correct value of the current in the "in-phase" potentiometer is indicated (to about 1 part in 10,000) by a sensitive reflecting electro-dynamic instrument which is connected in this circuit. The electro-dynamic instrument is not permanently calibrated (as is the case with the instrument used with the Drysdale potentiometer), but the current in the potentiometer wire is adjusted to its standard value (50 mA.) by means of direct current and a standard cell, in the same manner as if the potentiometer were being used on

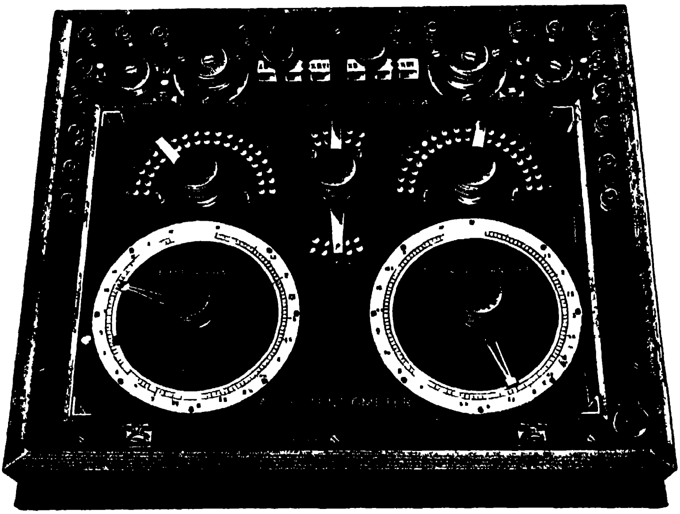


FIG. 289.—Co-ordinate Potentiometer
[H. Tinsley and Co.]

a direct-current circuit. The control (torsion head) of the electro-dynamic instrument is then so adjusted to bring the "spot" to a definite position on the scale, and this position therefore indicates the standard value of current in the potentiometer wire when the supply current is either direct or alternating.

The adjustment of the current in the "quadrature" potentiometer to its correct value and phase difference is effected by connecting in this circuit the primary winding of a standard mutual inductance and balancing the E.M.F. induced in the secondary winding against a definite potential difference of the "in-phase" potentiometer. If the mutual inductance is so constructed as to be free from eddy currents, the E.M.F. induced in its secondary will have a phase difference of 90° with respect to the current in the primary. Therefore, when this E.M.F. is balanced against the potential difference of the "in-phase" potentiometer, the current in the "quadrature" potentiometer has a phase difference of 90° with respect to that in the in-phase potentiometer.

By suitably choosing the value of the mutual inductance in relation to the frequency of the supply circuit, a simple relationship may be obtained between

induced E.M.F. and frequency. For example, if the mutual inductance is 0·3183 (= 0·1/π) H., the E.M.F. (E_2) induced in the secondary winding when the standard current (50 mA.) is passing in the primary winding is given by—

$$E_2 = 2\pi f M I_1 = 2\pi f \times (0{\cdot}1/\pi) \times 50 \times 10^{-3} = f/100.$$

Thus $E_2 = 0{\cdot}5$ V. for 50 frequency.

Therefore, if the supply frequency is 50, the potential slides of the "in-phase" potentiometer are set to 0·5 V. and the magnitude and phase of the current in the "quadrature" potentiometer are adjusted in succession until no deflection is obtained on the vibration galvanometer, the (alternating)

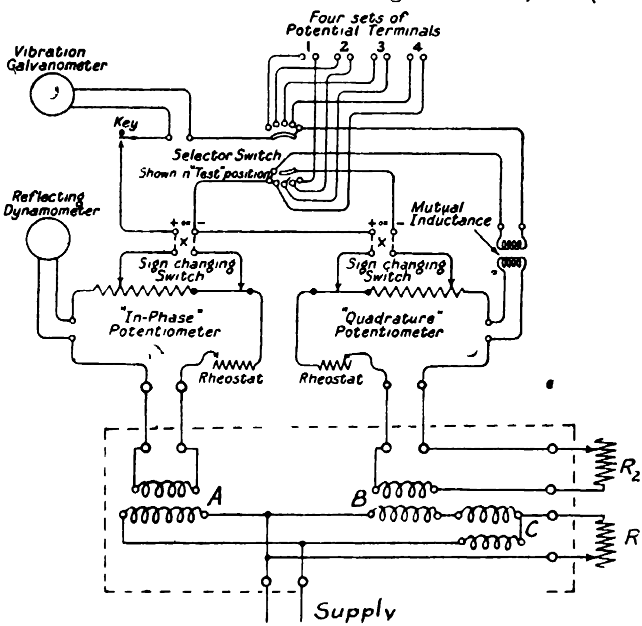


FIG. 290.—Connections of Co-ordinate Potentiometer

[NOTE.—The lower portion of the diagram (inside the dotted rectangle) refers to the isolating transformers A, B, and the phase-splitting transformer, C]

current in the "in-phase" potentiometer being maintained at the standard value.

The "unknown" E.M.F. to be determined is applied, *via* a vibration galvanometer, to both sets of potential slides (which are now connected in series with each other) and a balance is obtained by adjusting the potential slides, and, if necessary, changing the positions of the reversing switches, which are connected in series with them (see Fig. 290). The "in-phase" component of the E.M.F. is then given, in magnitude and sign, by the settings of the potential slides and reversing switch, respectively, of the "in-phase" potentiometer. Similarly, the "quadrature" component of the E.M.F. is given in magnitude and sign by the settings of the potential slides and reversing switch of the "quadrature" potentiometer. By means of shunts incorporated in the instrument, the "constants" of either set of potential slides may be changed to one-tenth normal, thereby enabling small magnitudes of either of the E.M.F. components to be accurately determined.

The magnitude of the "unknown" E.M.F. is, of course, given by

$$\sqrt{[(\text{in-phase E.M.F.})^2 + (\text{quadrature E.M.F.})^2]},$$

and its phase difference with respect to the current in the in-phase potentiometer wire (which is the quantity of reference for all E.M.F. measurements) is

$$\varphi = \tan^{-1}(\text{quadrature E.M.F.})/(\text{in-phase E.M.F.}).$$

Potentiometer method of measuring impedance. The impedance to be measured has a standard non-inductance resistance connected in series with it, and is supplied with current from the same source as the potentiometer. The potential differences at the terminals of the standard resistances and the impedance are then determined. Let these quantities, as determined by the Drysdale potentiometer, be $V_1\underline{/\varphi_1}$ and V_2/φ_2, respectively, where V_1, V_2, are the settings of the potential slides and φ_1. φ_2 the settings of the phase-shifting transformer. Then, if R_s is the value of the standard resistance, the impedance of the apparatus under test is given by

$$Z = \frac{R_s V_2}{V_1}\underline{/(\varphi_2 - \varphi_1)}$$

Whence $\qquad Z = R_s V_2/V_1$

Observe that, although the phase angles φ_1, φ_2 are not involved in the determination of the numerical value of the impedance, a knowledge of these angles enables the resistance and reactance to be calculated.

If the measurements are made on the Gall co-ordinate potentiometer and are denoted by $\pm \boldsymbol{e}_1 \pm j\boldsymbol{e}_2$, and $\pm \boldsymbol{e}'_1 \pm j\boldsymbol{e}'_2$, where $\pm \boldsymbol{e}_1 \pm \boldsymbol{e}'_1$ represent the settings of the potential slides of the "in-phase" potentiometer, and $\pm \boldsymbol{e}_2$, $\pm \boldsymbol{e}'_2$, the settings of the "quadrature" potentiometer when determining the voltages across the standard resistance and impedance, respectively, then

$$Z = R_s\sqrt{[(\boldsymbol{e}'^2_1 + \boldsymbol{e}'^2_2)/(\boldsymbol{e}_1^2 + \boldsymbol{e}_2^2)]}$$

Bridge methods of measuring inductance, capacity, and frequency. A large number of bridge methods employing alternating current have been devised for the measurement of inductance and capacity,* but we shall here only consider briefly one or two methods.

The general method of comparing inductances or capacities is by the **impedance bridge,** the theory of which was discussed in Chapter VI as an exercise on parallel circuits. With this method the value of unknown inductance or capacity is determined in terms of a standard known inductance or capacity respectively. An

* These methods are discussed in *Alternating-Current Bridge Measurements* (Hague), to which the student is referred for full details and proofs.

advantage possessed by the impedance bridge is that the frequency does not enter into the calculations of the results.

When a telephone is employed as a detector, difficulties may be experienced in obtaining an exact balance if the telephone is not at the same (earth) potential at the observer. This difficulty may be overcome by the Wagner earthing device, which enables the detector to be operated at the equivalent of earth potential without actually earthing any point of the bridge itself.

The application of this device to the impedance bridge (arranged for the comparison of capacities) is shown diagrammatically in Fig. 291. An adjustable auxiliary circuit, *e*, *f*, *g*, consisting of a condenser and a non-inductive resistance, is connected across the supply and the centre point of this circuit is earthed. The bridge is first approximately balanced (with switch *S* in position 1); the telephone is then transferred to points *d* and *f* (by placing switch *S* in position 2), and a subsidiary balance is obtained by adjusting the resistance and capacity of the auxiliary circuit. This balance indicates that the points *d* and *f* are at earth potential. A final balance is then obtained on the bridge with the switch *S* in position 1.

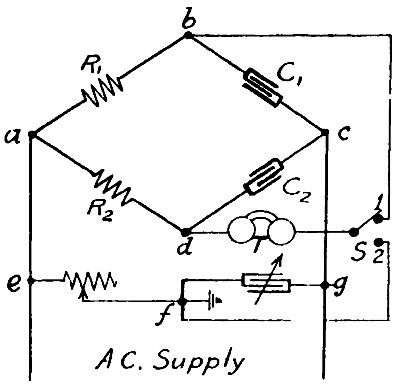


FIG. 291.—Wagner Earthing Device Applied to an A.C. Bridge

The method is applicable to all impedance bridges provided that the auxiliary circuit is similar in its make-up to the arms of the bridge across which it is connected

Anderson bridge. This bridge is convenient for the determination of inductance when a standard condenser is available. The value of the "unknown" inductance is obtained in terms of the capacity of the condenser, and the frequency is not involved in the calculation. A high degree of accuracy is obtainable if the bridge is used under favourable conditions.

The connections are shown in Fig. 292*a*, in which *LS* represents the inductance to be measured, *C* the standard condenser, *PQ* ratio coils, *R* a fixed resistance, and *r* a decade box of resistance coils. All the resistances should be non-inductive and suitable for alternating-current bridge measurements. In some cases it may

be desirable to include a variable non-inductive resistance in series with the "unknown" inductance, as this may facilitate the adjustments for balancing the bridge.

In carrying out a test, the ratio coils P, Q should be given equal values, and the value of r—together with the variable resistance, if any, in series with the inductance—adjusted until balance is obtained. Then, generally,

$$L = 10^{-6}\, CR[r(1 + Q/P) + Q] \quad . \quad . \quad . \quad (224)$$

and when $P = Q$,

$$L = 10^{-6}\, CR(2r + Q) \quad . \quad . \quad . \quad (224a)$$

where L is given in henries, C in microfarads, and P, Q, R, r in ohms.

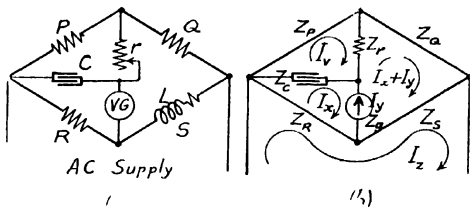


FIG. 292 —Anderson Bridge for Measurement of Inductance

[NOTE. If r has a high value the detector and source of supply should be interchanged in order to obtain the maximum sensitiveness of the bridge and detector.]

Theory of the Anderson bridge. The theory may be developed by the application of Kirchoff's laws to the network. Thus, if the currents in the meshes are represented by I_v, I_x, $I_x + I_y$, I_z, as shown in Fig. 292*b*, we have

$$I_v Z_P + (I_v - I_x - I_y)Z_r + (I_v - I_x)Z_c = 0$$
$$(I_x - I_z)Z_R + (I_x - I_v)Z_c + I_y Z_g = 0$$
$$(I_x + I_y - I_z)Z_s + I_y Z_g + (I_x + I_y - I_v)Z_r + (I_x + I_y)Z_Q = 0$$

Solving for the galvanometer current, I_y, we obtain

$$I_y = \frac{I_z[Z_R Z_Q(Z_c + Z_r) + Z_R Z_P(Z_Q + Z_r) - Z_c Z_P Z_s]}{\text{A function of the impedances}}$$

and since for balance $I_y = 0$, we have

$$Z_R Z_Q(Z_c + Z_r) + Z_R Z_P(Z_Q + Z_r) = Z_c Z_P Z_s$$

Now if $Z_R = R$, $Z_P = P$, $Z_Q = Q$, $Z_r = r$, $Z_c = -j(1/\omega C)$, $Z_s = S + j\omega L$, this equation reduces to

$$-j\frac{RQ}{\omega CP} + \frac{RrQ}{P} + R(Q + r) = -j\frac{S}{\omega C} + \frac{L}{C}$$

Whence, equating the in-phase components, we have

$$\begin{aligned} L &= C[Rr\,Q/P + R(Q + r)] \\ &= CR[r(1 + Q/P) + Q] \quad . \quad . \quad . \quad . \quad . \quad . \quad (224) \end{aligned}$$

and, equating the quadrature components, we have

$$RQ/\omega CP = S/\omega C$$

or

$$S = RQ/P$$

Campbell frequency bridge. This bridge is suitable for the measurement of frequencies preferably of the order of 1000 and upwards. In its simplest form it requires the use of a standard variable mutual inductance, a standard mica condenser of good quality, and a telephone (which is employed as a detector). The connections are shown in Fig. 293*a*. Balance is obtained by varying the mutual inductance. The reactance of the condenser is then equal to the reactance of the mutual inductance, and we have the relationship

$$\omega M = 1/\omega C$$

or

$$\omega = \sqrt{(1/MC)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

If the condenser is imperfect (i.e. it has dielectric losses) a sharp balance cannot be obtained with the simple arrangement shown in Fig. 293*a*. To compensate for the imperfection of the condenser, the arrangement shown in Fig. 293*b* may be employed. The imperfect condenser is represented by a capacity C and a low resistance ρ connected in series. This condenser is connected in series with a standard variable mutual inductance M—as in the simple form of bridge (Fig. 293*a*)—but in its primary and secondary circuits are also included the primary circuits of two other mutual inductances, M_1, M_2 (one of which should be variable), the secondary circuits of which are connected together.

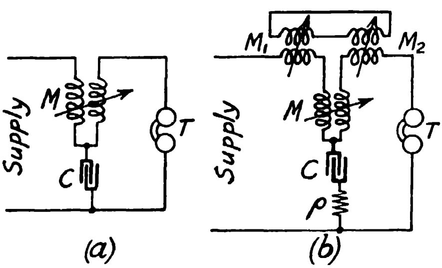


FIG. 293.—Campbell Bridge for Measurement of Frequency

Balance is obtained by varying M and either M_1 or M_2.

When a balance is obtained, we have

$$1/\omega^2 C = M + \omega^2 L M_1 M_2/(R^2 + \omega^2 L^2) \quad . \quad . \quad (226)$$

where M, M_1, M_2, denote the values (in henries) of the mutual inductances, C the capacity (in farads) of the condenser, and L, R the inductance and resistance, respectively, of the closed circuit formed by the secondary circuits of the auxiliary mutual inductances M_1, M_2.

Method of testing watt-hour and power factor meters at varying power factors. Two methods are available for polyphase meters and three methods for single-phase meters, two of the latter being similar to the methods employed with three-phase meters. The methods common to single-phase and polyphase meters are—

1 Supplying the current and potential coils of the meters from separate alternators, which are mechanically coupled together and have the same number of poles The stator of one alternator is

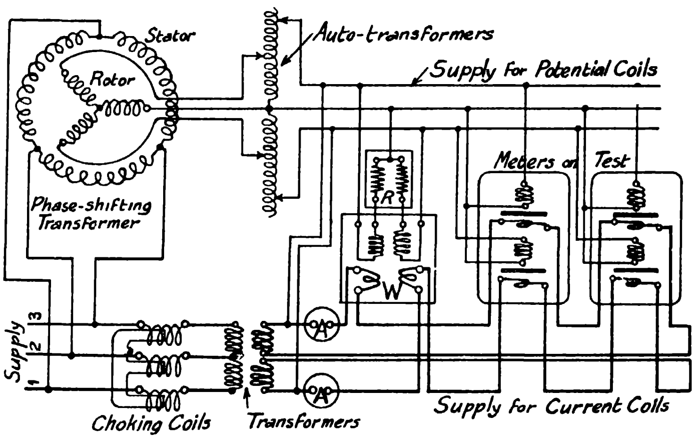


FIG. 294 —Connections for Testing Polyphase Watt Hour Meters on Fictitious Load

so constructed that its magnetic axis can be displaced relatively to that of the other alternator, and by these means any desired phase difference may be obtained between the E M.Fs. of the alternators. Thus, conditions equivalent to any desired power factor may be obtained in the meter circuits. The connections in the case of a single-phase meter have already been given in Fig. 281*d*.

2. Supplying the potential coils from the secondary winding of a phase-shifting transformer, the primary of which is excited from the system supplying the current coils.

A diagram of connections for this method is given in Fig. 294, from which it will be observed that the current coils of the meters are supplied through step-down transformers (and, if necessary, current transformers), and the potential circuits are supplied, through auto-transformers, from the secondary windings of the phase-shifting transformer. Regulation of current in the "current"

circuit of the meters is effected by means of a regulating choking coil connected in the primary circuit of the step-down transformer, and regulation of voltage for the potential circuits is effected by means of tappings on the auto-transformers.

The equivalent power in the meter circuits is determined by means of a standard electro-dynamic wattmeter, and the equivalent power factor is obtained either from the dial of the phase-shifting transformer or the volt-amperes in the meter circuits.

The equalizing connections between the current and potential circuits are for the purpose of ensuring that the fixed and moving coils of the wattmeter are at the same potential.

[NOTE. In both the above cases the meters are tested on a fictitious load, and the tests are effected with the expenditure of only a small amount of energy, viz. that necessary to supply the losses in the instruments and apparatus.]

Another method—which is applicable only to single-phase meters, and does not require the special apparatus which is necessary with the preceding methods—is based upon the phase relationship between the currents and voltages in a three-phase system. The connections are shown in Fig. 295*a*, and Fig. 295*b*, and vector diagrams are shown in Fig. 295*c*. The load should be non-inductive and may consist of banks of incandescent lamps.

The current coil of the meter under test, M, and that of the standard wattmeter, W, are connected in series with the line wire, No. 3, which is common to the two loads. When a power factor between unity and 0·5 is required, the potential circuits of the instruments are connected between this line (No. 3) and one of the other lines (Fig. 295*a*), according to whether a lagging or leading power factor is required. But when a power factor between 0·5 and zero is required, the potential circuits are connected across lines 1 and 2, as shown in Fig. 295*b*.

The vector diagram for these conditions is shown in Figs. 295*c*. The current, I, is the current coils of the instruments in the vector sum of the currents in the loads. The latter are represented by the vectors OI_A, OI_B—which are in phase with the line voltage vectors $OV_{1\text{-}3}$, $OV_{2\text{-}3}$, respectively—and the current in the instrument is represented by OI. Hence, if the loads are equal, OI bisects the angle between the vectors $OV_{1\text{-}3}$, $OV_{2\text{-}3}$, and is perpendicular to both $OV_{1\text{-}2}$ and $OV_{2\text{-}1}$. These conditions represent power factors (in the meter circuit) of 0·866 (lagging), 0·866 (leading), zero (lagging), zero (leading).

With unequal loads, the vector OI will occupy a position intermediate between the vectors $OV_{1\text{-}3}$, $OV_{2\text{-}3}$.

In the extreme case, when one load (B) is zero, OI is in phase with OV_{1-3}, and leads 60° with respect to OV_{2-1}. Hence these conditions represent power factors of unity and 0·5 (leading), according to whether the potential circuits are connected across lines 3 and 1, or across lines 3 and 2. In the other extreme case, when the load A is zero, the conditions are equivalent to power factors of unity and 0·5 (lagging), according to whether the potential circuits are connected across lines 3 and 2, or 3 and 1.

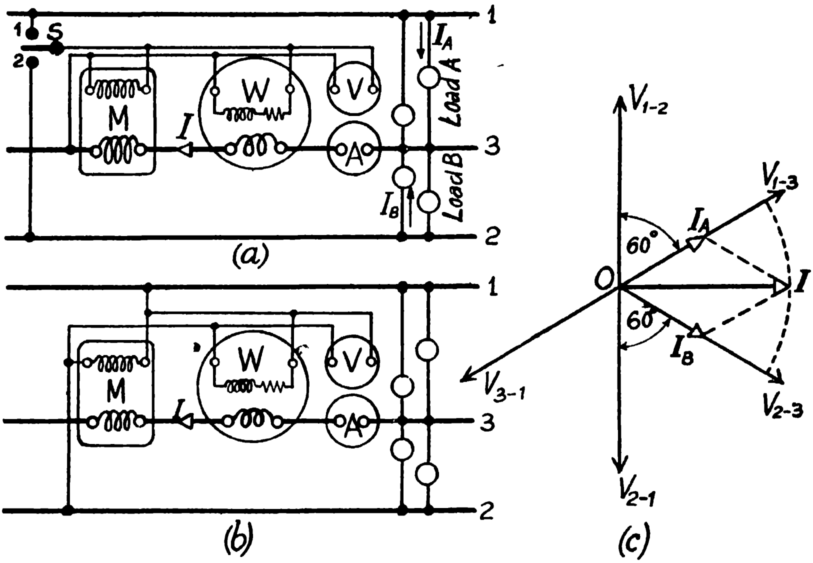


FIG 295 Connections for Testing Single Phase Watt Hour Meters at Various Power Factors

In all these cases it is assumed that the symmetry of the three-phase system is unaffected by the unbalanced loading.

If the direction of the phase rotation of the three-phase system is unknown, the phase of the current (i.e. whether lagging or leading) may be ascertained by a simple test, which consists of connecting an inductance temporarily in the pressure-coil circuit of the standard wattmeter, and noting the readings, with and without the inductance, for the same load conditions. If the reading with the inductance in circuit is larger than that without the inductance, the current is lagging; if the reading is smaller, the current is leading.

Determination of the phase sequence or direction of phase rotation in a three-phase system. A number of methods are available, some of which utilize the principles discussed in Chapter IX in connection with unbalanced star-connected circuits, while others—such as those employed in phase-rotation relays, and similar switchboard instruments—utilize the principle of the induction wattmeter.

A **simple phase-rotation indicator,** in which an unbalanced star-connected circuit is employed, requires only two *similar* incandescent lamps (which form a visual indicator) and either an inductance (preferably iron-cored) or a condenser. These are

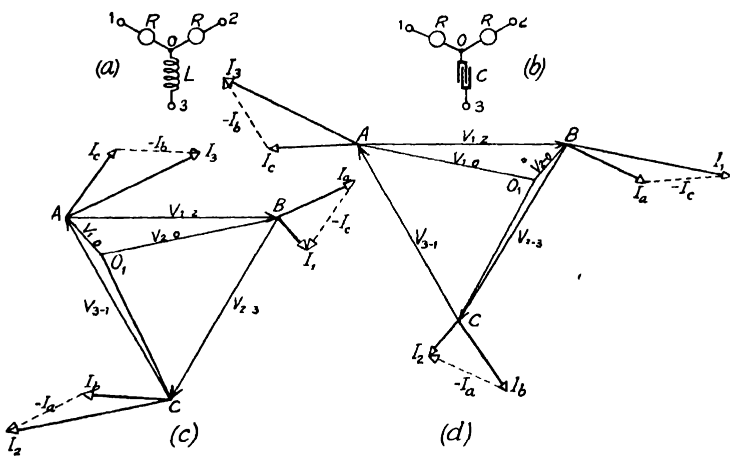


FIG. 296.—Circuit and Vector Diagrams for the Lamp-Resistance Method of Determining Direction of Phase Rotation

connected in star as shown in Figs. 296*a*, 296*b*. When either of these circuits is connected to a three-phase system, the lamps will have unequal voltages impressed upon them, and this feature is utilized to determine the direction of the phase rotation, as with the *inductive* circuit (Fig. 296*a*), the phase, or line wire, connected to the *bright* lamp is *lagging* with respect to that connected to the dim lamp, and with the *capacitive* circuit (Fig. 296*b*) the phase connected to the *bright* lamp is *leading* with respect to that connected to the dim lamp.

The vector diagrams are shown in Figs. 296*c*, 296*d*, and are drawn by the method employed in connection with Fig. 159. The line voltages are represented by the vector triangle *ABC*, the line currents (which are obtained from the phase currents of the

equivalent delta-connected circuit) are represented by the vectors OI_1, OI_2, OI_3.

Since the voltages across the lamps are in phase with the currents OI_1, OI_2, the potential of the neutral point O_1 is determined by drawing from the corners A and B of the triangle ABC, parallels AO_1, BO_1, to the vectors BI_1, CI_2, respectively, to meet at the point O_1.

[NOTE. The vectors BI_1, BI_2 represent the currents in the lamps A and B respectively.]

Then AO_1 represents the voltage across lamp A, BO_1 the voltage across lamp B, and CO_1 the voltage across the remaining branch (inductance or condenser) of the circuit.

Observe that with the inductive circuit (Figs. 296*a*, 296*c*), BO_1 is greater than AO_1; but that, with the capacitive circuit (Figs. 296*b*, 296*d*), AO_1 is greater than BO_1. Hence, since the vector diagrams have been drawn for clockwise phase rotation, the bright lamp is connected in the "*lagging* phase" when an *inductance* is employed (Fig. 296*a*), and in the "*leading* phase" when a *condenser* is employed (Fig. 296*b*); the terms "lagging phase" and "leading phase" referring to the phases in which the lamps are connected.

In a practical form of this indicator for 200/220 V., 50-cycle circuits, ordinary 20 W. or 30 W., 200 V., lamps are employed, and the condenser has a capacity of from 1 to 2 μF. The voltage impressed upon one lamp is about 80 per cent of the line voltage, and that impressed upon the other lamp is about 25 per cent of the supply voltage.

Example. Two equal non-inductive resistances and a condenser are connected in star, as in Fig. 296, and the combination is connected to a three-phase supply system. The capacity of the condenser is so chosen that the reactance of the condenser branch is equal to the resistance of one of the non-inductance branches Determine the voltages across the lamps and condenser.

Let R denote the resistance in each of the non-inductive branches. Then $-jR$ denotes the reactance of the condenser branch.

The line currents may be determined either by the indirect method, which involves the conversion of the star-connected circuit into the equivalent delta-connected circuit, or by the direct application of Kirchoff's laws to the star-connected circuit. As the former method introduces some features of interest for the circuit under consideration, it will be adopted in present case.

Thus, denoting the equivalent impedances between the line wires 1–2, 2–3 3–1 (Fig. 296*b*), as Z_a, Z_b, Z_c, taken in order, we have, from equation (76),

$$Z_a = R + R + R \,.\, R/(-jR) = R(2 + j1)$$

$$Z_b = R - jR - R \,.\, jR/R \qquad = R(1 - j2)$$

$$Z_c = -jR + R - jR \,.\, R/R \quad = R(1 - j2)$$

Hence, for clockwise phase rotation, the currents in the equivalent delta-connected circuit are

$$\dot{I}_a = \frac{V}{R}\frac{(1 + j0)}{(2 + j1)} = \frac{V}{R}(0{\cdot}4 - j0{\cdot}2)$$

$$\dot{I}_b = \frac{V}{R}\frac{(-0{\cdot}5 - j0{\cdot}866)}{(1 - j2)} = \frac{V}{R}(0{\cdot}246 - j0{\cdot}373)$$

$$\dot{I}_c = \frac{V}{R}\frac{(-0{\cdot}5 + j0{\cdot}866)}{(1 - j2)} = \frac{V}{R}(-0{\cdot}446 - j0{\cdot}027)$$

The currents in the star-connected circuit are then $\dot{I}_1 = \dot{I}_a - \dot{I}_c$, $\dot{I}_2 = \dot{I}_b - \dot{I}_a$, $\dot{I}_3 = \dot{I}_c - \dot{I}_b$, and the voltages across the branches of this circuit are $\dot{V}_{10} = R\dot{I}_1$, $\dot{V}_{20} = R\dot{I}_2$, $\dot{V}_{30} = -jR\dot{I}_3$.

Whence $\dot{V}_{10} = V(0{\cdot}846 - j0{\cdot}173)$

$$V_{10} = V\sqrt{(0{\cdot}846^2 + 0{\cdot}173^2)} = 0{\cdot}866\,V.$$

$$\dot{V}_{20} = V(-0{\cdot}154 - j0{\cdot}173)$$

$$V_{20} = V\sqrt{(0{\cdot}154^2 + 0{\cdot}173^2)} = 0{\cdot}232\,V.$$

$$\dot{V}_{30} = V(0{\cdot}346 + j0{\cdot}692)$$

Wattmeter method of determining the direction of phase rotation in a three-phase system. When the power in the system is being measured by the two-wattmeter method (using separate wattmeters) the phase rotation may be determined at the same time by taking two extra readings with a reactance (e.g. either a condenser or an inductance) inserted at the junction of the two pressure-coil circuits, as shown in Fig. 297. The star-connected circuit thus formed is similar to the circuit in Fig. 296, and, therefore, the changes in the magnitudes and phases of the voltages impressed upon the pressure coil circuits of the wattmeters, due to the insertion of the reactance, will depend upon the direction of the phase rotation of the three-phase system.

Vector diagrams representing the conditions when a condenser is connected in the pressure-coil circuits are shown in Fig. 297. A study of these diagrams will show that if A, B, denote the readings of the wattmeters when the connections are normal (A being the larger reading*), and A', B', denote the readings when the condenser is connected in the pressure-coil circuits, then if

$$A' \simeq A,\ B' < B;\ \text{or if } A' < A,\ B' > B,$$

* The special case when the two readings are equal is considered in the example which follows.

wattmeter B is connected in the "lagging" phase when the power factor is "lagging" and is in the "leading" phase when the factor is "leading."

To ascertain whether the power factor—or, more correctly, the

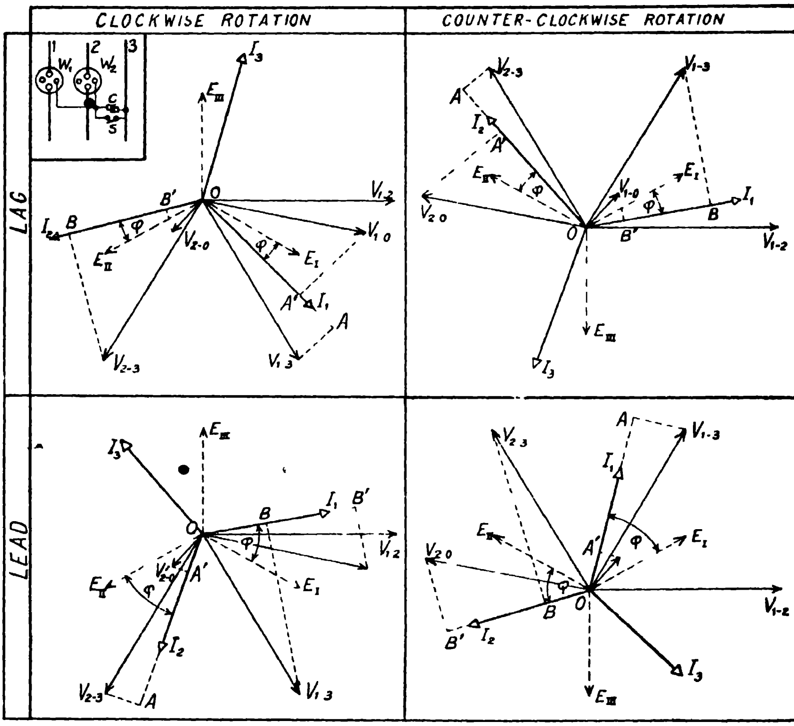


FIG. 297.—Vector and Circuit Diagrams for Two-Wattmeter Method of Determining Direction of Phase Rotation

current in the main circuit—is lagging or leading, the following simple test is applied*—

Open the common connection between current and pressure coils of the wattmeter which gives the larger reading when the connections

* This test depends upon the phase relationship between the line voltages and currents of a symmetrical three-phase system. Thus, with normal connections the phase difference between the currents in the coils of the wattmeter giving the larger reading is ($30° - \varphi$), and when the common end of the pressure coil is transferred to the other line the phase difference becomes ($90° - \varphi$) when the power factor is lagging and ($90° + \varphi$) when the power factor is leading.

are normal. Connect this end of the pressure coil to the line wire in which the second wattmeter is connected. If the new reading is *positive*, the current is *lagging*; if *negative*, the current is *leading*.

Experimental Verification. The following readings were obtained on two wattmeters connected, in accordance with the diagram in Fig. 297, to a three-phase circuit, the power factor of which could be varied as desired. The power factor was indicated by a power factor meter, and the direction of phase rotation was determined by an independent test—

Reading of Wattmeter Connected in Line No. 1	Reading of Wattmeter Connected in Line No. 2.	Power Factor.	Phase Rotation.
20	47	lagging	2–1–3
2*	35*	lagging	2–1–3
48	21	leading	2–1–3
17*	44*	leading	2–1–3
24	48	leading	1–2–3
46*	17*	leading	1–2–3
49	20	lagging	1–2–3
37*	3*	lagging	1–2–3

* These readings were taken with a condenser inserted in the potential circuit of wattmeters.

Example. Two similar wattmeters are connected in the usual manner for measuring the power in a three-phase circuit and provision is made for inserting a condenser in the pressure-coil circuits in the manner shown in Fig. 297. The reactance of the condenser is equal to the resistance of one of the pressure-coil circuits of the wattmeters.* Determine, for power factors of 1·0, 0·707, 0·5, 0·2, and both directions of phase rotation, the relationship between the readings of the wattmeters (i) when the connections are normal; (ii) when the condenser is connected in the pressure-coil circuits. The line voltage, current, and frequency are constant.

The voltages impressed upon the potential coils of the wattmeters when the condenser is in circuit are calculated as in the above example, and are given by

$$V_{1-0} = V\,(0{\cdot}846 - j0{\cdot}173),\quad V_{2-0} = V\,(-0{\cdot}154 - j{\cdot}0173),\ \text{for clock-}$$

wise phase rotation, and by

$$V'_{1-0} = V\,(0{\cdot}154 + j0{\cdot}173),\quad V'_{2-0} = V\,(-0{\cdot}846 + j{\cdot}0173)\ \text{for}$$

counter-clockwise phase rotation, the voltage between lines 1 and 2 being the quantity of reference.

Hence, for clockwise phase rotation and with the condenser in the pressure-coil circuits the reading (W'_1) on the wattmeter connected in line 1 is equal to $I_1 V_{1-0} \cos (18{\cdot}4° + \varphi)$, and that ($W'_2$) of the wattmeter connected in line 2 is $I_2 V_{2-0} \cos (18{\cdot}4° + \varphi)$, the plus sign being taken for lagging power factors. Similarly, for counter-clockwise phase rotation, the readings are: $W''_1 = I_1 V'_{1-0}$ $\cos (18{\cdot}4° + \varphi)$, $W''_2 = I_2 V'_{2-0} \cos (18{\cdot}4° + \varphi)$.

* If R is the resistance of the pressure-coil circuit, the capacity required is given by $C = 10^6/\omega R\ \mu F$. If the frequency is 50 and $R = 3{,}180$ O., $C = 1\ \mu F$.

The results are best expressed in tabular form—

Power Factor.	Phase Rotation	W_1 (without Condenser)	W'_1 (with Condenser)	W_2 (without Condenser).	W'_1 (with Condenser).
1·0	C	0·866 VI_1	0·712 VI_1	0·866 VI_2	0·22 VI_2
1·0	CC	0·866 VI_1	0·22 VI_1	0·866 VI_2	0·712 VI_2
0·707 (lag)	C	0·966 VI_1	0·398 VI_1	0·259 VI_2	0·104 VI_2
0·707 (lead)	C	0·259 VI_1	0·774 VI_1	0·966 VI_2	0·207 VI_2
0·707 (lag)	CC	0·259 VI_1	0·104 VI_1	0·966 VI_2	0·398 VI_2
0·707 (lead)	CC	0·966 VI_1	0·207 VI_1	0·259 VI_2	0·774 VI_2
0·5 (lag)	C	0·866 VI_1	0·174 VI_1	0	0·0466 VI_2
0·5 (lead)	C	0	0·65 VI_1	0·866 VI_2	0·174 VI_2
0·5 (lag)	CC	0	0·0466 VI_1	0·866 VI_2	0·174 VI_2
0·5 (lead)	CC	0·866 VI_1	0·174 VI_1	0	0·65 VI_2
0·2 (lag)	C	0·664 VI_1	−0·102 VI_1	−0·315 VI_2	−0·0274 VI_2
0·2 (lead)	C	−0·315 VI_1	0·433 VI_1	0·664 VI_2	0·116 VI_2
0·2 (lag)	CC	−0·315 VI_1	−0·0274 VI_1	0·664 VI_2	−0·102 VI_2
0·2 (lead)	CC	0·664 VI_1	0·116 VI_1	−0·315 VI_2	0·433 VI_2

Watt-hour meter method of determining direction of phase rotation in a three-phase system. For this method, an induction-type polyphase watt-hour meter with the terminals marked according to the B E S A specification is required. The tests applied

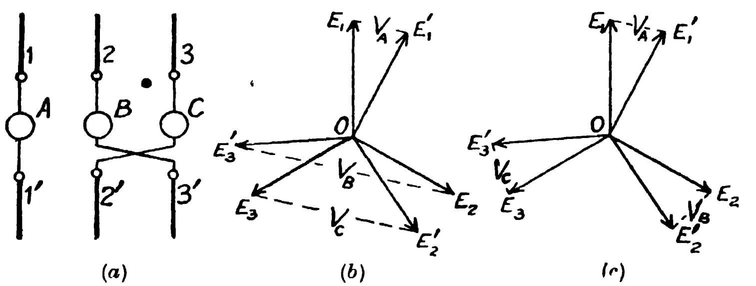


FIG. 298—Circuit and Vector Diagrams for the Comparison of Phase Rotation by Lamps

depend upon the power factor of the load connected to the meter, and are as follows—

1. When the load is non-inductive. Disconnect the "red" element of the meter from the circuit and connect a non-inductive resistance* in series with the potential or shunt circuit of the "blue" element. Take a reading. Next disconnect the "blue" element from the circuit and connect the non-inductive resistance in series with the shunt circuit of the "red" element and take a reading with the same load as before. Then the element which gives the higher reading is connected in the "leading" phase of the supply system.

* The value of the resistance should be of the order of 500 ohms per 100 volts of the normal operating voltage.

2. When the current is lagging. The test is similar to that for case (1), but the non-inductive resistance is not required.

Comparison of phase rotation of two three-phase systems. A simple method of comparing the phase rotation of two three-phase systems—e.g. two alternators which are to be operated in parallel—is to connect three similar lamps to the systems according to the scheme shown in Fig. 298*a*. Then, assuming the frequencies to be equal, if the systems have the same phase rotation, the lamps A and B—which are cross-connected in relation to the supply systems—will glow brightly, and lamp C will be either very dim or completely dark. On the other hand, if the phase rotations are not in the same direction, *all* the lamps will either glow with equal brightness or be completely dark.

Vector diagrams for these cases are shown in Figs. 298*b*, 298*c*, respectively The vectors OE_1, OE_2, OE_3, represent the equivalent phase voltages of one system and OE'_1, OE'_2, OE'_3, represent the equivalent phase voltages of the other system. Then, if the two systems have the same phase rotation and equal phase voltages (E), and θ is the phase difference between the systems, the voltages, V_A, V_B, V_C, impressed upon the lamps are

$$V_A = 2E \sin \tfrac{1}{2}\theta, \quad V_B = 2E \sin \tfrac{1}{2}(120° \pm \theta),$$
$$V_c = 2E \sin \tfrac{1}{2}(120° \mp \theta).$$

In the special case, when $\theta = 0$,

$$V_A = 0, \quad V_B = 2E \sin 60° = V, \quad V_c = 2E \sin 60° = V,$$

where $V(= \sqrt{3} \,.\, E)$ is the line voltage of each system

If the phase rotations are not the same, then

$$V_A = 2E \sin \tfrac{1}{2}\theta, \quad V_B = 2E \sin \tfrac{1}{2}\theta, \quad V_c = 2E \sin \tfrac{1}{2}\theta.$$

If the phase rotations are the same, but the frequencies differ slightly, then θ varies with respect to time and passes through a cycle of 360° during the time that one system is gaining or losing a cycle with respect to the other system. Hence, during the interval, the lamps A, B, C will each complete in succession the cycle dark-bright-dark-bright-dark.

The order in which the lamps B and C light up depends upon which system has the higher frequency. This feature may be utilized in connection with the synchronizing of small alternators, and was formerly applied by Siemens to synchroscopes.

Determination of ratio and phase angle of instrument transformers. These quantities may be determined directly by means of the Gall co-ordinate potentiometer (Fig. 289). In the case of a current transformer, the secondary winding is connected to the normal

instrument load, in which is included a standard non-inductive low resistance. A similar resistance is connected in the primary circuit. The potential differences across these resistances due to the primary and secondary currents are measured by the potentiometer, and from these measurements and a knowledge of the values of the standard resistances both the ratio and phase angle can be calculated.

In the case of a potential transformer, the secondary winding is connected to an instrument load and also to a potential divider from which a suitable voltage is obtained for the potentiometer. A similar potential divider is connected across the primary winding. The voltages are measured by the potentiometer, and the ratio and phase angle may be calculated from these measurements and a knowledge of the ratios of the potential dividers.

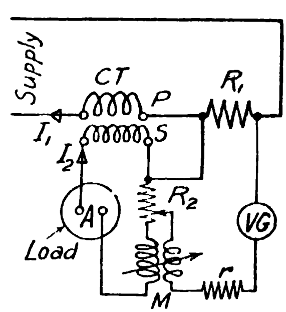


FIG. 299.—Method of Determining Ratio and Phase Angle of Current Transformer

When a co-ordinate potentiometer is not available, the methods now to be described may be adopted, the principles of which are based upon the potentiometer method of comprising E.M.Fs. A standard variable mutual inductance, standard non-inductive resistances, and a vibration galvanometer are required for the tests.

Ratio and phase angle of current transformer. The connections are shown in Fig. 299, in which R_1, R_2 represent standard non-inductive low resistances, M a standard variable mutual inductance, and VG a vibration galvanometer. R_2 and M are adjusted successively until there is no deflection of the galvanometer. Then, if the standard resistances are entirely non-inductive, the ratio is given by

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}\sqrt{(1 + \omega^2 M^2/R_2^2)} \quad . \quad . \quad (227)$$

and the phase angle β is given by

$$\beta = \tan^{-1} \omega M/R_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

Proofs. In the proofs we shall consider that the standard resistances R_1, R_2 have slight inductances L_1, L_2 respectively. Then, when the current in the vibration galvanometer is zero, we have

$$I_2 [(R_2 + j\omega L_2) - j\omega M] = I_1(R_1 + j\omega L_1)$$

i.e. $$\frac{I_1}{I_2} = \frac{R_2 + j\omega(L_2 - M)}{R_1 + j\omega L_1} = \frac{R_1R_2 + \omega^2 L_1(L_2 - M)}{R_1^2 + \omega^2 L_1^2} - j\omega\left[\frac{L_1R_2 - R_1(L_2 - M)}{R_1^2 + \omega^2 L_1^2}\right]$$

Whence
$$\frac{I_1}{I_2} = \sqrt{\left[\frac{R_2^2 + \omega^2(L_2 - M)^2}{R_1^2 + \omega^2 L_1^2}\right]}$$
$$= \frac{R_2}{R_1}\sqrt{\left[\frac{1 + \omega^2 (L_2 - M)^2/R_2^2}{1 + \omega^2 L_1^2/R_1^2}\right]} \qquad (227a)$$

which, when $L_1 = 0$, $L_2 = 0$, reduces to

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}\sqrt{(1 + \omega^2 M^2/R_2^2)}$$

Similarly, the phase angle, β, is given by

$$\beta = \tan^{-1}\frac{\omega L_1 R_2 - \omega R_1(L_2 - M)}{R_1 R_2 + \omega^2 L_1(L_2 - M)} \qquad (228a)$$

which, when $L_1 = 0$, $L_2 = 0$, reduces to

$$\beta = \tan^{-1}\omega M/R_2$$

Ratio and phase angle of potential transformer. The connections are shown in Fig. 300, in which a non-inductive high resistance R_1,

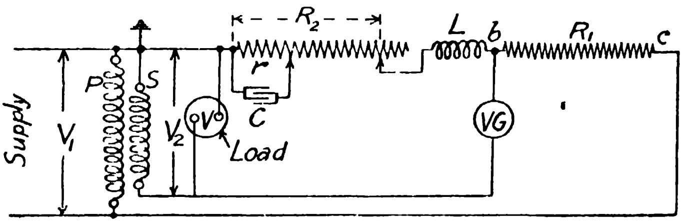


FIG 300—Method of Determining Ratio and Phase Angle of Potential Transformer

together with an inductance L and a smaller non-inductive resistance R_2 (a portion of which, r, is shunted by a condenser C), are connected across the primary winding. The secondary winding, to which the instrument load is connected, is reversed and connected through a vibration galvanometer across the resistance R_2, and the inductance L. Zero deflection on the galvanometer is obtained by successive adjustment of R_2, r, and, if necessary, C and L. Then, if the resistances R_1, R_2 are non-reactive, the ratio is given by

$$\frac{V_1}{V_2} = \frac{R_1 + R_2}{R_2}\sqrt{\left[\frac{1 + [\omega(L - Cr^2)/(R_1 + R_2)]^2}{1 + [\omega(L - Cr^2)/R_2]^2}\right]} \quad . \qquad (229)$$

and the phase angle, β, is given by

$$\beta = \tan^{-1}\omega R_1(L - Cr^2)/[R_2(R_1 + R_2) + \omega^2(L - Cr^2)^2] \quad . \qquad (230)$$

Proofs. The resistances R_1, R_2, are assumed to be without reactance. The impedance between the points a, b (Fig. 300), is given by

$$Z = [R_2 - r + r/(1 + \omega^2 C^2 r^2)] + j\omega[L - Cr^2/(1 + \omega^2 C^2 r^2)]$$

and if $\omega^2 C^2 r^2$ is negligible in comparison with unity,

$$Z = R_2 + j\omega(L - Cr^2)$$

Hence, when the current in the vibration galvanometer is zero, we have

$$V_2 = \frac{V_1 Z}{R_1 + Z}$$

i.e.
$$\frac{V_1}{V_2} = \frac{R_1 + Z}{Z} = \frac{R_1 + R_2 + j\omega(L - Cr^2)}{R_2 + j\omega(L - Cr^2)}$$

$$= \frac{R_2(R_1 + R_2) + \omega^2(L - Cr^2)^2}{R_2^2 + \omega^2(L - Cr^2)^2} + j\omega\left[\frac{R_1(L - Cr^2)}{R_2^2 + \omega^2(L - Cr^2)^2}\right]$$

Whence
$$\frac{V_1}{V_2} = \sqrt{\left[\frac{(R_1 + R_2)^2 + \omega^2(L - Cr^2)^2}{R_2^2 + \omega^2(L - Cr^2)^2}\right]}$$

$$= \frac{R_1 + R_2}{R_2}\sqrt{\left[\frac{1 + [\omega(L - Cr^2)/(R_1 + R_2)]^2}{1 + [\omega(L - Cr^2)/R_2]^2}\right]}$$

and
$$\tan\beta = \frac{\omega R_1(L - Cr^2)}{R_2(R_1 + R_2) + \omega^2(L - Cr^2)^2}$$

CHAPTER XVI

INITIAL (TRANSIENT) CONDITIONS IN SIMPLE ELECTRIC CIRCUITS

IN all the preceding discussions of electric circuits, particularly those given in Chapters III and IV, the relationship between impressed E.M.F. and current was obtained by *assuming* the law of variation of the current, e g. by assuming a sinusoidal current to be flowing in the circuit. The conditions relating to the sudden application of a sinusoidal E.M.F. to circuits possessing resistance, inductance, and capacity will now be investigated.

Relationship between impressed E.M.F. and initial current for a non-inductive circuit. Let the impressed E.M.F. be represented by the equation $e = E_m \sin \omega t$. Then, if R denotes the resistance of the circuit, the internal E M.F. (e_R) at any instant is given by $e_R = -Ri$, and this E M.F. must balance the impressed E.M.F.

Hence, for any value (e) of the impressed E.M F the current must equal e/R. Therefore, at whatever point on the E.M.F. wave the circuit is closed, the current rises instantly to the normal value corresponding to the value of the E M.F. For example, if the circuit is closed at the instant the impressed E.M.F. attains its maximum value, the current rises immediately from zero to its maximum value. Thus the conditions become permanent as soon as the circuit is closed and there are no transient phenomena

Relationship between impressed E.M.F. and initial current for a purely inductive circuit. Let the inductance of the circuit be L and the resistance be zero. Then at every instant the impressed E.M.F. balances the E.M F. of self-inductance, and the relationship between impressed E M F. and current is given by the equation

$$e = L\,di/dt = E_m \sin \omega t,$$

which may be re-arranged in the form

$$\frac{di}{dt} = \frac{e}{L} = \frac{E_m}{L} \sin \omega t.$$

Transposing, we have

$$di = (E_m/L) \sin \omega t \,.\, dt,$$

and, upon integrating this expression, we obtain

$$i = -(E_m/\omega L) \cos \omega t + A \qquad . \quad . \quad . \quad (231)$$

where A is the arbitrary constant of integration, the value of which is determined by the conditions existing at the instant the circuit is closed.

For example, let the circuit be closed when ωt has the value ψ. The current at this instant is zero. Hence, substituting for ωt and i in equation (231), we obtain

$$0 = -(E_m/\omega L)\cos\psi + A,$$

whence $$A = (E_m/\omega L)\cos\psi.$$

Therefore, the general equation for the current is

$$\begin{aligned} i &= -(E_m/\omega L)\cos\omega t + (E_m/\omega L)\cos\psi \\ &= (E_m/\omega L)(\cos\psi - \cos\omega t) \qquad . \quad (232) \end{aligned}$$

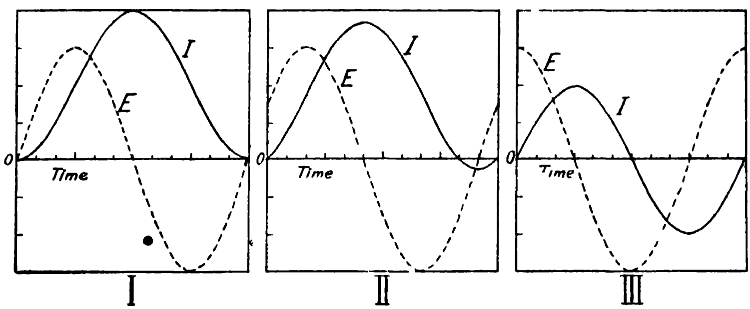


FIG. 301.—Wave-forms of Initial Current in Purely Inductive Circuit

[NOTE —The wave-form of the impressed E.M.F. is shown dotted]

It can be shown that, except for the special cases when $\psi = 0°$ and $\psi = 90°$, this equation represents a sinusoidal curve with its axis displaced from the abscissa axis (*see* curves I, II, III, Fig 301).

In the special case when $\psi = 0°$ the current curve does not cross the abscissa axis, and, therefore, the current never reverses (i.e. the current is pulsating instead of alternating).

In the other special case, when $\psi = 90°$, the current curve is symmetrical with respect to the abscissa axis, and the permanent conditions—as were considered in Chapter III—obtain from the instant at which the circuit is closed.

In Fig. 301 are shown the current wave-forms—calculated from equation (232)—corresponding to $\psi = 0°$, $\psi = 30°$, $\psi = 90°$, when the impressed E.M.F., at the instant of closing the circuit, has the values 0, 0·5 E_m, E_m.

Since equation (232) does not contain a transient term, the conditions represented by this equation are permanent, i.e. the succeeding current waves will be exact reproductions of the initial

wave, and the current will continue in the same form in which it started.

In practice, purely inductive circuits devoid of resistance and losses cannot be obtained, and with all highly inductive circuits the resistance, together with the losses in the iron core, produce damping effects, so that, a short time after the circuit is closed, the current wave-form assumes its normal shape.

Relationship between impressed E.M.F. and initial current for an inductive circuit (resistance and inductance in series). The general equation connecting impressed E.M.F and current is

$$e = Ri + L\,di/dt = E_m \sin \omega t,$$

which, when re-arranged, becomes

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_m}{L} \sin \omega t \qquad . \qquad . \qquad (233)$$

This is a linear differential equation of the first order, the general form of which is

$$\frac{dy}{dx} + Py = Q.$$

Such an equation is integrated by the factor $\varepsilon^{\int P dx}$, and the general solution* is

$$y\varepsilon^{\int P dx} = \int Q\varepsilon^{\int P dx}\, dx + c$$

or

$$y = \varepsilon^{-\int P dx}\int Q\varepsilon^{\int P dx}\, dx + c\varepsilon^{-\int P dx}$$

where c is the arbitrary constant of integration.

The general solution to equation (233) is therefore given by

$$i = \varepsilon^{-\int (R/L) dt}\frac{E_m}{L}\int \varepsilon^{\int (R/L) dt} \sin\ \omega t\ .\ dt + c\varepsilon^{-\int (R/L) dt}$$

Evaluating the supplementary integrals, we have

$$i = \varepsilon^{-(R/L)t}\frac{E_m}{L}\int \varepsilon^{(R/L)t} \sin \omega t\ .\ dt + c\varepsilon^{-(R/L)t}$$

The remaining integral is of the form

$$\int \varepsilon^{ax} \sin bx\ .\ dx,$$

which, when integrated, gives

$$\frac{(a \sin bx - b \cos bx)}{a^2 + b^2}\varepsilon^{ax}$$

* An alternative method of solution, in which the roots of an auxiliary equation are employed, is given on p. 448.

Hence the solution to equation (233) is

$$i = \frac{E_m}{L[(R/L)^2 + \omega^2]}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + c\varepsilon^{(R/L)t}$$

Simplifying, we have

$$i = \frac{E_m}{R^2 + \omega^2 L^2}(R\sin\omega t - \omega L\cos\omega t) + c\varepsilon^{(R/L)t}$$

$$= \frac{E_m}{\sqrt{(R^2 + \omega^2 L^2)}}\left\{\frac{R}{\sqrt{(R^2 + \omega^2 L^2)}}\sin\omega t - \frac{\omega L}{\sqrt{(R^2 + \omega^2 L^2)}}\cos\omega t\right\} + c\varepsilon^{(R/L)t}$$

$$= \frac{E_m}{\sqrt{(R^2 + \omega^2 L^2)}}\sin(\omega t - \varphi) + c\varepsilon^{(R/L)t} \quad . \quad . \quad (234)$$

$$= I_m\sin(\omega t - \varphi) + c\varepsilon^{(R/L)t} \quad . \quad . \quad (234a)$$

where $\qquad I_m = E_m/\sqrt{(R^2 + \omega^2 L^2)}$

and $\qquad \tan\varphi = \omega L/R$.

The first portion of equation (234) is the *particular* solution of equation (233), which is obtained when t has large values. This (particular) solution corresponds to the permanent state of the current in the circuit.

The second, or transient, term in equation (234) is the solution of the equation $L\,di/dt + Ri = 0$. This term takes into account the initial conditions in the circuit, viz., that initially the current is zero, and, therefore, when an E.M.F. is applied suddenly to the circuit, the current must start from zero at whatever point on the E.M.F. wave the circuit is closed.

The value of the arbitrary constant of integration (c) is determined by the conditions existing when the circuit is closed.

If the circuit is closed at a time t_1 after the E.M.F. has passed through its zero value (which corresponds to $t = 0$), the current at this instant (t_1) is zero, and, on substituting in equation (234a), we obtain

$$0 = I_m\sin(\omega t_1 - \varphi) + c\varepsilon^{(R/L)t_1}$$

whence $\qquad c = -I_m\,\varepsilon^{(R/L)t_1}\sin(\omega t_1 - \varphi)$.

Therefore the general equation for the current becomes

$$i = I_m\sin(\omega t - \varphi) - I_m\,\varepsilon^{(t_1 - t)R/L}\sin(\omega t_1 - \varphi) \quad . \quad (235)$$

Observe that at the instant of closing the circuit (i.e. $t = t_1$), the value of the exponential term is equal to that of the term representing the permanent conditions in the circuit.

This equation, except for the special case when $\omega t_1 = \varphi$, represents, when t is small, a series of unsymmetrical periodic curves. The dissymmetry gradually decreases with respect to time and ultimately vanishes (*see* Fig. 302).

When $\omega t_1 = \varphi$, the second, or exponential, term in equation (235) is zero and the permanent conditions exist (i.e. the current is sinusoidal) from the instant that the circuit is closed.

A graphical representation of equation (235), in the case of a particular circuit, is given in Fig. 302 (curve i), the applied E.M.F.

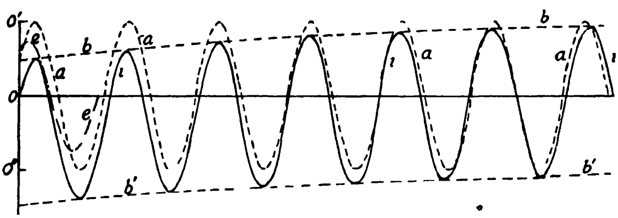


FIG. 302--Wave-forms of Initial Current in Circuit Containing Resistance and Inductance

being shown at e. The points on the current curve may be calculated directly by equation (235), but it is easier to obtain the current curve in the following manner—

First plot the current curve corresponding to the permanent conditions. This is a sine curve (a, Fig. 302) having a maximum value equal to $E_m/\sqrt{(R^2 + \omega^2 L^2)}$ and lagging φ with respect to the E.M.F. curve.

Next calculate a few points and draw the exponential curve (b). The calculated points are obtained from the equation

$$i' = I_m \varepsilon^{(t_1 - t)R/L} \sin(\omega t_1 - \varphi).$$

Finally, compound the curves according to equation (235)—i.e. subtract (b) from (a)—and obtain the resultant curve i.

[NOTE. Fig. 302 refers to the case in which L is constant. If, however, the magnetic circuit contains iron, the initial state of magnetization of the iron, together with the hysteresis loop, will have to be taken into consideration in determining the initial current and its wave-form. In this case a graphical construction, similar to that described in Chapter XI, will have to be employed.]

Relationship between impressed E.M.F. and initial current for a circuit containing only capacity. Let an E.M.F., represented by the equation $e = E_m \sin \omega t$, be applied to a condenser of capacity C.

Then, since the charging current of the condenser at any instant is equal to $C\,de/dt$, we have

$$
\begin{aligned}
i = C\frac{de}{dt} &= C\frac{d}{dt}E_m \sin \omega t \\
&= \omega C\, E_m \cos \omega t \\
&= \omega C\, E_m \sin(\omega t + \tfrac{1}{2}\pi)
\end{aligned}
$$

which is the equation (25) deduced in Chapter IV for the permanent conditions in the circuit.

Hence in this case—i.e. if the resistance is zero and there are no losses—there are no transient phenomena on closing the circuit.

If, however, the switch makes bad contact, so that sparking occurs at the contacts, transient conditions will occur, as the circuit is then equivalent to a condenser and a resistance (which possesses unstable characteristics) in series.

Relationship between impressed E.M.F. and initial current for a series circuit containing capacity and resistance. Let a sinusoidal E.M.F.—represented by $e = E_m \sin \omega t$—be applied to a series circuit containing a condenser, of capacity C, and a resistance R. Then the general equation of the circuit is

$$e = Ri + (1/C)\int i\,dt.$$

Differentiating with respect to time, we obtain

$$\frac{de}{dt} = R\frac{di}{dt} + \frac{i}{C}$$

Transposing and re-arranging terms, we have

$$\frac{di}{dt} + \frac{1}{RC}i = \frac{1}{R}\frac{de}{dt} = \frac{\omega E_m}{R}\cos \omega t,$$

which is a linear differential equation of the first order.

The solution of this equation is obtained in exactly the same manner as that for an inductive circuit (*see* p. 508), and is

$$i = \frac{E_m}{\sqrt{[R^2 + (1/\omega C)^2]}}\sin(\omega t + \varphi) + A\varepsilon^{t/RC} \quad . \quad . \quad (236)$$

$$= I_m \sin(\omega t + \varphi) + A\varepsilon^{t/RC} \quad . \quad . \quad . \quad . \quad . \quad (236a)$$

This equation differs from equation (234*a*) in the time constant and the sign of the phase-angle φ. Except in the special case when the circuit is closed at the instant the impressed E.M.F. is passing through zero, the initial current waves are represented by a series of unsymmetrical waves similar to those shown in Fig. 302.

Relationship between impressed E.M.F. and initial current for a series circuit containing capacity, resistance, and inductance. Let the applied E.M.F. be represented by the equation $e = E_m \sin \omega t$, and let R, L, C denote the resistance, inductance, and capacity, respectively, of the circuit Then the general equation of the circuit is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e = E_m \sin \omega t \quad . \quad . \quad . \quad (237)$$

Differentiating throughout in order to clear the integral, we obtain

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = \frac{de}{dt} = \omega E_m \cos \omega t \quad . \quad (238)$$

Differentiating again, we obtain

$$R\frac{d^2i}{dt^2} + L\frac{d^3i}{dt^3} + \frac{1}{C}\frac{di}{dt} = -\omega^2 E_m \sin \omega t$$

Substituting from equation (237), we have

$$L\frac{d^3i}{dt^3} + R\frac{d^2i}{dt^2} + \frac{1}{C}\frac{di}{dt} = -\omega^2\left(Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt\right)$$

Differentiating again for the purpose of clearing the integral, and re-arranging, we have

$$CL\frac{d^4i}{dt^4} + CR\frac{d^3i}{dt^3} + (1 + \omega^2 CL)\frac{d^2i}{dt^2} + \omega^2 CR\frac{di}{dt} + \omega^2 i = 0$$

The solution to this equation may be expressed in the form

$$i = A_1\varepsilon^{m_1 t} + A_2\varepsilon^{m_2 t} + A_3\varepsilon^{m_3 t} + A_4\varepsilon^{m_4 t}$$

where A_1, A_2, A_3, A_4 are constants and m_1, m_2, m_3, m_4 are the roots of the equation

$$CLx^4 + CRx^3 + (1 + \omega^2 CL)x^2 + \omega^2 CRx + \omega^2 = 0$$

The roots are readily obtained, since this equation can be factorized thus

$$(x^2 + \omega^2)(CLx^2 + CRx + 1) = 0$$

Whence the roots are

$$m_1 = \omega\sqrt{-1}, \qquad m_2 = -\omega\sqrt{-1}$$

$$m_3 = -\frac{R}{2L} + \frac{\sqrt{(R^2 - 4L/C)}}{2L}, \quad m_4 = -\frac{R}{2L} - \frac{\sqrt{(R^2 - 4L/C)}}{2L}$$

Therefore the general solution is

$$i = A_1\varepsilon^{j\omega t} + A_2\varepsilon^{-j\omega t} + \varepsilon^{-(R/2L)t}\left(A_3\varepsilon^{t\sqrt{(R^2-4L/C)}/2L} + A_4\varepsilon^{-t\sqrt{(R^2-4L/C)}/2L}\right) \quad . \quad . \quad (239)$$

The first terms represent the particular solution and correspond to the permanent conditions in the circuit ; the last term represents the transient conditions. This term corresponds to the solution of the equation

$$L\, di^2/dt^2 + R\, di/dt + i/C = 0$$

Observe that the transient term may take one of three forms, according to whether CR^2 is $> = < 4L$.

The particular solution—i.e. $i = A_1\varepsilon^{j\omega t} + A_2\varepsilon^{j\omega t}$— reduces to

$$\begin{aligned} i &= A_1(\cos \omega t + j \sin \omega t) + A_2 (\cos \omega t - j \sin \omega t) \\ &= (A_1 + A_2) \cos \omega t + j(A_1 - A_2) \sin \omega t \\ &= A \cos \omega t + B \sin \omega t \quad . \quad . \quad . \quad . \quad (240) \end{aligned}$$

where $A = A_1 + A_2$, $B = j(B_1 + B_2)$.

The constants A and B are evaluated by substituting in equation (238) the value of i given by equation (240). Thus

$$\begin{aligned} di/dt &= -\omega A \sin \omega t + \omega B \cos \omega t, \\ d^2i/dt^2 &= -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t \end{aligned}$$

Whence, from equation (238), we have

$$\begin{aligned} &-\omega^2 L(A \cos \omega t + B \sin \omega t) + \omega R(B \cos \omega t - A \sin \omega t) \\ &+ (A \cos \omega t + B \sin \omega t)/C = \omega E_m \cos \omega t \end{aligned}$$

Equating sine and cosine terms we have

(i) $$-\omega^2 LB - \omega RA + B/C = 0$$

(ii) $$-\omega^2 LA + \omega RB + A/C = \omega E_m$$

from which $$A = \frac{E_m(1/\omega C - \omega L)}{(1/\omega C - \omega L)^2 + R^2}$$

$$B = \frac{E_m R}{(1/\omega C - \omega L)^2 + R^2}$$

Substituting these quantities in equation (240), we have

$$i = \frac{E_m}{R^2 + (1/\omega C - \omega L)^2}\left[\left(\frac{1}{\omega C} - \omega L\right) \cos \omega t + R \sin \omega t\right]$$

or, if $\omega L > 1/\omega C$, as is usually the case,

$$i = \frac{E_m}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}} \sin(\omega t - \varphi) \quad . \quad (241)$$

$$= I_m \sin(\omega t - \varphi) \quad . \quad . \quad . \quad . \quad . \quad (241a)$$

where $\quad I_m = E_m/\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}$

and $\quad \varphi = \tan^{-1}(\omega L - 1/\omega C)/R.$

This equation is, of course, identical with that deduced in Chapter VI by assuming permanent conditions in the circuit.

The general equation (239) may, therefore, be written in the form

$$i = I_m \sin(\omega t - \varphi) + \varepsilon^{(R/2L)t}\left(A_3\varepsilon^{t\sqrt{(R^2-4L/C)}/2L} + A_4\varepsilon^{-t\sqrt{(R^2-4L/C)}/2L}\right) \quad . \quad . \quad (242)$$

The evaluation of the transient term of this equation is made according to whether R is $\gtreqless 2\sqrt{(L/C)}$.* Each case will be considered separately.

Case I. $R < 2\sqrt{(L/C)}$. This case is more interesting than the others, as the conditions correspond to an oscillatory discharge when a condenser discharges through an inductive resistance, the constants of which satisfy the above relationship, i.e. $R^2 < 4L/C$.

Denote $\sqrt{(4L/C - R^2)}/2L$ by λ. Then we may write

$$\sqrt{(R^2 - 4L/C)}/2L = j\lambda.$$

Accordingly, the transient term becomes

$$\varepsilon^{(R/2L)t}(A_3\varepsilon^{j\lambda t} + A_4\varepsilon^{-j\lambda t}) = A'\varepsilon^{-(R/2L)t}\sin(\lambda t + \varphi')$$

and equation (242) takes the form

$$i = I_m \sin(\omega t - \varphi) + A'\varepsilon^{-(R/2L)t}\sin(\lambda t + \varphi') \qquad (243)$$

where the constants A', φ', include the constants A_3, A_4, and are to be determined from the initial conditions.

As two constants are involved, we must have two equations connecting the initial conditions. One equation connects current and time; the other connects charge (q) and time. The latter is obtained from the former, since $q = \int i\,dt$. Hence, by integrating equation (243) we have†

$$q = -(I_m/\omega)\cos(\omega t - \varphi)$$
$$+ \varepsilon^{(R/2L)t}A'\sqrt{(LC)}\sin(\lambda t + \varphi' + \varphi'') \quad . \quad (244)$$

where $\quad \varphi'' = \tan^{-1}\sqrt{(4L/C - R^2)}/R.$

* The quantity $\sqrt{(L/C)}$ is called the "surge impedance" (sometimes the "natural impedance") of the circuit, and is of importance in connection with surges and similar disturbances.

† NOTE. $\int e^{ax}\sin bx = e^{ax}\left(\dfrac{a\sin bx - b\cos bx}{a^2 + b^2}\right)$

$$= e^{ax}\left(\frac{\sin[bx - \tan^{-1}(b/a)]}{\sqrt{(a^2 + b^2)}}\right).$$

If the condenser is initially uncharged, then, at the instant (t_1) of closing the circuit, $i = 0$, $q = 0$. Hence, substituting in equations (243), (244), we have

$$0 = I_m \sin(\omega t_1 - \varphi) + A' \varepsilon^{(R/2L)t_1} \sin(\lambda t_1 + \varphi')$$

and

$$0 = -(I_m/\omega)\cos(\omega t_1 - \varphi) + \varepsilon^{(R/2L)t_1} A' \sqrt{(LC)} \sin(\lambda t_1 + \varphi' + \varphi'')$$

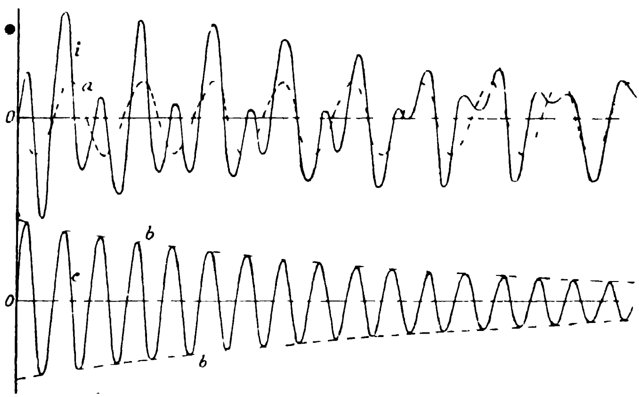


FIG. 303.—Wave-forms of Initial Current in Circuit, Containing Resistance, Inductance, and Capacity in Series

[NOTE. The dotted sine curve, a, represents the current wave-form corresponding to the steady-state, or permanent, conditions.]

Whence

$$\varphi' = \cot^{-1}\left(\frac{2\cot(\omega t_1 - \varphi) + \omega CR}{\omega\sqrt{(4LC - C^2R^2)}}\right) - \lambda t_1$$

$$A' = -I_m \varepsilon^{-(R/2L)t_1} \sin(\omega t_1 - \varphi) / \sin\left[\cot^{-1}\left(\frac{2\cot(\omega t_1 - \varphi) + \omega CR}{\omega\sqrt{(4LC - C^2R^2)}}\right)\right]$$

$$= -\frac{2I_m \varepsilon^{-(R/2L)t}}{\omega\sqrt{(4LC - C^2R^2)}} \sqrt{[(\omega^2 LC - 1)\sin^2(\omega t_1 - \varphi) + \tfrac{1}{2}\omega CR \sin 2(\omega t_1 - \varphi) + 1]}$$

Substituting these values in equation (243), we have

$$i = I_m \sin(\omega t - \varphi)$$
$$-\frac{2I_m\sqrt{[(\omega^2LC - 1)\sin^2(\omega t_1 - \varphi) + \tfrac{1}{2}\omega CR \sin 2(\omega t_1 - \varphi) + 1]}}{\omega\sqrt{(4LC - C^2R^2)}}$$
$$\varepsilon^{-(t-t_1)R/2L} \sin\left[\lambda(t - t_1) + \cot^{-1}\left(\frac{2\cot(\omega t_1 - \varphi) + \omega CR}{\omega\sqrt{(4LC - C^2R^2)}}\right)\right] \quad (245)$$

This equation shows that the initial wave-forms of the current may be of a very complex nature, and examples are given in Fig. 303

Case II. $R = 2\sqrt{(L/C)}$. This corresponds to the critical case for the discharge of a condenser through an inductive resistance as the discharge is then just non-oscillatory.

The transient term of equation (242) now takes the form

$$A_3\varepsilon^{-(R/2L)t} + A_4\varepsilon^{-(R/2L)t}$$

or

$$\varepsilon^{-(R/2L)t}(A_3 + A_4 t)$$

and accordingly equation (242) becomes

$$i = I_m \sin(\omega t - \varphi) + \varepsilon^{-(R/2L)t}(A_3 + A_4 t) \qquad . \qquad . \qquad (246)$$

The constants A_3, A_4, must be determined from the initial conditions.

By integrating equation (246) we obtain

$$\int i\,dt = q = -\frac{I_m}{\omega}\cos(\omega t - \varphi) - 2\frac{L}{R}\varepsilon^{-t(R/2L)}\left(A_3 + A_4 t + 2\frac{L}{R}A_4\right)$$

If initially the condenser is uncharged, then at the instant (t_1) of closing the circuit, we have

$$0 = I_m \sin(\omega t_1 - \varphi) + (A_3 + A_4 t_1)\varepsilon^{-(R/2L)t_1}$$

and

$$0 = -\frac{I_m}{\omega}\cos(\omega t_1 - \varphi) - 2\frac{L}{R}\left[A_3 + A_4\left(t_1 + 2\frac{L}{R}\right)\right]\varepsilon^{-(R/2L)t_1}$$

whence

$$A_3 = -I_m\varepsilon^{(R/2L)t_1}\left[\left(1 + \frac{R}{2L}t_1\right)\sin(\omega t_1 - \varphi) - \frac{R^2 t_1}{4\omega L^2}\cos(\omega t_1 - \varphi)\right]$$

$$A_4 = \frac{R}{2L}I_m\varepsilon^{(R/2L)t_1}\left[\sin(\omega t_1 - \varphi) - \frac{R}{2\omega L}\cos(\omega t_1 - \varphi)\right]$$

EXAMPLES

THIS collection of examples includes questions set at the following examinations—

University of London, B.Sc. (Eng.)* Reference *L.U.*

City and Guilds of London Institute, Electrical Engineering.—Reference *C. & G.*

The examples have been grouped according to subject matter and chapter.

I.—GENERAL PRINCIPLES AND FORMS OF REPRESENTATION (CHAPTERS I AND II)

1. Write down (*a*) the maximum value, (*b*) the root-mean-square value, (*c*) the frequency of the following—

(1) $100 \sin 500t$; (2) $(A + B) \sin (3\omega t - \theta + \varphi)$; (3) $P \cos \omega t - Q \sin \omega t$; (4) $A \sin(\omega t + \tan^{-1} b/a)$.

Write down also the phase difference with respect to $B \sin(\omega t - \alpha)$ in cases (3) and (4).

[*Ans.* Maximum values—(1) 100, (2) $(A + B)$, (3) $\sqrt{(P^2 + Q^2)}$, (4) A. Frequencies—(1) 79·7, $3\omega/2\pi$, $\omega/2\pi$, $\omega/2\pi$. Phase differences—(3) $\alpha - \tan^{-1}P/Q$, (4) $\alpha + \tan^{-1}b/a$.]

2. Plot curves, in rectangular co-ordinates, for one cycle of the quantities $a = 10 \sin(\omega t + 10°)$, $b = 5 \sin(\omega t - 30°)$, and determine graphically the curve representing the sum of these quantities.

[*Ans.* Equation to curve: $c = 14·2 \sin(\omega t - 3·1°)$.]

3. What is the justification for assuming a sinusoidal time variation of the alternating quantities met with in electrical engineering. In the case of an alternating current of sinusoidal form and 10^5 frequency the R.M.S. value is 0·01 A. Write down the equation for the instantaneous value, measuring time from a maximum positive value. What will be the instantaneous value after a lapse of (*a*) $0·2 \times 10^{-5}$ seconds, (*b*) $2·7 \times 10^{-5}$ seconds? (*L.U.*, 1925.)

[*Ans.* Equation—$i = 0·01414 \cos 628{,}560t$; (*a*) 0·00437, (*b*) − 0·00437.]

4. What is meant by the root-mean-square value of an alternating quantity? Show that a hot-wire instrument indicates the R.M.S. value of an alternating current or E.M.F. Why is the R.M.S. value used in practice in preference to the mean value?

5. Define "form factor" in connection with alternating E.M.F. wave forms. Calculate this factor in the case of (i) a V-shaped wave, (ii) a sine wave. For what purposes is a knowledge of form factor required? (*L.U.*, 1914.)

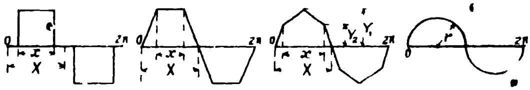
[*Ans.* (i) 1·155, (ii) 1·11.]

6. An alternating current has the following values after equal time intervals beginning at zero: 0, 3, 4, 4·5, 5·5, 8, 10, 6, 0, − 3, − 4, −4·5, − 5·5, − 8, − 10, − 6, 0. Find the R.M.S. value and form factor.

[*Ans.* R.M.S value = 5·8 A. Form factor = 1·145.]

* The Engineering Examination Papers of the University of London are published regularly by the University of London Press.

7. Calculate the form factor of the wave-forms shown in the accompanying diagram.



(a) (b) (c) (d)

[*Ans.* (a) $\sqrt{(X/x)}$; (b) $\frac{2}{\sqrt{3}}\sqrt{\left[1-\left(\frac{x}{X+x}\right)^2\right]}$

(c) $\frac{2}{\sqrt{3}}\sqrt{\left[\frac{X^2Y_2^2+xXY_1(Y_1+Y_2)}{(xY_1+XY_2)^2}\right]}$; (d) 1·04

8. Calculate the form factor of the flux wave-form shown in Fig. 188, p. 335.

[*Ans.* Form factor = 1.055.]

9. Calculate the R.M.S. value and form factor of the exciting current wave-form shown in Fig. 187, p. 334.

[*Ans.* R.M.S. value = 1·94 A. Form factor = 1·18.

10. Calculate the "winding distribution factor" (or "breadth coefficient") for the winding of a single-phase alternator having three full-pitch coils per pair of poles located in slots 20° apart (i.e. the winding is equivalent to the coils shown in Fig. 6, the angles α, β, being each equal to 20°), the flux distribution in the air gap being sinusoidal.

[NOTE —The "winding distribution factor" is the ratio: Vector sum of E.M.Fs. (in coils connected in series)/Arithmetic sum of these E.M.Fs.]

[*Ans.* $\frac{1}{3}(1 + 2\cos 20°) = 0{\cdot}96$.]

11. Show that if a conductor transmits simultaneously a sinusoidal alternating current of maximum value I_m, and a direct current, I_D, the I^2R loss in the conductor is equal to $\sqrt{(I_D^2 + \frac{1}{2}I_m^2)}/(I_D + I_m/\sqrt{2})$ of that which would be obtained if the two currents were transmitted over the same distance by separate conductors, the current densities being the same in the two cases.

12. Upon what does the validity of vector diagrams as a means of representing electric and magnetic phenomena rest? Mention the chief cases in which they are useful representations of such phenomena, and discuss how far they are strictly true in these cases. (*L.U.*, 1917.)

13. What is a vector quantity and how is it represented by a vector? Under what conditions may alternating quantities be represented by vectors? Show how the vector sum of (a) two quantities, (b) three quantities is determined.

What is a vector diagram, and how are the vectors in such a diagram distinguished? Draw to scale a vector diagram for a circuit consisting of an inductive resistance (R = 5 O., L = 0·2 H.) supplied at 100 V., 50 frequency.

14. Explain the method of representing a vector quantity by the j notation, and show the method may be used in calculations of alternating quantities.

Two circuits, the impedances of which are given by (a) $8 - j7$, (b) $5 + j6$, are connected in parallel across a 100 V. supply. Calculate the current passing through each circuit, and the total, or line, current. Find also the angle of phase difference between the currents and the applied P.D. (*L.U.*, 1923.)

[*Ans.* $I_A = 9{\cdot}43$ A., ${}_A = 41{\cdot}2°$ (leading), $I_B = 12{\cdot}8$ A., $\varphi_B = 50{\cdot}2°$ (lagging), $I = 15{\cdot}75$ A., $\varphi = 13{\cdot}3°$ (lagging).]

II.—INDUCTANCE AND CAPACITY (CHAPTERS III AND IV)

1. A circuit consists of two portions, AB and BC. The voltage across AB is 60 V., that across BC is 100 V., and their phase difference is 45°. What is the voltage between the terminals A and C, and what is its phase difference with respect to the voltage across BC.

[*Ans.* E = 148·6 V., φ = 17°. (leading).]

2. Define (*a*) inductance, (*b*) reactance, (*c*) impedance. Calculate the approximate values of these quantities for an air-core coil 50 cm. long and 10 cm. diameter wound with 1,000 turns of wire having a resistance of 5 O., the frequency of the applied voltage being 50 cycles per second. (*C. & G.*, 1924.)

[*Ans* (*a*) 0 0197 H., (*b*) 6·2 O., (c) 7·96 O.]

3. Prove the relationship between impressed E.M.F. and current for an inductive circuit which is carrying a sinusoidal alternating current.

Such a circuit having a resistance of 15 O. and an inductance of 0·03 H. is connected to a 200 V., 50-cycle supply system. Determine the current and the phase difference between current and impressed E.M.F.

[*Ans.* 11·28 A. Phase difference = 32·2° (lagging).]

4. An inductive circuit has a resistance of 60 O. and an inductance o 0·36 H., and is supplied at 220 V., 50 frequency. Calculate (*a*) the reactance of the circuit ; (*b*) the impedance ; (*c*) the current ; (*d*) the phase difference between impressed E.M.F. and current.

[*Ans.* X = 113·4 O., Z = 128·3 O, I = 1·715 A., φ = 62·1° (lagging.)]

5. Define mutual inductance.

The reluctance of the core of a certain transformer is 0·002. Find the coefficient of mutual inductance between the primary and secondary coils which have 1250 and 40 turns respectively, neglecting magnetic leakage.

[*Ans.* 0·313 H.]

6. A solenoid is wound with 1068 turns in a length of 60·4 cm. and is fitted with an internal search having 242 turns and a mean cross section of 16·2 sq. cm., the search coil occupying a symmetrical position at the centre of the solenoid. Find the E.M.F. induced in the search coil when an alternating current of 2 A. at 50 frequency is passed through the solenoid.

[*Ans.* 0·546 V.]

7. A certain choking coil with a large air-gap has a winding divided into two sections, each consisting of 150 turns of a resistance of 0·25 O. If the impedance of one section is 8 O., what will be the impedance of the whole winding from end to end ? What would be the impedance of the two sections if connected in parallel ?

[*Ans.* 32 O (approx.), 8 O. (approx.).]

8. A coil of insulated copper wire, consisting of 500 turns, having a mean diameter of 25 cm., is short-circuited. The conductor is 1 mm. diameter. It is placed horizontally in the earth's field (H_v = 0·44), and is then turned over through two right angles. What quantity of electricity flows round the circuit, given ρ = 1·8 microhms per cm. cube ? [NOTE.—H_v denotes the vertical component of the earth's magnetic force.] (*C. & G.*, 1924.)

[*Ans.* $2{\cdot}4 \times 10^{-4}$ coulombs.]

9. The air gap under the pole face of a series motor is 0·06 in. in length and has an effective area of 80 sq. in. The exciting winding has 5 turns and a resistance of 0.005 O. Assuming the reluctance of the iron portions of the

magnetic circuit to be one-fifth of the reluctance of the air gap, calculate the voltage at the terminals of the series coil when a current of 150 (R.M.S.) at 25 cycles is passing. (*C. & G.*, 1920.)

[*Ans.* 20·8 V.]

10. Deduce the relation between the electrostatic C.G.S. unit of capacity and the microfarad. Calculate the capacity, in microfarads, of a tubular condenser 18·5 cm. long, the outside diameter of the inner tube being 3·45 cm., and the inside diameter of the outer tube being 3·7 cm. The dielectric is air, and fringing at the ends may be neglected. (*L.U.*, 1921.)

[*Ans* 0·000148 μF.]

11. What is meant by the "charging current" of a condenser connected to an alternating current supply system? Draw a diagram showing relationship between charging current and impressed E.M.F.

Calculate the charging current of a 15 μF. condenser when supplied at 220 V., 50 frequency.

[*Ans.* 1·037 A.]

12. Two condensers having capacities of 10 μF. and 20 μF. are connected (i) in parallel, (ii) in series, and the combination is supplied at 250 V., 50 frequency. Determine the line currents in each case.

[*Ans.* (i) 2·355 A., (ii) 0·525 A.]

13. What capacity condenser must be connected in series with a 50 V., 30 W lamp in order that the lamp may receive its normal current when the lamp and condenser are connected to a 110 V., 50-cycle supply? (*C. & G.*, 1920.)

[*Ans.* 19·5 μF.]

14. Deduce from first principles a formula for the capacity of a concentric cable in terms of the radii of the conductors and the specific inductive capacity of the dielectric. Determine the capacity per mile of such a cable if the diameter of the inner conductor is 1·5 cm., the specific inductive capacity of the dielectric is 2·5, and the thickness of dielectric is 0·5 cm. (*L.U.*, 1916.)

[*Ans* 0·438 μF.]

15. Calculate the inductance and capacity per mile of a concentric cable when the diameters of the inner conductor and inner insulation are 3 mm. and 10 mm. respectively, and the specific inductive capacity of the insulator is 2·5. Prove the formula you employ. (*L.U.*, 1917.)

[*Ans.* L = 0·69 mH., C = 1·85 μF.]

16. Mathematical formulæ show that the effects of inductance and of capacity in a circuit are mutually opposed and tend to neutralize each other. Show how this interaction takes place physically. (*L.U.*, 1917.)

III.—POWER AND POWER FACTOR (CHAPTER IV)

1. What is meant by the "power factor" of a circuit?

Show how the power factor of a circuit may be determined experimentally, and draw a diagram of connections.

Calculate the power factor of an inductive resistance which takes a current of 5 A. and a power of 100 W. at 100 V.

[*Ans.* 0·2.]

2. What is the power factor of the circuit in Example 13 (consisting of the 30 W., 50 V. lamp and the 19·5 μF. condenser)?

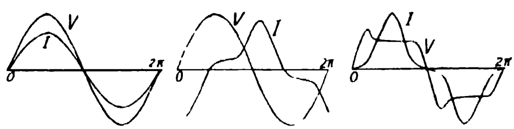
[*Ans.* 0·455 (leading).]

3. Calculate the current in, and the power factor of, a circuit of which the resistance is 5 O. and the inductance is 0·01 H. when the supply pressure is 200 V., 50 frequency.

[*Ans.* 33·8 A., cos φ = 0·845.]

4. Define the term "power factor" as applied to alternating-current circuits. What power factor does one usually find for the following: (*a*) a water resistance, (*b*) the primary circuit of an alternating-current transformer when the secondary is unloaded, (*c*) a hand-fed alternating-current arc? Illustrate each case by means of rectangular co-ordinate diagrams of wave-forms. Assume the wave of the supply electromotive force to be sinusoidal. (*L.U.*, 1922.)

[*Ans.* (*a*) 1·0, (*b*) 0·1 to 0·2, (*c*) 1·0. See accompanying diagrams.]



(*a*) (*b*) (*c*)

5. A circuit contains three coils, *A*, *B*, *C*, in series When 1 A. direct current is passed through them, the voltage drop across each, taken in order, is 5, 2, 10. When an alternating current of 1 A. and of frequency 50 is passed, the voltages are 5·6, 6, and 11. Find the power factor and power dissipated in each wire. (*L.U.*, 1924.)

[*Ans.* Cos φ 0·8 P_A = 5 W., P_B = 2 W P_C = 10 W.]

IV.—SERIES AND PARALLEL CIRCUITS (CHAPTERS VI AND VII)

1. If the alternating current supplied to a single-phase circuit under full-load lags 45° behind the voltage, and the inductance is maintained constant, prove that the full-load cannot be exceeded, however much the resistance of the circuit is varied. (*L.U.*, 1919.)

2. A condenser of 10 μF. capacity is connected in series with a coil having an inductance of 250 mH. and a resistance of 400 O. An alternating E.M.F. of 100 V., sine wave-form, and 50 frequency is applied to this circuit. What current will flow through it. (*L.U.*, 1921.)

[*Ans.* 0·215 A.]

3. A resistance of 12 O. and an inductance of 0·18 H. are connected in series and supplied at 500 V., 40 frequency. Find the current and power factor.

What values of resistance and inductance, when connected in parallel, will take the same current, at the same power factor, as the above circuit? (*L.U.*, 1912.)

[*Ans.* 10·65 A., cos φ 0·255; 183 O., 0·194 H.]

4. Two inductive resistances, having reactances of 5 O. and 2 O., and resistances of 3 O. and 4 O., respectively, are connected in parallel and supplied at 100 V. Calculate the line current.

[*Ans.* 38 A.]

5. An alternating-current circuit includes two sections, *AB* and *BC*, in series. The section *AB* consists of two branches in parallel. The first of

these is formed of a non-inductive resistance of 60 O. in series with a condenser of 50 μF.; while the second consists of a resistance of 60 O. having an inductance of 250 mH. The section *BC* consists of a resistance of 100 O. having an inductance of 300 mH. The frequency is 50 cycles per second and the voltage across Section *AB* is 500 V. What is the voltage across *BC*? (*L.U.*, 1914.)

[*Ans.* 955 V.]

6. What is meant by the resonance frequency of an alternating current circuit? Deduce expressions for the resonance frequency of (1) a series circuit, and (2) a parallel circuit. Show by diagrams how the line current and power factor of such circuits, supplied at constant voltage and varying frequency, vary with the supply frequency.

An inductive coil having a resistance of 35 O. and an inductance of 0·5 H. forms part of a series circuit, for which the resonance frequency is 55 cycles per second. If the supply frequency is 50 cycles and the voltage is 100 V., determine (1) the line current, (2) the power factor, and (3) the voltage across the inductive coil.

[*Ans.* (1) 2·08A., (2) 0·728 (leading), (3) 335V.]

7. A condenser, the losses in which are negligible, is connected in series with an air choking coil having a self-inductance of 0·1 H. across the terminals of an alternator, the voltage of which is maintained constant at 100 V. The frequency is gradually increased until at 250 cycles per second the current reaches a maximum value of 50 A. Calculate the capacity of the condenser and the ammeter reading at 200 cycles per second. (*C. & G.*, 1925.)

[*Ans.* 4·05 μF., 1·42 A.]

8. The two branches of a parallel circuit consist of (i) an inductive resistance, and (ii) a shunted condenser connected in series with an inductionless resistance. The inductive resistance has an impedance of 60 O. and the ratio of reactance to resistance is 3. The condenser has a capacity of 10 μF., and is shunted by a resistance of 0·1 megohm. The value of the inductionless resistance connected in series with the shunted condenser is 100 O. The supply pressure is 200 V., and the frequency is 50 cycles per second.

Determine (*a*) the current in the two branches, (*b*) the line current, (*c*) the power supplied

[*Ans.* (*a*) I_I 0·6 A., I_{II} - 3·33 A. (*b*) I = 2·87 A. (*c*) P = 246·4 W.]

9. Show how condensers are connected for improving power factor. It is desired to install a condenser to obtain 200 k.V.A. (leading) on a 600 V. system at a frequency of 50 cycles per second. If each element has a capacity of 1 μF., how many elements will be needed? (*C. & G.*, 1924.)

[*Ans.* 1765 elements.]

10 A single-phase motor takes a current of 80 A. on full-load when its power factor is 0·83. What capacity must be placed in parallel with it to make the power factor unity upon a 500 V., 50 frequency circuit? (*L.U.*, 1917.)

[*Ans.* 286 μF.]

11. A condenser is placed in parallel with two inductive loads, one of 20 A. at 30° lag, and one of 40 A. at 60° lag. What must be the current in the condenser so that the current from the external circuit shall be at unity power factor? (*C. & G.*, 1925.)

[*Ans.* 44·5 A.]

12. A choking coil of negligible resistance, when connected to a 500 V. 50 cycle circuit takes 1 A. at 0·8 power factor. What capacity must be

placed in parallel with it in order to make the power factor of the combination equal to unity? (*L.U.*, 1921.)

[*Ans.* 3·82 μF.]

13. An air-cored choking coil is subjected to an alternating voltage of 100 V. The current taken is 0·1 A. and the power factor 0·2 when the frequency is 50. Find the capacity which, if placed in parallel with the coil, will cause the main current to be a minimum. What will be the impedance of this parallel combination (*a*) for currents of frequency 50, (*b*) for currents of frequency 40 ? (*L.U.*, 1914.)

[*Ans.* Capacity = 3·14 μF. Impedances: (*a*) 5,000 O., (*b*) 1,940 O.]

14 Define the terms "admittance," "conductance," and "susceptance," with reference to alternating-current circuits. Calculate their respective values for a circuit consisting of a resistance of 20 O. in series with an inductance of 0·07 H. when the frequency is 50 cycles per second. (*C. & G.*, 1925.)

[*Ans* Y = 0·0336, g = 0·0226, b = 0·0248.]

15. The total load on a factory is 1,000 kW. at a power factor of 0·95 lagging. If 800 kW. of the load has a power factor of 0·8 lagging, what is the power factor of the remaining portion of the load ?

[*Ans* 0·59 (leading).]

16. An alternator supplies current to two single-phase motors located some distance apart, each motor taking 7·5 kW. at a power factor of 0·8. The resistance of the line wires between the alternator and one motor is 0·2 O. and that of the line wires between the motors is also 0·2 O. Determine the voltage at the alternator in order that the voltage at the further motor may be 100 V.

[*Ans.* 145·8 V.]

17. A 200 V., 50 cycle alternator is loaded with two choking coils which may be connected either in series or parallel. One coil has a resistance of 1·5 O. and an inductance of 60 mH.; the other coil has a resistance of 2·5 O. and an inductance of 30 mH. Determine the current in the circuits and the power factor of the load (i) when the coils are connected in series, (ii) when they are in parallel.

[*Ans.* (i) I = 7·04 A., $\cos\varphi$ = 0·14. (ii) I_1 = 10·65 A., I_2 = 20·7 A., I = 31·2 A.; $\cos\varphi$ = 0·2]

18. The load on a single-phase alternating-current supply system is 100 kW. at a power factor of 0·71 (lagging). If phase advancing apparatus is available for parallel connection, taking leading current at a power factor of 0·1, what must be its load in kV.A. if the power factor of the whole system is to be raised to (*a*) 0·8, (*b*) 0·9, and (*c*) 0·95.

[*Ans.* (*a*) 24·15 kV.A., (*b*) 51 kV.A., (*c*) 66·5 kV.A.]

19. A load of 2,500 kW. at a power factor of 0·8 is supplied by two alternators operating in parallel. If the output of one machine is 1,000 kW. at a power factor of 0·95 lagging, at what output is the other machine operating ? (*L.U.*, 1917.)

[*Ans.* 1,500 kW. at $\cos\varphi$ 0·697 (lagging).]

20. Two transformers, *A* and *B*, have a ratio of transformation of 3,300 to 220. When run on short circuit, *A* took 600 W. at 100 V. on the primary to make 230 A. flow in the secondary; *B* took 1,100 W. at 80 V. to make the same current (230 A.) flow in its secondary. The two transformers are run in parallel on the same primary and secondary bus-bars, with a total load of

100 kW. at 0·8 power factor. Find approximately how the load will be divided between the transformers. (*L.U.*, 1922.)

[*Ans* $P_A = 31$ kW. (270 A. at $\cos\varphi = 0·52$); $P_B = 69$ kW. (332 A at $\cos\varphi = 0·95$).]

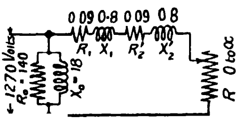
HINT. The equivalent impedance of each transformer is calculated from data of the short-circuit test. Thus the primary current corresponding to a secondary current of 230 A. 230/(3300/220) = 15 3 A. approximately. Equivalent impedance of *A* (referred to primary) is $Z_{1A}' = 100/15·3 = 6·5$ O. Equivalent resistance of *A* (referred to primary) is $R_{1A}' = 600/15·3^2 = 2·55$ O. Equivalent impedance of *A* (referred to secondary winding) $= 6·5/(3300/220)^2 = 0·029$ O. Phase angle of impedance is $\varphi_A = \cos^{-1}(R_{1A}'/Z_{1A}')$.

The parallel-connected transformers and the load are equivalent to a simple series-parallel circuit supplied at the secondary voltage (220 V.), the parallel branches consisting of the equivalent impedances (referred to the secondary winding) of the transformers, and the series portion consisting of the load.

21. Two transformers, *A* and *B*, of equal rating share a load of 180 kW. at 0·9 power factor (lagging). At full-load the voltage drop due to resistance in transformer *A* is 1 per cent of the normal terminal voltage, and that due to reactance 6 per cent. The corresponding figures for transformer *B* are 2 per cent and 5 per cent respectively. Find the load in kW. on each transformer. (*L.U.*, 1925.)

[*Ans.* $P_A = 101·7$ kW., $P_B = 78·3$ kW.]

22. Show that the performance of a polyphase induction motor operating with constant applied voltage and frequency can be approximately represented by the equivalent circuit shown in the accompanying diagram. Point out the assumptions made and develop a vector diagram for the circuit which enables the line and load currents to be determined directly. With the values given—which relate to one winding of a star connected three-phase motor—determine (*a*) the maximum power input, (*b*) the maximum power expended in the resistance *R*, (*c*) the value of *R* which gives the maximum power factor (*L.U.*, 1925.)



[*Ans.* Vector diagram is similar to that shown in Fig. 68. (*a*) 516 kW., (*b*) 445 kW., (*c*) 5·6 O.]

V.—POLYPHASE CIRCUITS (CHAPTERS VIII AND IX)

1. Discuss the theory of the measurement of three-phase power in (1) balanced, and (2) unbalanced circuits, and mention the difficulties which arise in practice in making the measurements. Show how a single-phase wattmeter can be used to measure the power in a three-phase mesh-connected balanced load when the neutral point of the system is not available. (*L.U.*, 1924.)

2. Describe the two wattmeter method of measuring the power supply to a three-phase system, and show how the two wattmeters may be combined to give a single-pointer instrument.

Prove that if the two wattmeters read accurately over the range of frequency necessary, the indications can be used to measure the power supplied by any three-wire system with alternating, direct, or any form of pulsating current. (*L.U.*, 1925.)

3. Establish and explain fully a method of determining the total power, and the power factor, of a three-phase balanced circuit, from the readings of a single wattmeter. If the current and voltage are also measured, write down an alternative expression for the power factor. To what causes would you attribute any discrepancy between the values obtained by the two methods in a given test? (*L.U.*, 1914.)

4. Explain, with the aid of vector and connection diagrams, how you would measure the power input to a three-phase motor. The readings of two wattmeters properly connected for this test are respectively 2·5 kW. and 0·25 kW., the latter reading being obtained after reversing the connections of the current coil. Find the power and power factor. (*C. & G.*, 1924.)

[*Ans.* $P = 2{\cdot}25$ kW., $\cos\varphi = 0{\cdot}427$.]

5. Two three-phase alternators are running in parallel and supplying power to a transmission line symmetrically loaded. Two wattmeters properly connected give readings of 300 and 900 kW. respectively, and both wattmeter deflections follow the variations of the load. The readings of ammeters connected in the line and in the circuits of the two alternators are in the ratio 5 : 3 : 4. Determine how much load each alternator is taking, and the readings that would be indicated by each of four wattmeters properly connected in pairs in the circuit of each alternator. (*L.U.*, 1919.)

[*Ans.* 272 kW., 928 kW.; 506 kW., − 234 kW., 516 kW., 412 kW.]

6. The primaries of three single-phase transformers are to be delta-connected to a three-phase supply with 20,000 V. between lines, and the secondaries star-connected to give 2,000 V. between lines. Draw a diagram of connections and give the voltage ratio of the transformers. (*L.U.*, 1919.)

[*Ans.* 20,000/1,155.]

7. Two single-phase transformers are used to convert a balanced three-phase into a balanced two-phase pressure. The pressure on the three-phase side is 6,600 V. and that on the two-phase side is 2,200 V. Find the number of turns in each transformer when the induced pressure per turn is 10 V. (*L.U.*, 1919.)

[*Ans.* Transformer *A* (Fig. 140): $N_1 = 660$, $N_2 = 220$; *B*, $N_1 = 572$, $N_2 = 220$.]

8. Describe and give the theory of a method of transforming from two phase to three phase, using static transformers. Deduce the numerical relationships between the currents and voltages associated with the two-phase and three-phase systems. Show that if the load on one side of the transformer is balanced it will also be balanced on the other side. Ignore magnetizing currents and losses. (*L.U.*, 1923.)

9. Two electric furnaces are supplied with single-phase current at 80 V. from a three-phase 11,000 V. system by means of two single-phase Scott-connected transformers with similar secondary windings. When the load on one transformer is 500 kW. and the load on the other is 800 kW., what current will flow in each of the three-phase lines (i) at unity power factor, (ii) at 0·5 power factor? Neglect the phase displacement and losses in the transformers. (*C. & G.*)

[*Ans.* Two solutions are possible: (*a*) 800 kW. load on transformer *A* (Fig. 140), (1) 77 A., 77 A., 52·5 A.; (2) 154 A., 154 A., 105 A.; (*b*) 800 kW load on transformer *B* (Fig. 140), (1) 62 A., 62 A., 83·5 A.; (2) 124 A., 124 A., 167 A.]

10. Determine the relative voltages of the secondary windings of the three-phase/nine-phase transformer connections shown in Fig. 136.

[*Ans* (*A*)—(a_1) 2·88 *V*, (b_1) 2·53 *V* · (*B*)—(a_1) 2·53 *V*, (b_1) 1·88 *V*; where *V* is the voltage between adjacent terminals of the load.]

11. Determine the relative voltages of the secondary windings of the three-phase/twelve-phase transformer connections shown in Fig. 137.

[*Ans.* (*A*)—(a_1, b_1) 3·34 *V*, (c_1) 3·86 *V*; (*B*)—(a_1, b_1) 3·34 *V*, (c_1, d_1) 1·03 *V*; (*C*)—(a_1, b_1) 3·73 *V*; (*D*)—(a_1 b_1) 2·73 *V*; where *V* is the voltage between adjacent terminals of the load.]

12. A three-phase induction motor is supplied from mains at 440 volts. It is coupled to a pump taking 50 B.H.P. The efficiency of the motor is 92 per cent and its power factor is 90 per cent. What is the current in each phase? (*C. & G.*, 1925.)

[*Ans.* 59 A.]

13. Two three-phase generators operating in parallel supply a non inductive load of 500 kW. at 3,300 volts. If the output from one machine is 200 kW. at 0·75 power factor, lagging, what is the output from the other machine. (*L.U.*, 1916.)

[*Ans* 300 kW., $\cos\varphi$ 0·866 (leading).]

14. A power station supplies a three-phase current of 100 A. at unity power factor from bus-bars at 10,000 V. between phases, to a point 3 miles away through a cable 0·035 sq. in. cross section per core. What is (*a*) the power sent out, (*b*) the power received, (*c*) the percentage loss, (*d*) the annual cost of the losses at 0·5 pence per unit for a constant supply, given ρ 0·7 × 10^{-6} ohm per inch cube? (*C. & G.*, 1925.)

[*Ans.* (*a*) 1,730 kW., (*b*) 1,616 kW., (*c*) 6·58 per cent, (*d*) £2,080.]

15. A three-phase transmission system carrying currents of the same wave-form supplies current to a delta-connected winding. Determine an expression giving the value of the current at any instant in an arm of the delta in terms of the line currents, there being no circulating current in the delta. Give the analytic and graphic solutions for the case when the wave-form is sinusoidal. (*L.U.*, 1925.)

16. Show that when the fall of voltage due to resistance and reactance is small compared with the line voltage, the fall of voltage along a three-phase transmission line per ampere per mile is given by the expression $\sqrt{3}(R\cos\varphi + X\sin\varphi)$, where R is the resistance per mile of conductor, X the reactance per mile of conductor, and $\cos\varphi$ the power factor of the load.

Find the fall of voltage along a three-phase transmission line, the line pressure at the load being 30,000 V. and the length of the line being 30 miles, when 5,000 kV.A. are delivered at a power factor of 0·8, the current lagging. The resistance and reactance per mile are 0·72 O. and 0·6 O. respectively. (*L.U.*, 1925.)

[*Ans.* 4,700 V.]

17. A three-phase, 50-cycle, generating station supplies an inductive load of 5,000 kW. at a power factor of 0·7 by means of an overhead transmission line 5 miles long, with conductors symmetrically arranged. The resistance per mile of each wire is 0·61 O., and the self-inductance per mile of the loop formed by any two conductors taken together is 0·0035 H. The pressure at the receiving end is maintained constant at 10,000 V. If a condenser is connected across the load to increase the power factor at the receiving end from 0·7 to 0·9, calculate (*a*) the value of capacity per phase of condenser, (*b*) the station voltage when the condenser is in use, (*c*) the station voltage when the condenser is disconnected. (*C. & G.*, 1924.)

[*Ans.* (*a*) 28·7 μF., (*b*) 12,220 V., (*c*) 12,950 V.]

18. What load at 0·8 power factor, lagging, can be delivered by a three-phase line 5 miles long with a pressure drop of 10 per cent. The station voltage is 11,000 V., and the resistance and reactance per mile of line are 0·09 O. and 0·08 O. respectively. (*C. & G.*, 1923.)

[*Ans.* 14,400 kW.]

19. An unbalanced star-connected load is supplied from a three-phase, three-wire system at a line voltage of 100 V. The current taken by one

branch (A) is 20 A. at a power factor of 0·8 lagging, and that taken by a second branch (B) is 10 A. at a power factor of 0·75 (lagging). Determine the current in, and the power factor of, the third branch, C. Also determine the total power supplied to the load and the readings of two wattmeters connected, according to the two-wattmeter method, with the current coils in the branches A, B.

[*Ans.* $I_C = 16{\cdot}5$ A., $\cos\varphi_C = 0{\cdot}993$. $P = 1762$ W., $P_A = 782$ W., $P_B = 980$ W.]

20. A combined power and lighting load is supplied by a three-phase, four-wire distribution system. The three-phase motor load absorbs 1,000 kW. at a power factor of 0·8, while the lamps between the outers and the neutral take 200, 300, and 400 kW. respectively. Calculate the current in each of the four wires when the supply pressure between outers is 400 volts. (*C. & G.*, 1924.)

[*Ans.* 2,540 A., 2,940 A., 3,350 A. Neutral current, 752 A.]

21. A four-wire, three-phase system has connected to it three-phase motors of 500 kW. capacity and 0·9 power factor, and a lighting load between each phase and the neutral wire of 300, 200, and 100 kW. respectively. What will be the current in the neutral wire and each of the lines, the voltage between lines being 400 V.? (*L.U.*, 1917.)

[*Ans.* 748 A. (neutral), 2,090 A., 1,620 A., 1,205 A.]

22. Determine the necessary cross-section of each wire of a three-phase transmission line, 2 miles in length, designed to deliver 500 kW. at 6,000 V., unity power factor and 95 per cent efficiency. The specific resistance of the material used is 0·7 microhm per inch cube. (*L.U.*, 1923.)

[*Ans.* 0·023 sq. in.]

23. Neglecting the pressure drop due to the impedance of the line, calculate the station output and its power factor when the three-phase load consists of an over-excited rotary converter giving 500 kW. at 0·975 power factor (leading), a 400 h.p. induction motor operating at a power factor of 0·85 (lagging), and a 500 kV.A. transformer with a power factor of 0·95 (lagging). Make your own assumptions as to the efficiency of each portion of the load. (*L.U.*, 1919.)

[*Ans.* 1,274 kW. at $\cos\varphi = 0{\cdot}944$. Note.—Assumed efficiencies: Rotary converter 94 per cent, induction motor 93 per cent, transformer 98 per cent.]

24. Prove that in a symmetrically arranged three-phase transmission line the inductive drop and resistance drop between any two wires is the same as that which would occur if half the total power were transmitted at the same voltage and frequency along two of the wires.

An overhead three-phase line consists of three wires, each 0·8 in. in diameter, spaced 4 ft. apart. The current flowing per wire is 300 A. at 50 cycles. Calculate the resistance and inductive drop per mile of line; the specific resistance of copper is 0·67 microhm per inch cube. (*L.U.*, 1921.)

[*Ans.* $RI = 20{\cdot}3$ V., $XI = 15{\cdot}3$ V.]

25. A three-phase generator supplies power to an unbalanced star-connected resistance load. Assuming the star point of the generator only to be earthed, show how to find the potential of the star point of the load. (*L.U.*, 1924.)

26. A three-phase supply at 400 V., 50 frequency, and 0·9 power factor is required for a factory 1·3 miles from a generating station. The total power at the factory is 500 kW. If the voltage lost in transmission is 10 per cent of the received voltage, calculate the necessary cross-section of each conductor. Assume the resistance of a cable 1 mile long and 1 sq. in. in cross-section is

0·43 O. The effects of inductance and capacity may be neglected (*L.U.*, 1924.)

[*Ans.*, 1·95 sq. in.]

27. A sub-station receives, over a three-phase line, 5,000 kW. at a frequency of 60 and a voltage of 33,000 V. from a station 20 miles distant. The power factor of the load is 0·85 lagging. Each line wire has a resistance of 0·683 O. and a reactance of 0·734 O. per mile. Assuming a balanced load, find (*a*) the P.D. at the generating station, (*b*) the power lost in the line, (*c*) the percentage voltage regulation, (*d*) the power factor at the generator. (*L.U.*, 1924.)

[*Ans.* (*a*) 36,470 V., (*b*) 432 kW., (c) 10·5 per cent, (*d*) $\cos\varphi_g = 0\cdot84$]

28. 1,000 kW. of three-phase power are to be delivered over a distance of 20 miles to a star-connected receiving circuit, the power factor of which is 0·85. The voltage between lines at the receiving end is 20,000, and the frequency is 50. What must be the voltage at the transmitting end if the resistance per mile of each line conductor is 1·5 O. and the inductance per mile is 0·0015 H. ? (*L.U.*, 1918.)

[*Ans.* 21,810 V.]

29. The measured capacity between any two cores of a three-phase cable is 5 μF. Calculate how many kV.A. are needed to keep the cable charged when connected to 10,000 V., 50 cycle, three-phase bus-bars. (*C. & G.*, 1924.)

[*Ans.* 472 kV.A.]

30. A symmetrical three-core cable surrounded by an earthed metal sheath is connected to a three-phase supply. The frequency is 50 and the voltage between line 20,000 V. The capacities of the cable are represented by the following data: the capacity between the sheath and the three conductors bunched together is 0·5 μF., the capacity between one conductor and the other two connected to the sheath is 0·6 μF. Find the charging current in each core, and prove the formula you use. (*L.U.*, 1915.)

[*Ans.* 2·96 A.]

VI.—NON-SINUSOIDAL WAVE FORMS (CHAPTER XI)

1. An alternating current is made up of several sinusoidal components of amplitudes I_1, I_3, I_5, . . . and frequencies f, $3f$, $5f$, ... Prove the expression for the effective value of the amperes as given by the reading of an accurate hot-wire ammeter. (*L.U.*, 1917.)

2. An E.M.F. $e = 100 \sin \omega t + 8 \sin 3\,\omega t$ is applied to a circuit which has a resistance of 1 O., an inductance of 0·02 H., and a capacity of 60 μF. A hot-wire ammeter is connected in series with the circuit, and a hot-wire voltmeter is connected to the terminals. Calculate the ammeter and voltmeter reading and the power supplied to the circuit. Given $\omega = 300$. (*L.U.*, 1922.)

[*Ans* 71 V., 5·18 A 26·8 W.]

3. An alternating current represented by $i = 10 \sin \omega t$ is superimposed upon a direct current of 80 A. What is the R.M.S. value of the resultant current ?

[*Ans.* 80·25 A]

4. An E.M.F. represented by the equation $e = 150 \sin 314t + 50 \sin 942t$ is applied to a condenser having a capacity of 20 μF. What is the R.M.S. value of the charging current ? (*C. & G.*, 1920.)

[*Ans.* 0·942 A.]

5. Estimate the R.M.S. value of an alternating current of irregular wave shape in terms of the fundamental component and the harmonics. An alternating pressure is represented by $v = 1{,}000 \sin \omega t + 250 \sin 3\omega t + 200 \sin 5\omega t$. Estimate the reading that will be given by an electrostatic voltmeter connected to the circuit. (*L.U.*, 1924.)

[*Ans.* 742 V.]

6. A condenser of 1·5 μF. capacity is supplied with a voltage having a wave form $e = 1{,}000 \sin \omega t + 350 \sin 3\omega t + 270 \sin 5\omega t$, the frequency being 80 cycles per second. Calculate the current taken, as measured on an ammeter. If energy is being taken from the supply circuit at the same time, how do the harmonics affect the power factor ? (*L.U.*, 1911.)

[*Ans.* 1·06 A.]

7. A star-connected, three-phase alternator, the phase E.M.Fs. of which are symmetrical but non-sinusoidal, supplies a balanced star-connected load. Show that if the E.M.F. wave-form contains 3rd, 9th, 15th, etc., harmonics, a difference of potential will exist between the neutral points of generator and load, and that if these neutral points are connected by a fourth line wire the current in this wire is made up of components having frequencies 3, 9, 15, etc., times the fundamental frequency.

VII.—MAGNETIC CIRCUITS (CHAPTER XI)

1. Define "form factor" and prove that the hysteresis loss in a transformer to which a given R.M.S. voltage at a given frequency is applied increases when the form factor of the applied voltage decreases, and vice versa. (*L.U.*, 1919.)

2. The connection between the magnetizing current and flux for a particular alternating-current electromagnet is shown by the following table, which gives values for one-half of the magnetization loop, the complete loop showing the usual symmetry with respect to the axes—

Flux (kilolines)	0	30	75	120	150	179	176	165	138	102	60	0
Magnetizing Current (A.)	5·5	6·25	8	10·5	14·25	20·5	13·5	6·8	0	−3	−45	−5·5

Plot the loop on squared paper, and by its aid deduce and plot the wave-form of the magnetizing current when the flux follows a sine law, the amplitude of the flux being equal to the maximum value of the flux in the above magnetization loop. (*L.U.*, 1921.)

3. A transformer has its primary winding connected to mains whose voltage varies according to the sine law at a frequency of 50. The secondary coil has 50 turns and gives 100 V. when on open circuit. The section of the transformer core is 20 sq. in. Determine the maximum value of the flux density in the core. Prove the formula used. (*L.U.*, 1916.)

[*Ans.* B_m = 7,000 lines per sq. cm.]

4. A choking coil is required to give a reactive voltage drop of 100 V. when carrying a current of 10 A. at 60 frequency. The winding consists of 200 turns of 0·013 sq. in. copper wire, the mean length of turn being 16 in. The laminated iron portion of the magnetic circuit has a mean length of 18 in. and a magnetic cross section of 6 sq. in. Determine the length of the air gap if its effective cross section is 10 sq. in. Determine also the effective resistance, reactance, impedance, and power factor of the coil. The specific loss (hysteresis and eddy-currents) in the core at the working flux density and 60 cycles is 1·6 W. per kg.

[*Ans.* Length of air gap = 1·22 cm. (= 0·48 in.), R_{eff} = 0·395 O. ; X = 10 O. Z_{eff} = 10·01 O ; $\cos\varphi$ = 0·394.]

5. Calculate the total length of air gap in a choking coil having 300 turns through which 100 A. flow at a frequency of 50 cycles per second, and which is required to produce an inductive drop of 100 V. Assume that the maximum flux density in the gap is 10,000 lines per sq. cm. and that the iron needs 10 per cent of the total ampere-turns. Draw a sketch of such a coil. (*C. & G.*, 1923.)

[*Ans.* 4·8 cm.]

6. A single-phase core-type transformer has the following dimensions: cross-section of each core 515 sq. cm., distance between axes of cores 53 cm., cross-section of each yoke 650 sq. cm., distance between axes of yokes 96 cm. Calculate the flux produced by 1,330 ampere-turns having given the following relationship between B and H for the magnetic circuit—

B . .	6,000	9,000	10,500	13,750	
H . .	1·3	2·6	5·9	18·6	(*L.U.*, 1923.)

[*Ans.* $\Phi_m = 5{\cdot}7 \times 10^6$ lines.

7. Calculate the no-load current for a single-phase transformer having given the following data: Primary voltage 2,200, supply frequency 50, no-load loss 500 W., number of turns in primary winding 1,200, magnetic cross-section of core 35 sq. in., magnetic length of core 80 in., permeability (at flux density corresponding to normal primary voltage) 1,200.

[*Ans.* 0·37 A.]

8. A transformer for 10 kV.A., 2,200 V. (primary), 50 cycles, takes 100 W. at rated voltage and frequency when its secondary is open. If the magnetizing current is 90 per cent of the exciting current, what is the no-load power factor of the transformer? What is the exciting current in amperes and as a percentage of the rated full-load current? (*L.U.*, 1918.)

[*Ans.* $\cos\varphi_0 = 0{\cdot}435$. $I_0 = 0{\cdot}105$ A. or 2·3 per cent.]

VIII.—MEASURING INSTRUMENTS AND MEASUREMENTS (CHAPTERS XII, XIII, XV)

1. How do wave shape and frequency affect the readings of soft-iron alternating-current ammeters and voltmeters? Explain carefully the nature of the errors when instruments calibrated on a sinusoidal wave are used (*a*) on a peaked, and (*b*) on a flat-topped wave. (*L.U.*, 1917.)

2. What is meant by the root-mean-square value of an alternate current or voltage? The voltage between two terminals has a maximum value of 150 V., and varies according to the sine law. A hot-wire voltmeter is connected across these terminals. What voltage will it read? What will be the reading if the voltmeter is joined to the terminals through an inductive conductor the resistance of which is 100 O. and the inductance 500 mH.? Assume the frequency is 50 and the resistance of the voltmeter is 500 O. (*L.U.*, 1916.)

[*Ans.* 106·1 V; 35·6 V.]

3. A soft iron voltmeter for a maximum reading of 120 V. has an inductance of 0·6 H. and a total resistance of 2,400 O. The instrument is calibrated to read correctly on a 60 cycle circuit What must be the specification of a series resistance to be used with the voltmeter so that its range is increased five-fold? (*L.U.*, 1923.)

[*Ans.* $R_1 = 9{,}676$ O.]

4 The relationship between the inductance, current, and position of the moving system of a 2 A. moving iron ammeter is—

Scale Reading (Amp.) .	0·8	1·0	1·2	1·4	1·6	1·8	2·0
Deflection (Degrees) .	16	26	36·5	49·5	61·5	74·5	86·5
Inductance (μH.) .	573·2	574·2	575·2	576·6	577·8	578·8	579·5

Deduce an expression for the deflecting torque in terms of the rate of change of inductance with position of moving system, and calculate the deflecting torque at scale readings of 1 A. and 2 A. (*L.U.*, 1925.)

[*Ans.* 0·028 gm.-cm., 0·053 gm.-cm.]

5. An alternating current voltmeter with a maximum scale reading of 50 V. has a resistance of 500 O. and an inductance of 0·09 H. The magnetizing coil is wound with 50 O. of copper wire and the remainder of the circuit is a series non-inductive resistance. What additional apparatus is needed to make this instrument read correctly both on direct current and alternating-current circuits of 50 frequency? (*L.U.*, 1924.)

[*Ans.* A condenser of capacity 0·185 μF. connected in parallel with the series resistance.]

6. Obtain an expression for the law of the electrostatic voltmeter in terms of the strength of the control and the rate of change of capacity with deflection. Point out the requirements for a uniform scale, and show that in order to gain the maximum sensitivity for a low reading voltmeter the clearances between the vanes should be as small as possible. (*L.U.*, 1924.)

[*Ans.* For uniform scale: $dC/d\theta = k/\theta$, or $C = k \log_e \theta$, where C is the capacity corresponding to the deflection θ. See also p. 391.]

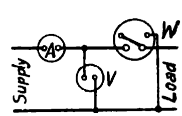
7. Describe, giving appropriate sketches, the construction of the shielded-pole type of induction ammeter, pointing out the advantages and disadvantages of this type over the other alternating-current ammeters. Upon what general conditions does the production of the necessary rotating or sliding field depend? Show how these conditions are satisfied in the instrument described.

Give a vector diagram showing the relation between the current and the fluxes. How will the torque exerted on the moving system vary with the current and with the frequency? (*L.U.*, 1924–5.)

8. Discuss the causes of inaccuracy, other than the power consumed within the instrument, in a moving-coil dynamometer type (electrodynamic) wattmeter.

Under what conditions are these errors most important, and how can they be reduced by careful design? How can allowances be made for the residual errors? (*L.U.*, 1924.)

9. A wattmeter is connected up as in the accompanying diagram to measure the power supplied to an alternating-current circuit.



The following readings were taken: $A = 2{\cdot}4$, $V = 200$, $W = 22$. The resistance of the pressure circuit of the wattmeter is 8,000 O., that of the current coil 0·46 O., and that of the voltmeter 2,000 O. What are the true watts supplied to the load, and what is its power factor? Neglect the inductances of the instruments. (*L.U.*, 1924.)

[*Ans.* 17 W.]

10. State the correction factor for a dynamometer (electro-dynamic) type of wattmeter. Give a method of measuring accurately the power in a circuit in which the power factor is $< 0{\cdot}1$. (*C. & G.*, 1923.)

11. The following data refer to an electro-dynamic wattmeter having a current range of 3 A. and a pressure range of 2,500 V.

Resistance of pressure-coil circuit, 25,000 O.; inductance of pressure coil circuit, 0·532 μH.; resistance of fixed coils, 5·5 O.

Determine the error when this instrument is used on a 100 cycle circuit at power factors of (*a*) 0·9, and (*b*) 0·15 (lagging), the current being 3 A.

[*Ans.* (*a*) $6{\cdot}4 \times 10^{-5}$ per cent, (*b*) $8{\cdot}85 \times 10^{-4}$ per cent.]

12. An electro-dynamic wattmeter has a shunt coil with a resistance of 750 O. and a series resistance of 2,250 O. A condenser of 1 μF. capacity is arranged so that it can be shunted across the series resistance. If two readings of the wattmeter are taken, W_1 without the condenser shunt and W_2 with the condenser shunt connected, determine a formula by which the power factor of the circuit in which the power is being measured can be found in terms of these readings. Frequency 50 cycles. (*L.U.*, 1922.)

[*Ans.* $\tan \varphi = 2{\cdot}12 - 1{\cdot}955\ W_2/W_1$]

13. Describe the construction and give the principles of operation of an induction type watt-hour (or energy) meter. Point out the arrangements made to obtain accuracy at varying power factors, and discuss the extent to which the accuracy is dependent upon constancy of frequency. (*L.U.*, 1925.)

14. Describe, with sketches, some type of power factor meter for use on single-phase circuits, giving an account of the theory of the instrument. Discuss the effects of variations of voltage and frequency on the accuracy of the readings. (*L.U.*, 1924–5.)

15. What precautions are necessary to secure accuracy in the design of three-phase power-factor indicators ? What are the special difficulties that occur in making a successful single-phase instrument ? (*L.U.*, 1917.)

16. In what senses is the term " power factor " used in connection with a three-phase unbalanced supply system ? What is the correct definition under these circumstances ? Describe an instrument for measuring the power factor of such a system, giving a diagram of connections and pointing out wherein the instrument satisfies, or fails to satisfy, the theoretically correct conditions. (*L.U.*, 1924.)

17. Describe, with sketches and diagrams, the construction of a good form of oscillograph for delineating the wave shape of an alternating current, and state what precautions must be taken in the design of the instrument if an accurate record of the wave shape is to be obtained. (*L.U.*, 1924.)

18. Describe, with sketches, the construction of a modern vibration galvanometer. What are its advantages and disadvantages for uses in alternating-current bridge measurements compared with a telephone receiver. Explain the effect of damping on the sensitivity of the galvanometer. (*L.U.*, 1924.)

19. Describe two methods, not involving the uses of a wattmeter, for measuring the power expended in an alternating-current circuit. What special precautions must be taken when using these methods if an accurate result is to be obtained ? Prove any formula used. (*L.U.*, 1924.)

20. Prove the law of the quadrant electrometer for both steady and alternating voltages, and show how this instrument may be used to measure the inductance of an ammeter shunt or other low resistance of the order of 0·001 O. (*L.U.*, 1925.)

21. When using an electrostatic wattmeter to measure the dielectric loss in a leaky condenser at high voltage, the following readings were obtained: Voltage applied to the condenser, 30,000 V. ; voltage between the moving vanes of the wattmeter and the earthed end of the high voltage supply, 250 V.; voltage across the 130 O. resistance connected between the quadrants, 26·8 V.; deflection of the wattmeter (scale divisions), – 37 ; constant of the wattmeter, 3·52.

A step-up transformer was used to supply the 30,000 V. and the vanes of the wattmeter were connected to a tapping brought out from the secondary winding of this transformer near the earthed end.

Calculate the power factor and the dielectric loss of the condenser at 30,000 V.

Prove any formulæ used, and indicate the circumstances under which the wattmeter deflection is likely to be negative. (*L.U.*, 1924.)

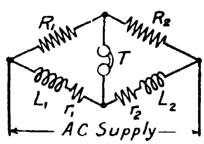
[*Ans* $P = 205{\cdot}6$ W. ; $\cos\varphi = 0{\cdot}0332$.]

Note. The deflection will be negative if the term in equation (221) representing the power loss in the series resistance R is greater than the power being measured.

22. Explain how an alternating pressure can be compared with a continuous pressure, and hence how an alternating-current voltmeter can ultimately be referred to the datum of the voltage of a standard cell. Describe clearly the steps and the kind of instruments necessary. (*C. & G.*, 1924.)

23. Why is it necessary when using an alternating-current potentiometer to maintain a source of supply of unvarying frequency ? Describe the instrument and its accessories, and compare the accuracy possible with it and with a direct-current potentiometer. (*L.U.*, 1924.)

24. Describe the wattmeter method of determining the iron losses in a sample of transformer stampings. Show how the copper loss in the magnetizing winding may be excluded from the wattmeter reading. Explain how the hysteresis and eddy-current losses may be separately determined from the test results. (*L.U.*, 1925.)



25. In a simple inductance bridge (shown in the accompanying diagram) intended for the measurement of inductance at telephone frequencies, trouble is frequently experienced owing to the residual reactance of the resistances. Show that if in the arms R_1, R_2, the ratio of resistance to residual reactance is the same for each, no error is introduced. (*L.U.*, 1924.)

26. An alternating-current bridge is arranged as follows : The arms AB and BC consist of non-inductive resistances of 100 O., the arms BE and CD of non-inductive variable resistances, the arm EC of a condenser of capacity 1 μF., the arm DA of an inductive resistance. The alternating-current source is connected to A and C and the telephones to E and D. A balance is obtained when the resistance of the arm CD is 50 O., and the arm BE 2,500 O.

Calculate the resistance and inductance of the arm DA.

If there are harmonics in the wave-form of the source of alternating current, what will be the effect ? (*L.U.*, 1925.)

[*Ans.* $L = 0{\cdot}255$ H., $R = 50$ O.]

27. Explain, with a diagram of connections, an electrical method of measuring the frequency of an alternating current in terms of the capacity of a condenser and the mutual inductance between two circuits, using a telephone or vibration galvanometer as an indicator. What precautions must be taken if this method is to be accurate for telephonic frequencies ? (*L.U.*, 1924.)

IX.—TRANSFORMERS AND MISCELLANEOUS (CHAPTERS XIV AND XVI.)

1. Describe a method of determining experimentally the ratio and phase angle of an instrument potential transformer. Explain the principle of the method.

Assuming a non-inductive load, discuss the variation of ratio and phase angle with variation of voltage and frequency. (*L.U.*, 1925.)

2. Develop a simple expression for the pressure-drop in a transformer. If the copper loss in a transformer is 1 per cent of the full load output at unity

power factor, and the inductive pressure is 3 per cent of the normal pressure, find the regulation at full-load when the power factor is 0·8. (*C. & G.*, 1923.)

[*Ans.* (i) Pressure drop = $I(R \cos \varphi + X \sin \varphi)$, where R and X are equivalent resistance and reactance, respectively, of the transformer, and $\cos \varphi$ is the power factor of the load (ii) 2·6 per cent.]

3. Explain what is meant by (*a*) equivalent resistance, (*b*) leakage reactance of a transformer. Calculate these values from the following test figures obtained on the primary side of a step-down transformer, the secondary being short-circuited. Applied P.D. 40 V., current 60 A., power input 800 W. (*C. & G.*, 1925.)

[*Ans.* Equivalent resistance (referred to primary) 0·222 O. Equivalent reactance (referred to primary) = 0·63 O]

4. A 50-cycle single-phase transformer has a ratio of transformation of 5 to 1 and a full-load secondary current of 200 A. The resistance and reactance of the primary are 0·8 O. and 2·5 O. respectively, and the corresponding values for the secondary are 0·04 O. and 0·1 O. If the low-voltage winding is short-circuited, what voltage must be applied to the other winding so that full-load current may be obtained in the former? Neglect the no-load current. (*L.U.*, 1923.)

[*Ans.* 212 V.]

5. A coil of inductance 20 mH. and resistance 2·2 O. has applied to it an alternating E.M.F. of 50 frequency. Near the coil there is an eddy-current path of inductance 0·5 μH. and resistance 50 μO. The mutual inductance between this path and the coil is 10 μH.

Find the percentage change in the equivalent resistance and inductance of the coil caused by the eddy currents. (*L.U.*, 1924.)

[*Ans.* Per cent increase in equivalent resistance = 0·83; per cent decrease in equivalent inductance = 0·91.

HINT. The solution is obtained from the equations connecting (i) the primary and secondary currents, (ii) the E.M.Fs. in the primary circuit. Thus—

$$\text{(i)}\ \dot{I}_2 Z_2 = -j\omega M\ \dot{I}_1,$$
$$\text{(ii)}\ \dot{E} - j\omega M\ \dot{I}_2 = \dot{I}_1 \dot{Z}_1,$$

where the subscripts 1, 2, refer to the primary (coil) and secondary (eddy-current path) circuits respectively, and M is the mutual inductance between them.

6. A condenser of 10 μF. capacity charged to 20,000 V. is suddenly discharged through an inductance of 0·001 H. Find the maximum current and the frequency of the resultant oscillation. (*L.U.*, 1924.)

[*Ans.* I_m = 2000 A., f = 1592. NOTE $i = E\sqrt{(C/L)} \sin (t/\sqrt{(LC)})$.]

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